Minimally doubled fermions and topology in 2D

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Work in collaboration with Johannes H. Weber:
arXiv:2003.10803 [PRD] and forthcoming

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In memoriam Artan Boriçi (19.4.1965-25.3.2021)

One-month overlap at PSI in October 1999
→ Artan got me interested in overlap fermions

Three-week visit to JSC in early summer 2009
→ Artan got me interested in minimally doubled fermions

http://www.gazetadita.al/covid-i-merr-jeten-fizikantit-dhe-anetarit-te-akademise-se-shkencave-artan-borici

Quick intro: Naive and Wilson fermions

- **Naive fermions**

$$D_{\text{nai}}(x,y) = \sum_\mu \gamma_\mu \nabla_\mu(x,y) + m\delta_{x,y}$$

$$D_{\text{nai}}(p) = \mathrm{i}\sum_\mu \gamma_\mu \frac{1}{a}\sin(ap_\mu) + m$$

$$= \mathrm{i}\sum_\mu \gamma_\mu \bar{p}_\mu + m \quad \text{with} \quad \bar{p}_\mu \equiv \frac{1}{a}\sin(ap_\mu)$$

- **Wilson fermions**

$$D_{\text{W}}(x,y) = \sum_\mu \gamma_\mu \nabla_\mu(x,y) - \frac{ar}{2}\sum_\mu \triangle_\mu(x,y) + m\delta_{x,y}$$

$$D_{\text{W}}(p) = \mathrm{i}\sum_\mu \gamma_\mu \frac{1}{a}\sin(ap_\mu) + ar\sum_\mu \{1-\cos(ap_\mu)\} + m$$

$$= \mathrm{i}\sum_\mu \gamma_\mu \bar{p}_\mu + \frac{ar}{2}\sum_\mu \hat{p}_\mu^2 + m \quad \text{with} \quad \hat{p}_\mu \equiv \frac{2}{a}\sin(\frac{ap_\mu}{2})$$

Quick intro: Karsten-Wilczek and Borici-Creutz fermions

- **Karsten-Wilczek fermions**

$$D_{\mathrm{KW}}(x,y) = \sum_\mu \gamma_\mu \nabla_\mu(x,y) - \mathrm{i}\frac{ar}{2}\gamma_4 \sum_{i=1:3} \triangle_i(x,y) + m\delta_{x,y}$$

$$D_{\mathrm{KW}}(p) = \mathrm{i}\sum_\mu \gamma_\mu \frac{1}{a}\sin(ap_\mu) + \mathrm{i}ar\gamma_4 \sum_{i=1:3}\{1-\cos(ap_i)\} + m$$

$$= \mathrm{i}\sum_\mu \gamma_\mu \bar{p}_\mu + \mathrm{i}\frac{ar}{2}\gamma_4 \sum_{i=1:3} \hat{p}_i^2 + m$$

- **Borici-Creutz fermions**

$$D_{\mathrm{BC}}(x,y) = \sum_\mu \gamma_\mu \nabla_\mu(x,y) - \mathrm{i}\frac{ar}{2}\sum_\mu \gamma'_\mu \triangle_\mu(x,y) + m\delta_{x,y}$$

$$D_{\mathrm{BC}}(p) = \mathrm{i}\sum_\mu \gamma_\mu \bar{p}_\mu + \mathrm{i}ar\sum_\mu \gamma'_\mu\{1-\cos(ap_\mu)\} + m$$

$$= \mathrm{i}\sum_\mu \gamma_\mu \bar{p}_\mu + \mathrm{i}\frac{ar}{2}\sum_\mu \gamma'_\mu \hat{p}_\mu^2 + m \quad \text{with} \quad \gamma'_\mu \equiv \Gamma\gamma_\mu\Gamma, \Gamma \equiv \frac{1}{\sqrt{d}}\sum_\mu \gamma_\mu$$

Testbed Schwinger Model: QED in 2D with any N_f

$\mathrm{SM}(N_f\!=\!0)$ simulated with Metropolis/overrelax/instanton-hit/P-hit [arXiv:1203.2560 and T. Eichhorn Thu 13:00 Algo.], topological charge autocorrelation $O(1)$ at any β.

