Spectral reconstruction of an inclusive rate in the two-dimensional O(3) model

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Lattice '21
Zoom/Gather@MIT, Boston, MA
Jul. 27th, 2021

Scattering amplitudes from spectral functions

JB, M. T. Hansen, Phys. Rev. D100 (2019)

For the process $\gamma^* \to \pi + \pi$

Consider the spectral function:

$$C(t) = \langle \pi(\boldsymbol{p}_1) | \hat{\pi}(\boldsymbol{p}_2) \mathrm{e}^{-\hat{H}t} \, \hat{J}_{\mathrm{em}}(0) | 0 \rangle$$
$$= \int_0^\infty \mathrm{d}\omega \, \mathrm{e}^{-\omega t} \rho_{\boldsymbol{p}_1 \boldsymbol{p}_2}(\omega)$$

'Smear' with a particular kernel of width ϵ

$$\rho_{\epsilon, \boldsymbol{p}_1 \boldsymbol{p}_2}(E) = \int_0^\infty \mathrm{d}\omega \, K_\epsilon(E - \omega) \, \rho_{\boldsymbol{p}_1 \boldsymbol{p}_2}(\omega)$$
$$K_\epsilon(x) = \frac{\epsilon}{x^2 + \epsilon^2} + \frac{ix}{x^2 + \epsilon^2} = \frac{i}{x + i\epsilon}$$

Scattering amplitudes from spectral functions

Apply LSZ reduction:

$$_{\mathrm{out}}\langle \pi(\boldsymbol{p}_1)\pi(\boldsymbol{p}_2)|\hat{J}_{\mathrm{em}}(0)|0\rangle = \frac{2E(p_2)}{Z_\pi^{1/2}} \lim_{\epsilon\to 0^+} \epsilon \rho_{\epsilon,\boldsymbol{p}_1\boldsymbol{p}_2}(E(\boldsymbol{p}_2))$$

Spectral function has an on-shell pole, which must be ‘amputated’

- No reliance on finite-volume, applicable above arbitrary thresholds 😁
- Energies/momenta fixed with varying L and a 😁
- Generalizes to arbitrary (inclusive, exclusive) amplitudes (in principle) 😁
- Requires large volume, solution of ill-posed inverse problem 😟

Spectral Reconstruction

Backus, Gilbert ‘68, ‘70
F. Pijpers, M. Thompson ‘92
M. R. Hansen, A. Lupo, N. Tantalo, PRD99 (2019)

Linear ansatz:

$$\hat{\rho}_\epsilon(E) = \sum_{t=a}^{t_{\max}} q_t(\epsilon, E)\, C(t), \qquad \hat{\delta}_\epsilon(E,\omega) = \sum_{t=a}^{t_{\max}} q_t(\epsilon, E)\, \mathrm{e}^{-\omega t}$$

Two criteria when choosing $\{q_t(\epsilon, E)\}$

- Accuracy: $$A[q] = \int_{E_0}^{\infty} \mathrm{d}\omega \left\{ \delta_\epsilon(E-\omega) - \hat{\delta}_\epsilon(E,\omega) \right\}^2$$
- Precision: $$B[q] = \mathrm{Var}\left\{ \rho_\epsilon(E) \right\}$$

Optimal coeffs minimize:

$$G_\lambda[q] = (1-\lambda) A[q] + \lambda B[q]$$

Controlled Test

JB, M. W. Hansen, M. T. Hansen, (M. R. Hansen), A. Patella, N. Tantalo, *in prep.*

2d O(3)-model: …, M. Lüscher, U. Wolff, Nucl. Phys. B339 (1990),...

$$S[\sigma] = -\beta \sum_{x,\mu} \sigma(x) \cdot \sigma(x+\hat{\mu}), \qquad \sigma(x) \in \mathbb{R}^3, |\sigma(x)| = 1$$

Conserved (global) current:

$$j_\mu^a = \beta \epsilon^{abc} \sigma^b(x) \hat{\partial}_\mu \sigma^c(x)$$

Massive single-particle states. Target process: inclusive rate for $j \to X$

$$\rho(E) = \sum_\alpha \delta(\boldsymbol{P}_\alpha)\delta(E - E_\alpha) \left| {}_{\text{out}}\langle \alpha | \hat{j}(0) | 0 \rangle \right|^2$$

Controlled Test

Spectral function from current-current correlator:

$$C(t) = \int \mathrm{d}\boldsymbol{x} \, \langle j_1^a(x) j_1^a(0) \rangle = \int_0^\infty \mathrm{d}\omega \, \mathrm{e}^{-\omega t} \rho(\omega)$$

All (even) particle-number sectors contribute

$$\rho(E) = \sum_{n=1}^{\infty} \rho^{(2n)}(E), \qquad \rho^{(2n)}(E) = 0 \,\mathrm{for}\, E < 2nm$$

Identical analysis: R-ratio from vector-vector correlator in QCD!