Wilson gauge action per site:

$s_{\text{wil}}(x) = 1-\mathrm{Re}(U(x)) = 1-\cos(\theta(x))$

Plaquette at position $x=(x_1,x_2)$:

$U(x) = U_1(x)U_2(x+e_1)U_1^\dagger(x+e_2)U_2^\dagger(x)$

$U(x) = \exp(\mathrm{i}\theta(x))$

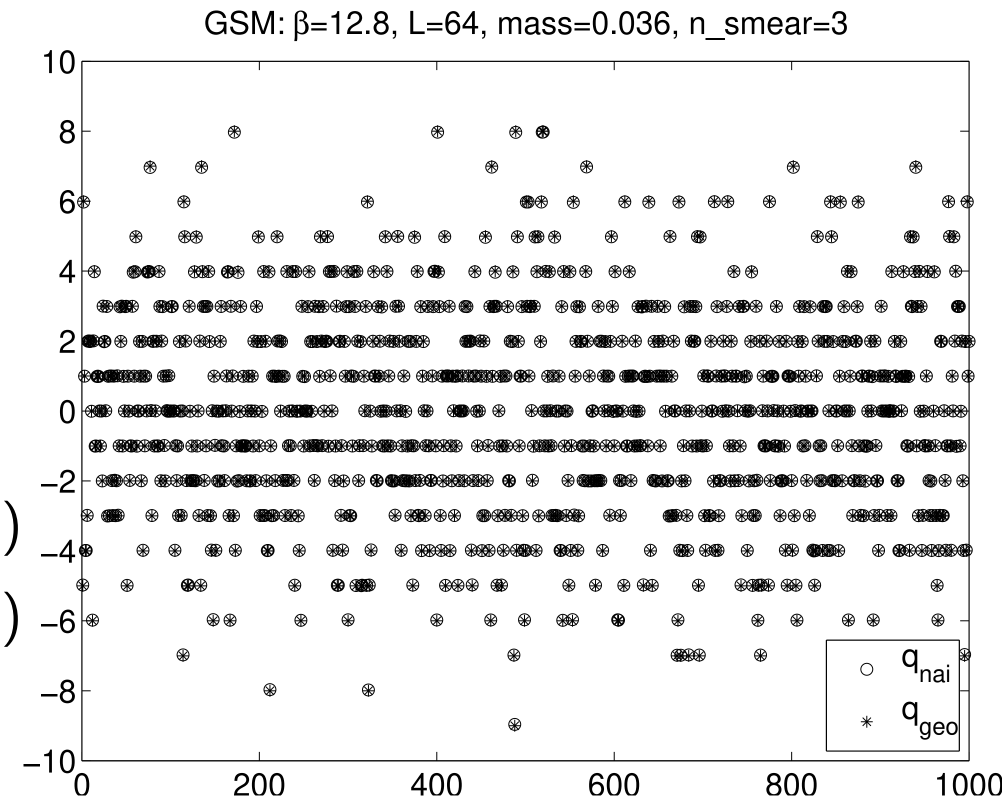
Two gluonic topological charges:

$q_{\text{nai}}^{(3)} = \sum \sin(\theta^{(3)}(x))/(2\pi) \in \mathbf{R}$ ("naive")

$q_{\text{geo}}^{(3)} = \sum \theta^{(3)}(x)/(2\pi) \in \mathbf{Z}$ ("geometric")

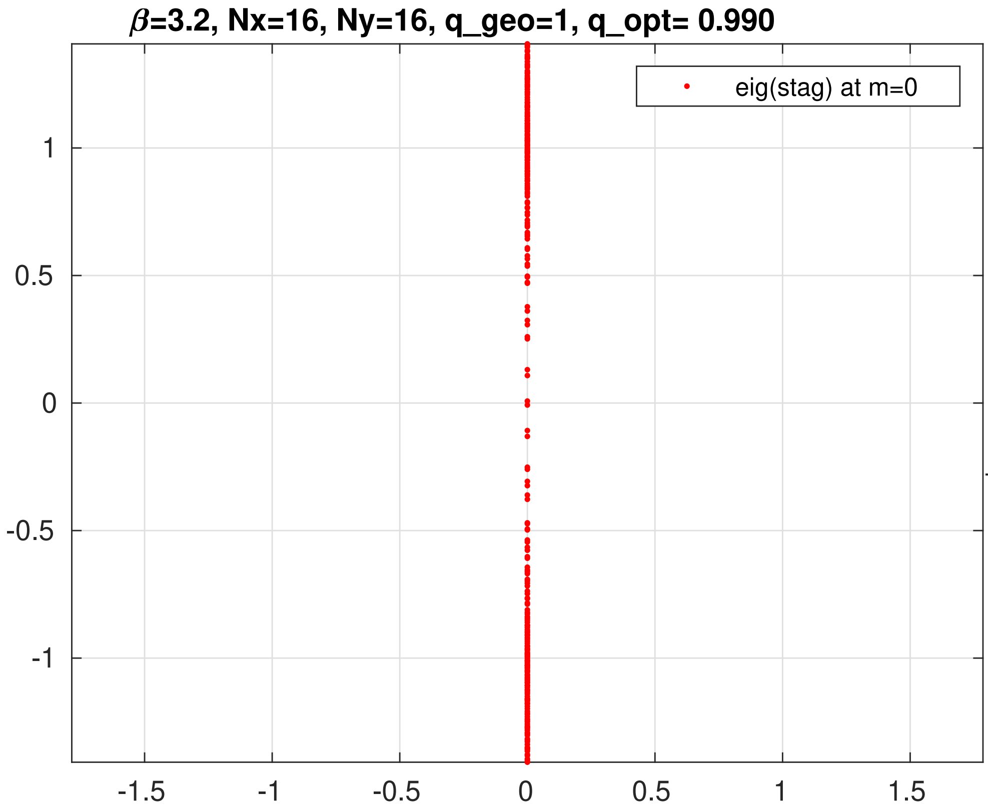
$\theta^{(3)}$ plaquette angle after 3 smearings

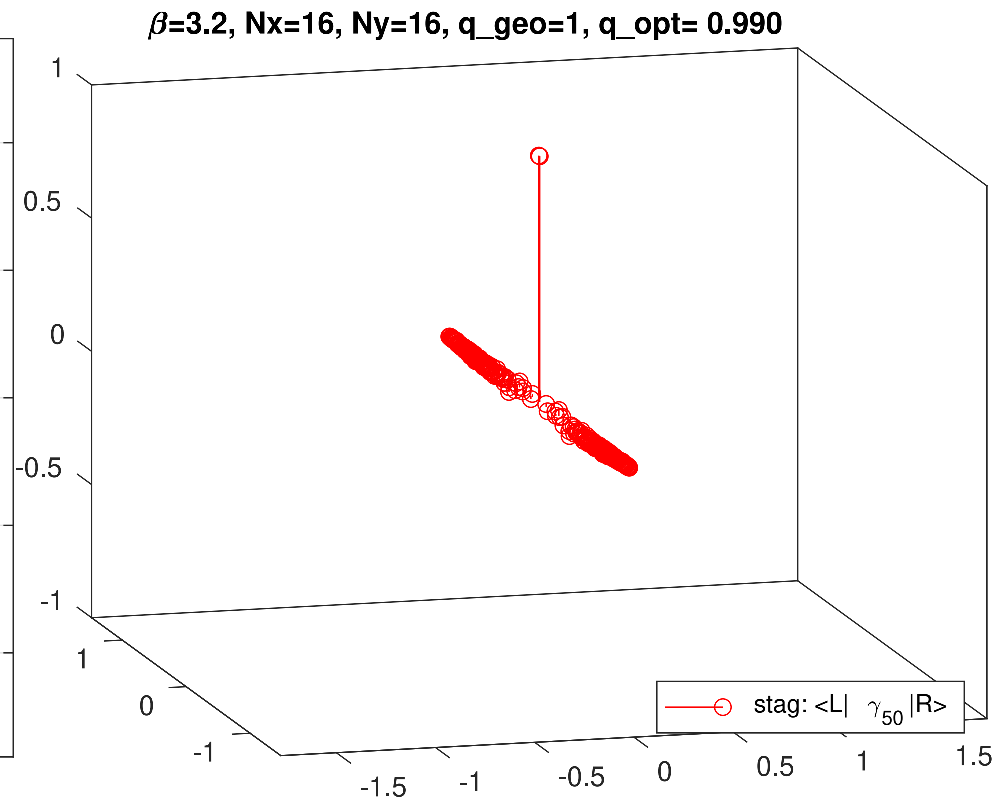
$q_{\text{opt}}(x)$ is clover-leaf version of $q_{\text{nai}}(x)$



All subsequent plots on one 16^2 configuration at $\beta=3.2$ to keep PDF leightweight. All fermion operators use 1 step of $\rho=0.25$ stout-smearing [Morningstar Peardon 2003].

Eigenvalues and topology with staggered fermions





$2|q|$ would-be zero-modes (changes to $4|q|$ in 4D), remnant chiral symmetry $U(1)_\epsilon$