$$\rho(\omega) = \frac{R(\omega)}{12\pi^2}, \qquad R(\omega) = \frac{\sigma(\mathrm{e}^+\mathrm{e}^- \to \mathrm{hadrons})}{4\pi\alpha_{\mathrm{em}}(\omega)^2/(3\omega^2)}$$

Controlled Test

ID	$(L/a) \times (T/a)$	am	mL	mT
A1	640×320	0.076669(20)	49	25
A2	640×320	0.0623644(66)	40	20
A3	640×320	0.0453605(96)	29	15
A4	1280×640	0.0259551(49)	33	17
A5	1920×960	0.0176980(47)	34	17
A6	2880×1440	0.0112925(45)	33	16
B1	160×320	0.0623644(66)	10	20
B2	320×320		20	
B3	480×320		30	
B4	640×320		40	
B5	800×320		48	
B6	960×320		58	
C1	320×160	0.0623644(66)	20	10
C2	320×240			15
C3	320×320			20
C4	320×480			30

Simulations with Wolff two-cluster algorithm:

- A1-A6: continuum limit
- B1-B6: finite-L effects
- C1-C4: finite-T effects

Controlled Test

Five smearing kernels $\delta_\epsilon(E-\omega)$:

$$\delta_\epsilon^{\mathsf{g}}(x) = \frac{1}{\sqrt{2\pi}\epsilon}\exp\left[-\frac{x^2}{2\epsilon^2}\right], \qquad \delta_\epsilon^{\mathsf{c0}}(x) = \frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2},$$

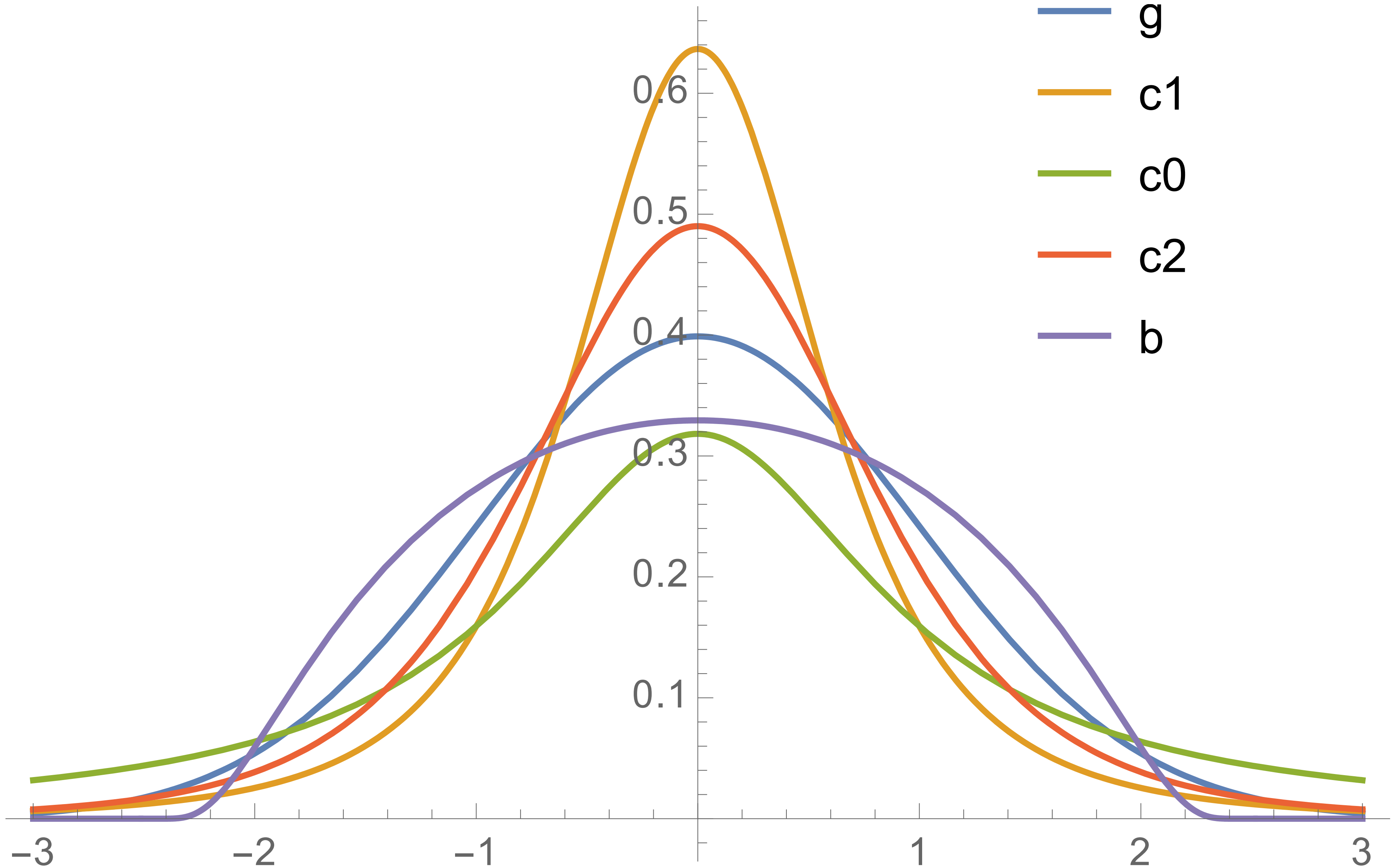
$$\delta_\epsilon^{\mathsf{c1}}(x) = \frac{2}{\pi}\frac{\epsilon^3}{(x^2+\epsilon^2)^2}, \qquad \delta_\epsilon^{\mathsf{c2}}(x) = \frac{8}{3\pi}\frac{\epsilon^5}{(x^2+\epsilon^2)^3},$$