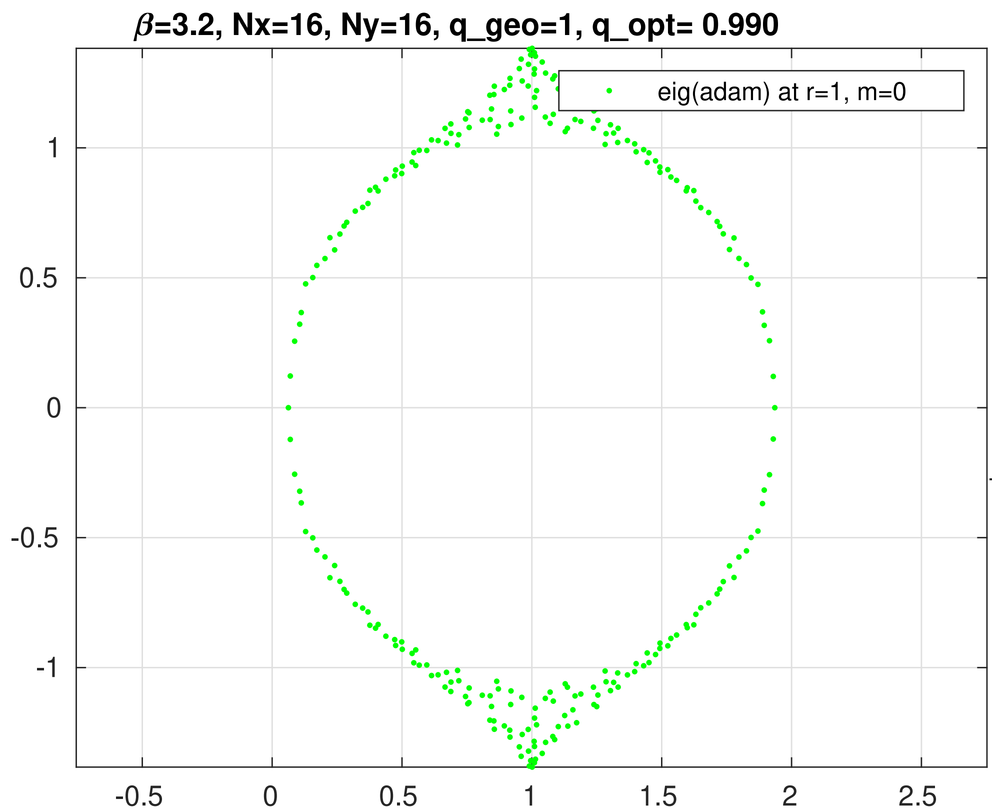
$\epsilon \equiv \gamma_5 \otimes \xi_5$ not sensitive to topology (see "backup pages" for meaning of $\gamma_\mu \otimes \xi_\nu$)

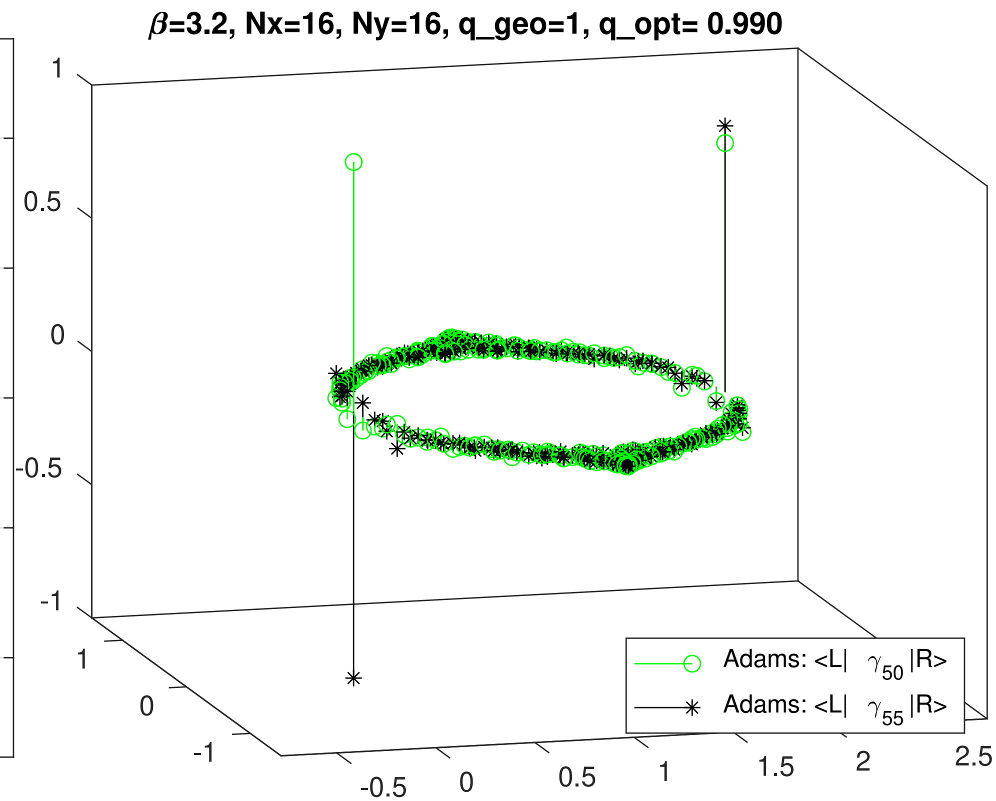
$\Gamma_5 \simeq \gamma_5 \otimes 1$ crafted to "turn around" chirality of wrong mode (both point upwards)

$\Xi_5 \simeq 1 \otimes \xi_5$ not sensitive to topology (Γ_5 and Ξ_5 depend on gauge-field U)