$$\delta_\epsilon^{\mathsf{b}}(x) = \frac{1}{N\epsilon}b(x/\epsilon), \quad b(y) = \begin{cases} \exp\left(-\frac{1}{1-y^2}\right), & -1<y<1, \\ 0, & \text{otherwise}, \end{cases}$$

Width of ‘b’ and ‘c2’ adjusted to coincide with Gaussian second moment.

Controlled Test

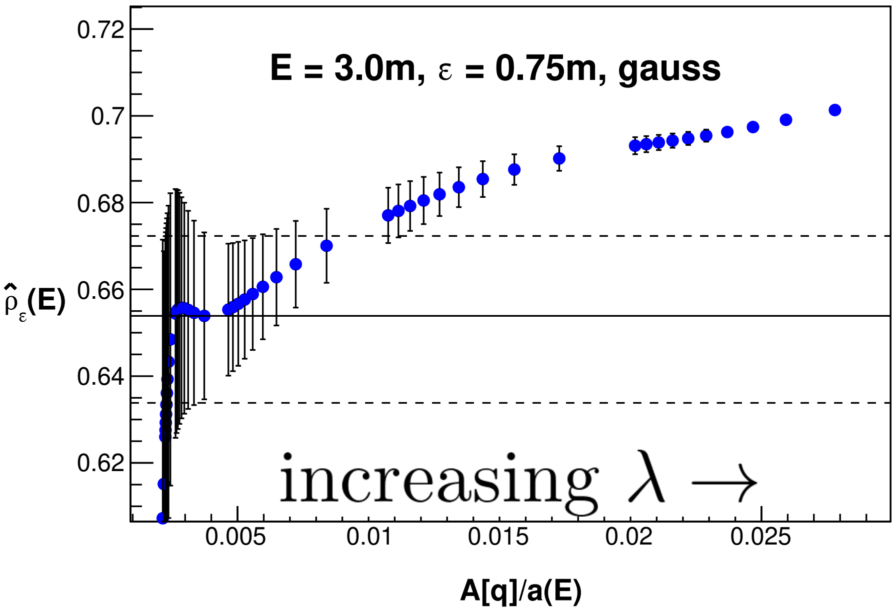
Five smearing kernels $\delta_\epsilon(E-\omega)$:



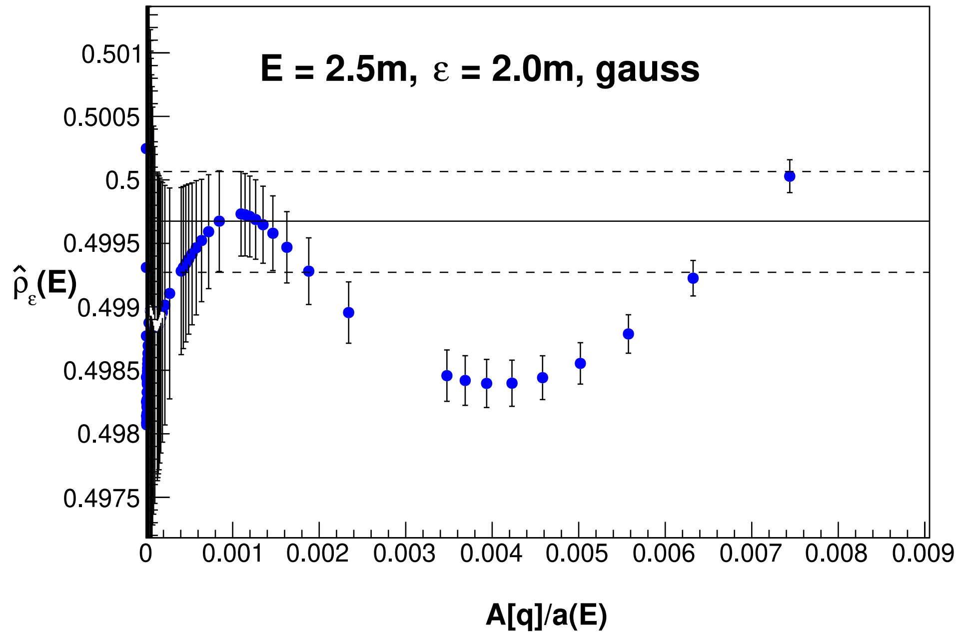
Spectral Reconstruction

$$G_\lambda[q] = (1-\lambda)A[q] + \lambda B[q]$$

Tradeoff parameter (λ) balances systematic (A) and statistical (B) error



‘Difficult’