$1 \equiv 1 \otimes 1$ not sensitive to topology

Eigenvalues and topology with Adams fermions





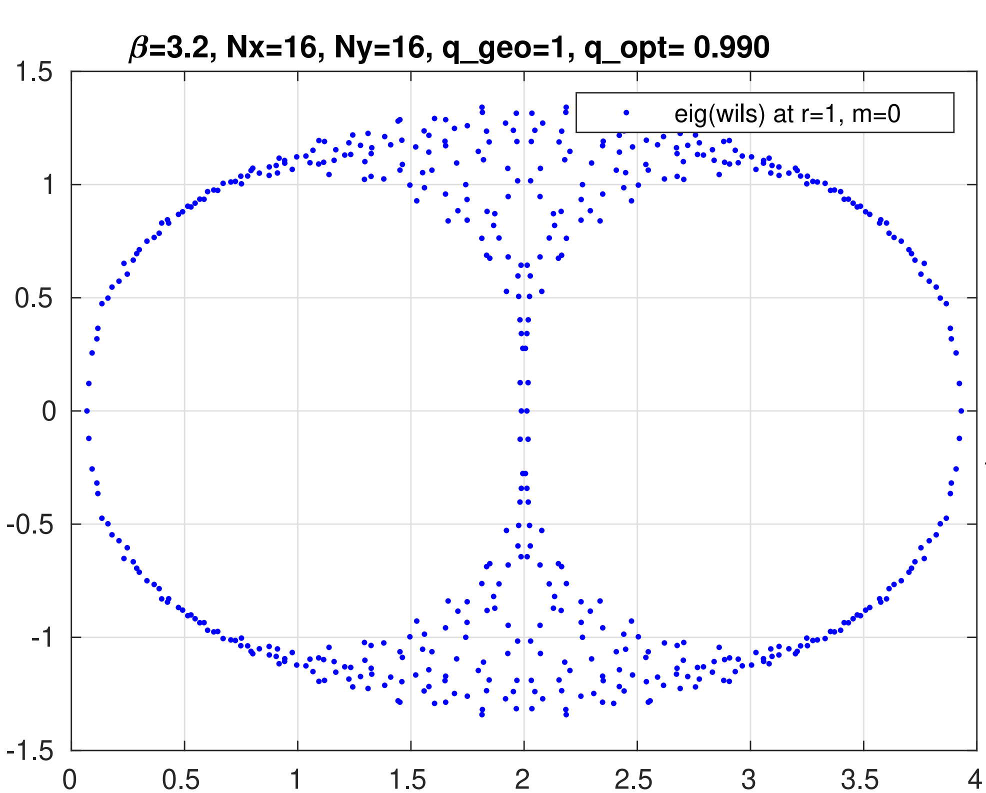
$$D_{\text{Adams}} = \sum_\mu \eta_\mu \nabla_\mu + r[1 \otimes \xi_5 + 1 \otimes 1] \quad \begin{pmatrix} \text{respects } \epsilon\text{-hermiticity} \\ \text{breaks chiral symmetry} \end{pmatrix}$$

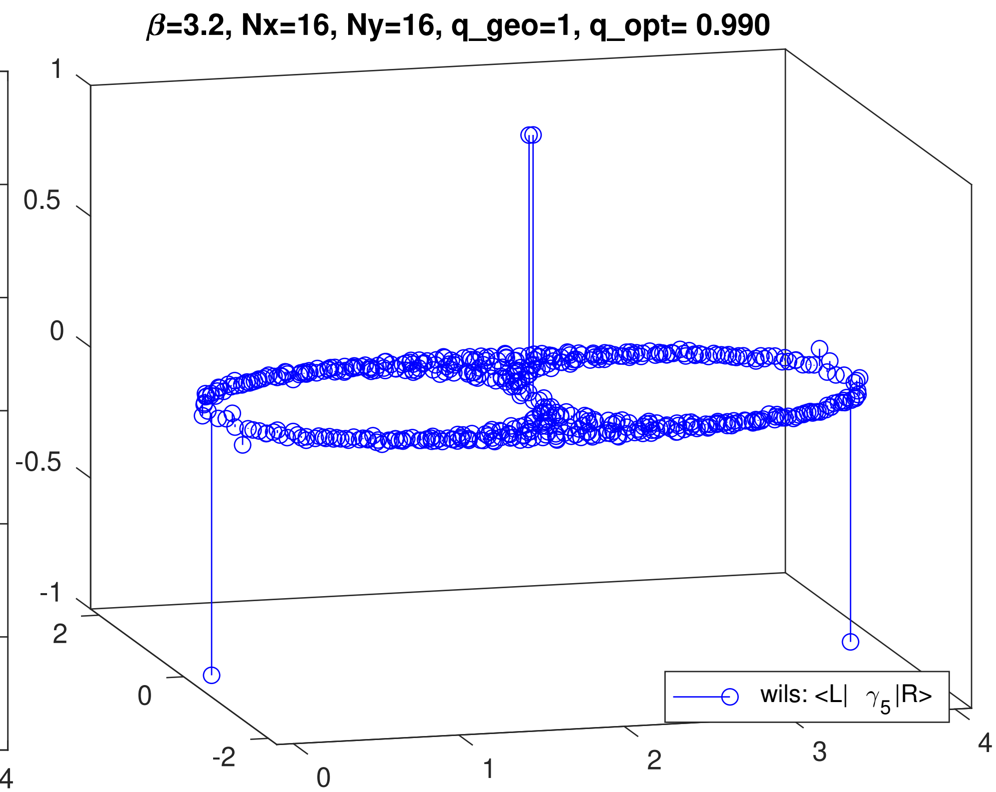
- $|q|$ would-be zero-modes in physical branch (become $2|q|$ in 4D) [Adams 2009]

$\Gamma_5 \simeq \gamma_5 \otimes 1$ still produces 2 upward pointing modes (one physical, one doubler)

$\epsilon \equiv \gamma_5 \otimes \gamma_5$ now produces 2 oppositely oriented modes (one physical, one doubler)

Eigenvalues and topology with Wilson fermions

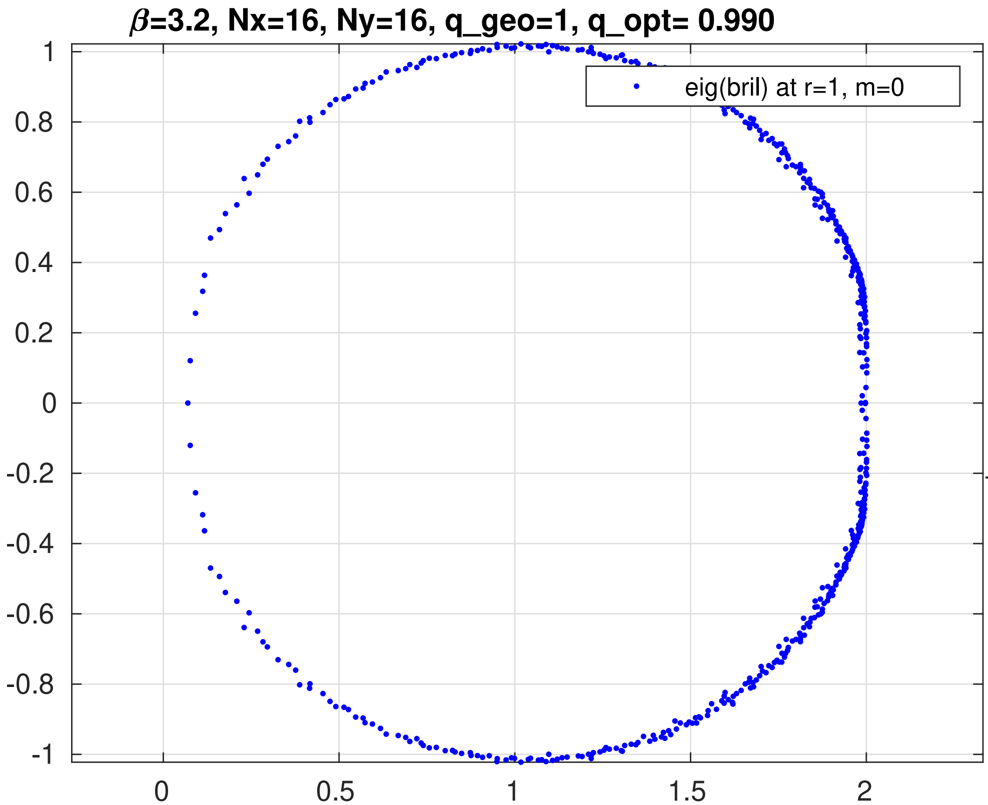


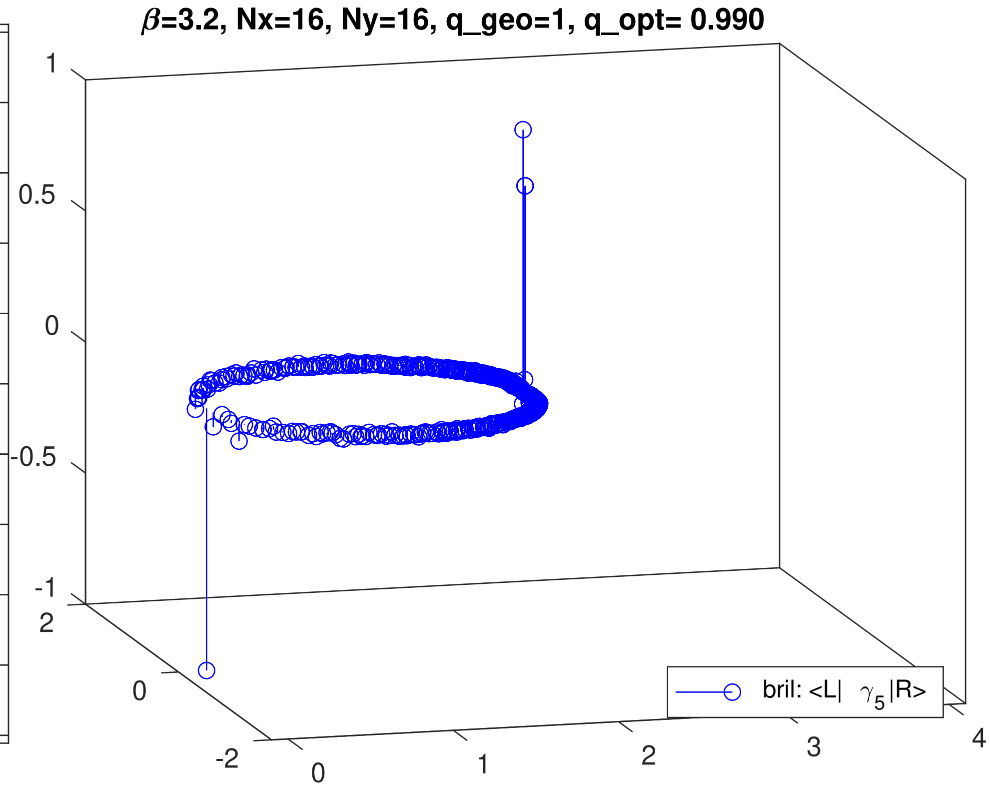


- $|q|$ would-be zero-modes in physical branch (unchanged in 4D), additive mass shift
- L/R-eigenmode sandwich $\langle L|.|R\rangle$ for non-chiral D [Hip et al 2001]

Proposal: central-branch fermions realize 2 species in 2D (6 in 4D) with same-chirality and remnant $U(1)$ symmetry [Misumi Yumoto 2005].