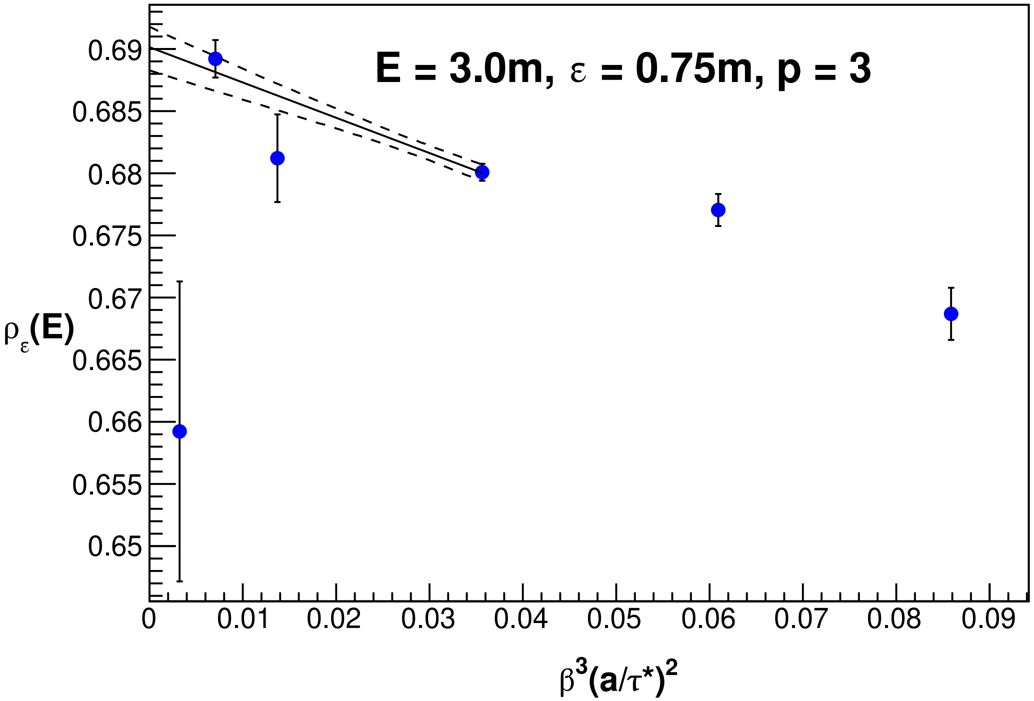
‘Easy’

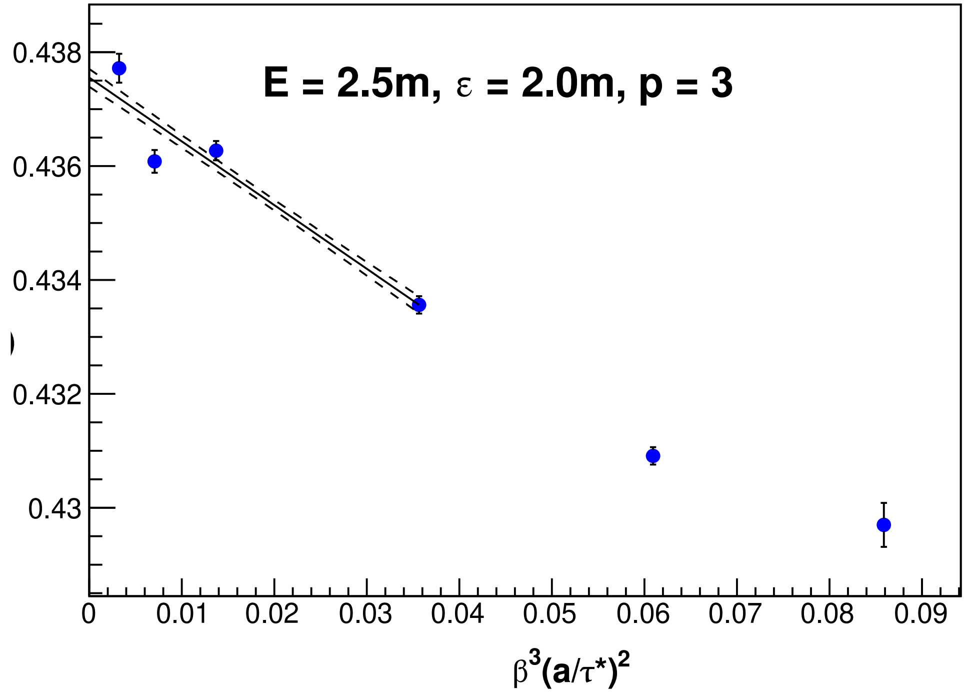
Plateau indicates statistics-limited regime, automatically selected. No further sys. error estimate.