Eigenvalues and topology with Brillouin fermions



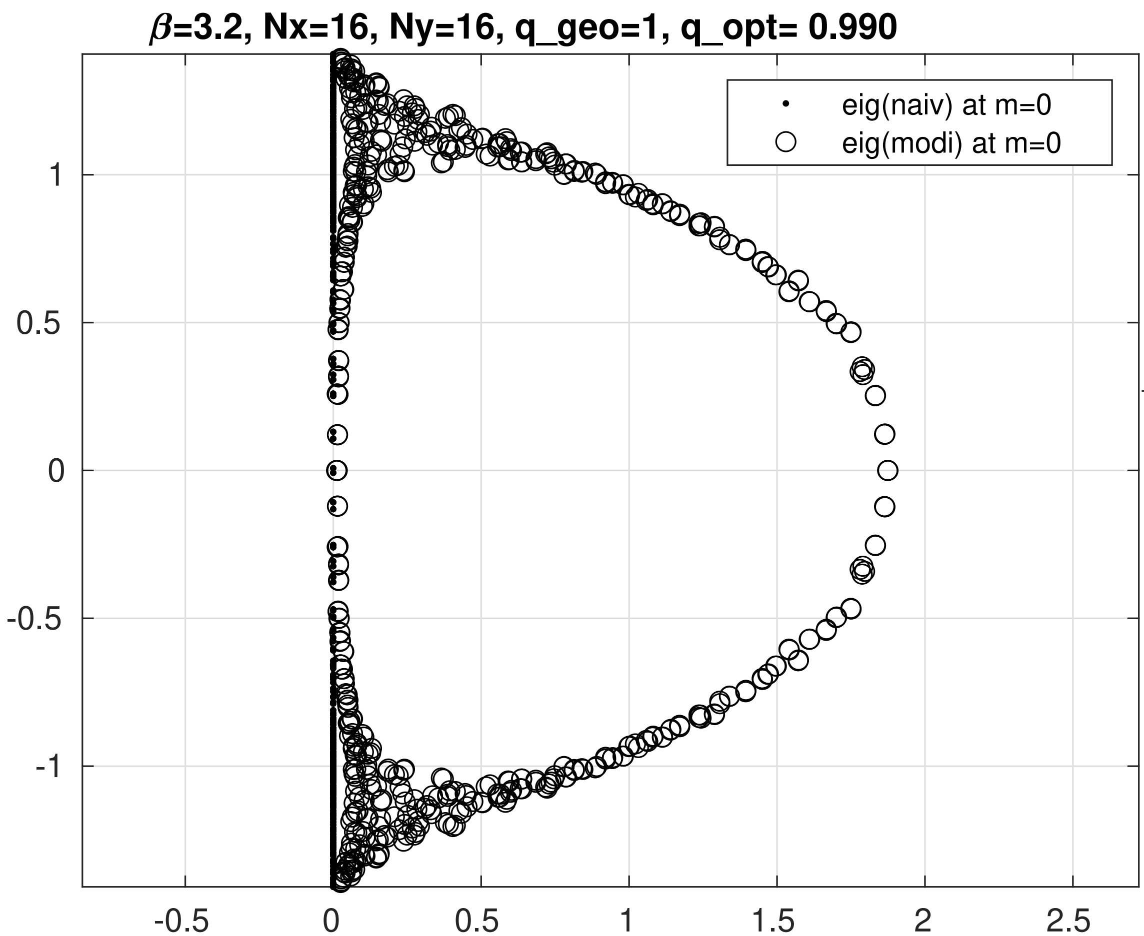


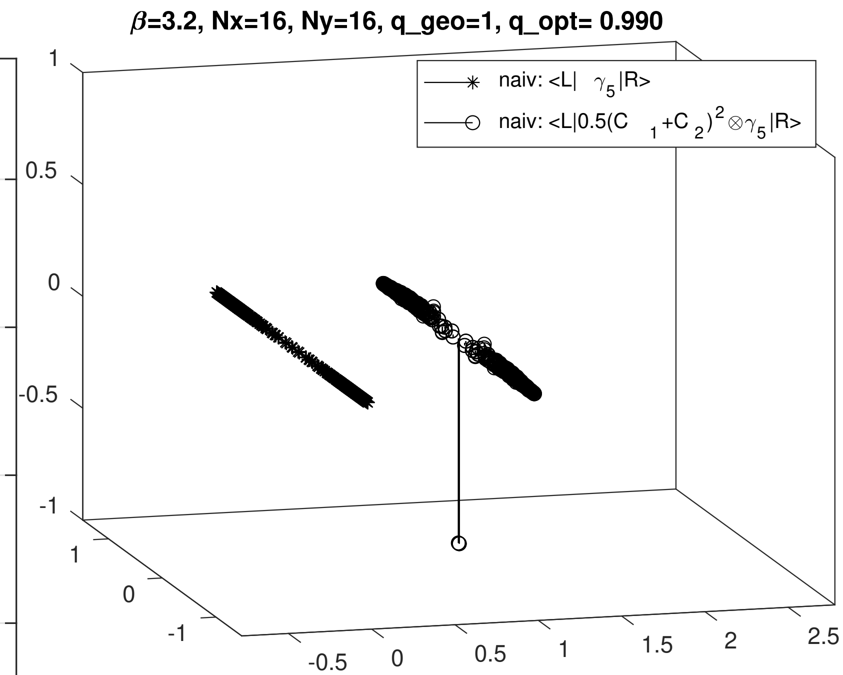