Continuum Limit

Long History! For spectral quantities: … , J. Balog, F. Niedermayer, P. Weisz, Nucl. Phys. B824 (2010)

$$\lim_{a\to 0} E(a) = E^{\rm phys} + A\beta^3 a^2 (1 + \frac{r}{\beta} + \frac{c}{\beta^2} + \dots)$$



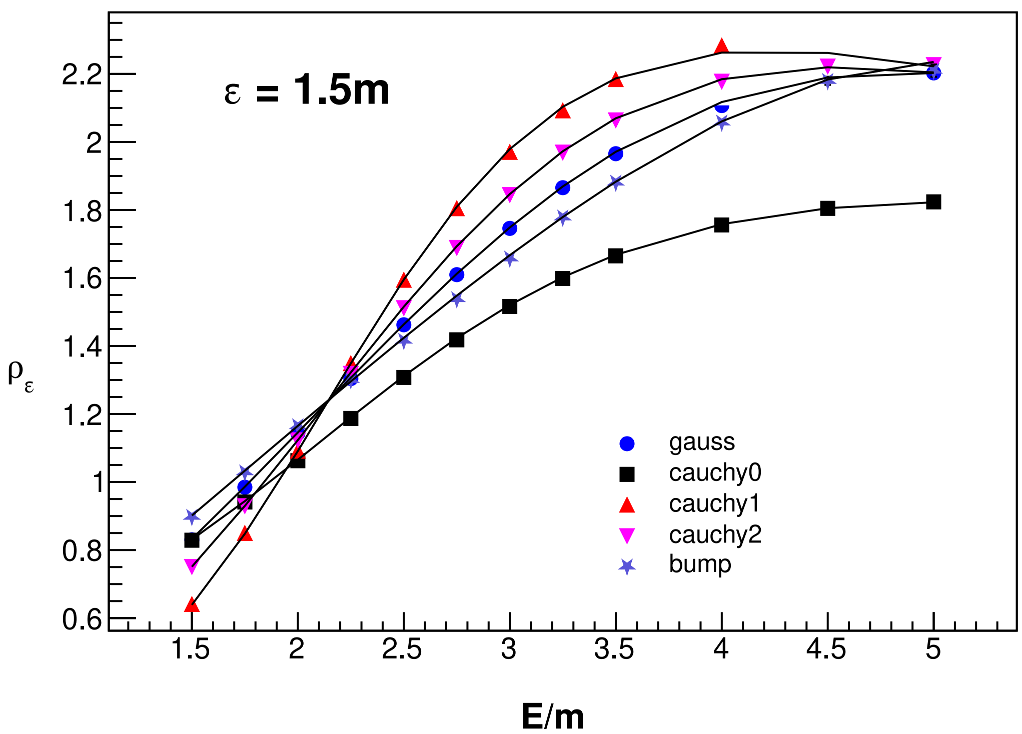


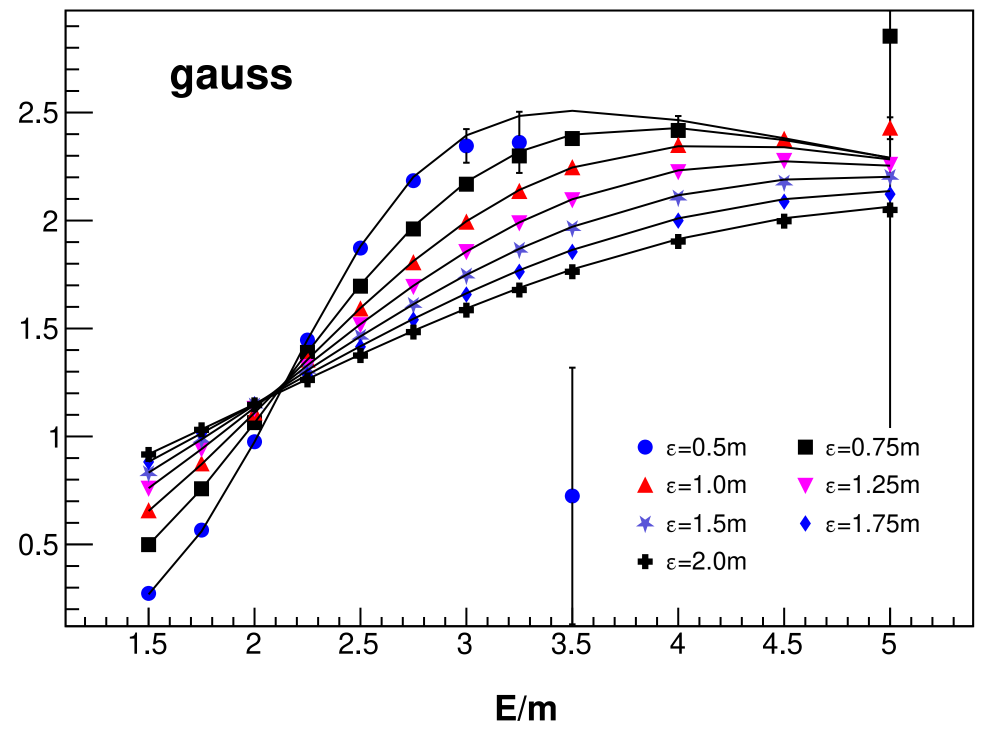
Not applicable here.

Sys. error estimate from three (arbitrary) fit forms: $\beta^p (a/m)^2, \quad p = 0,\, 3,\, 6$

Results: fixed smearing width

PRELIMINARY: systematic error estimates not finalized.





Solid lines: exact smeared spectral function, using N=2 and 4 particle contributions.

Results: extrapolation to zero width

All kernels have the same $O(\epsilon^2)$ coefficient (up to a sign):

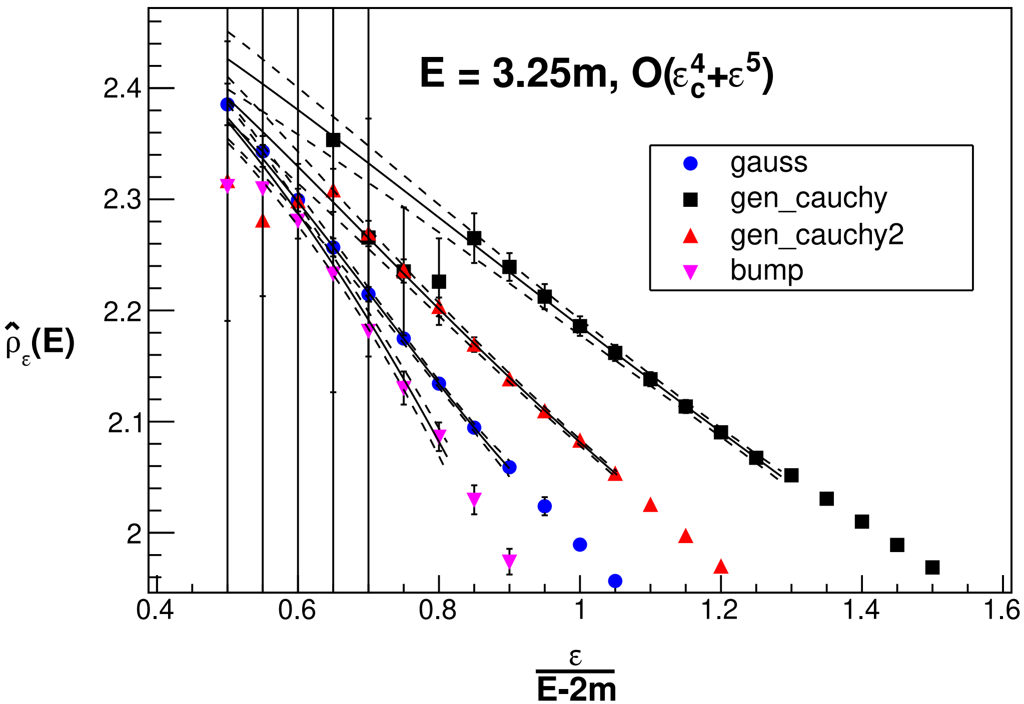
$$\rho_\epsilon^{\mathsf{x}}(E) = \rho(E) + \sum_{k=1}^{\infty} w_k^{(\mathsf{x})} a_k(E) \epsilon^k \,,$$

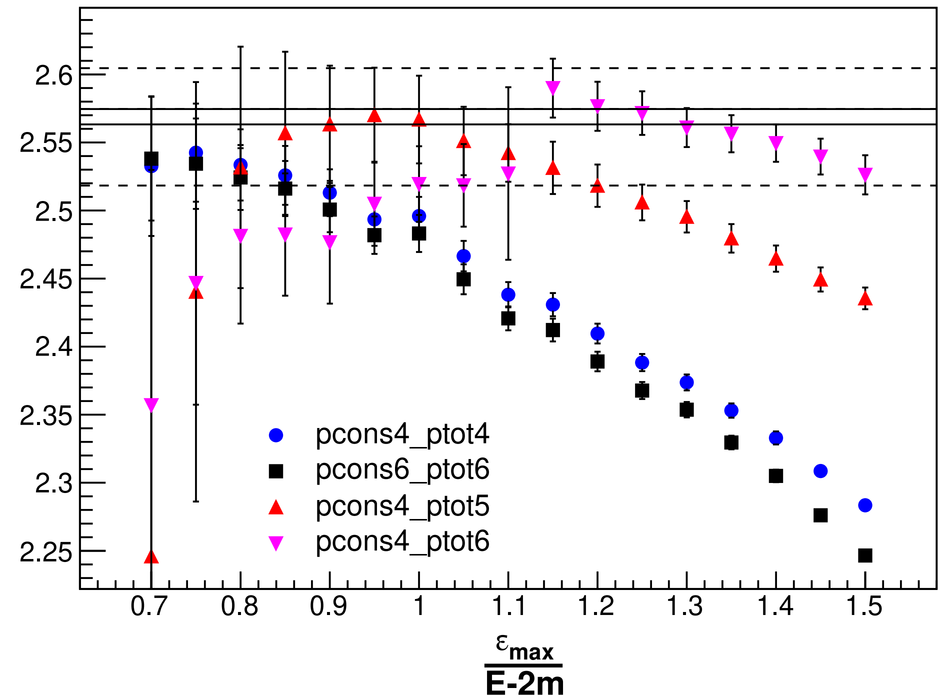
x	$w_1^{(\mathsf{x})}$	$w_2^{(\mathsf{x})}$	$w_3^{(\mathsf{x})}$	$w_4^{(\mathsf{x})}$
g	0	-1	0	1
c0	1	1	1	1
c1	0	-1	-2	-3
c2	0	-1	0	9
b	0	-1	0	2.119

Expansion coefficients worked out to arbitrary orders. (Numerically for ‘bump’)

Results: extrapolation to zero width

Fits to determine $\{a_k(E)\}$:

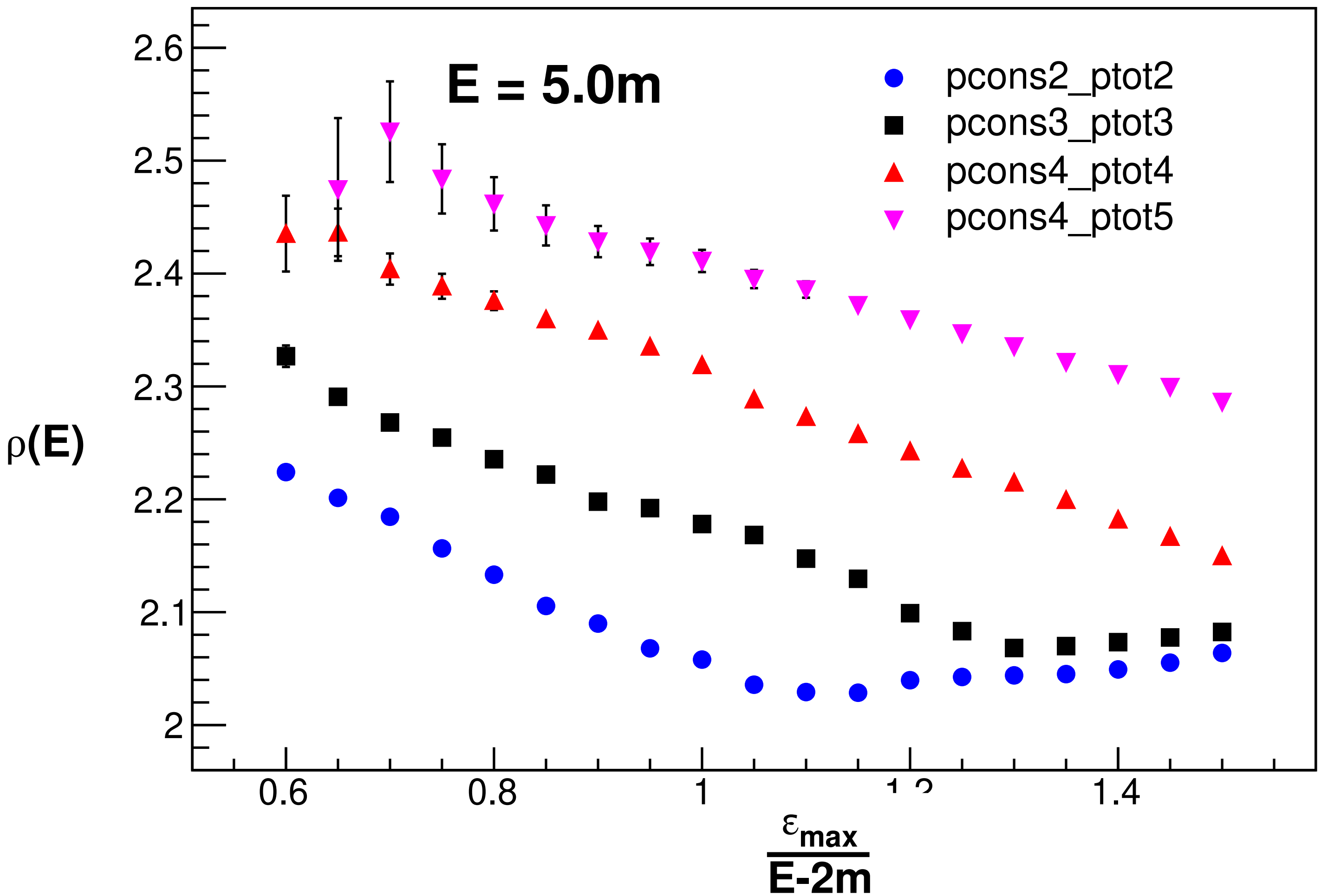




'Cauchy0' kernel not useful due to linear approach.

Results: extrapolation to zero width

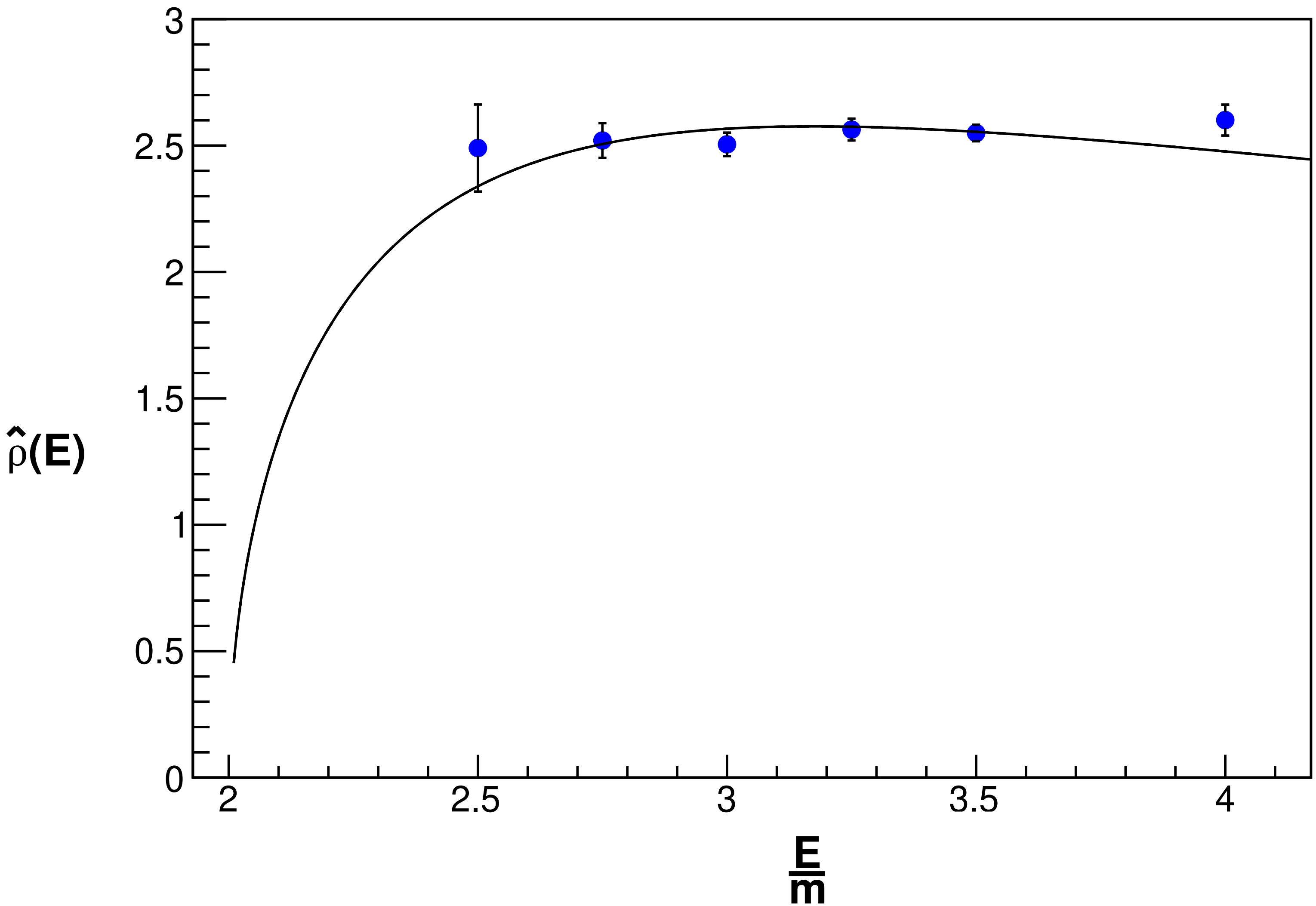
Fits to determine $\{a_k(E)\}$:



Large dependence on extrap. form above E=4m

Results: extrapolation to zero width

Elastic region only:



Extrapolation systematics not yet controlled above E=4m. Below E=2.5m requires too small widths.

Conclusions

- Spectral functions are an (in principle) alternative to the finite-volume approach to scattering amplitudes. Different way to construct observables in lattice field theory!
- Our approach to the inverse problem is controlled and flexible, works with arbitrary smearing kernels and basis functions. More work to effective take eps→0 limit.
- Stay tuned for exclusive amplitudes in the 2d O(3) model!
- Application to QCD:
  - ➔ Signal-to-noise and precision similar to vector-vector correlator
  - ➔ Masterfield paradigm ideal: don't miss Marco Ce + Patrick Fritzsch talks (Friday)