$D_{\rm B} = \sum_\mu \gamma_\mu \nabla_\mu - \frac{r}{2}\triangle$ like Wilson but ∇_μ and $\triangle$ with hypercubic stencil

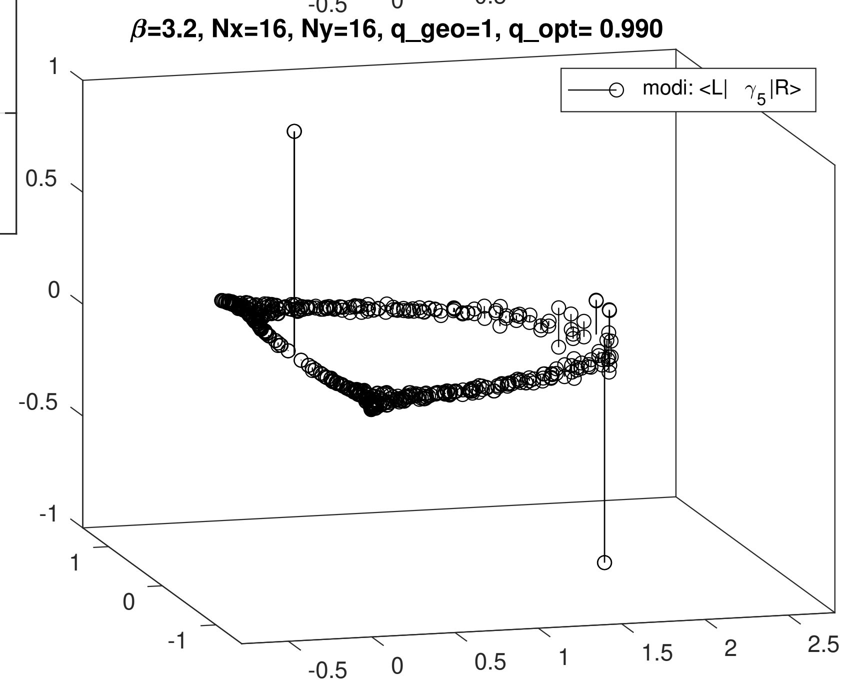
- $|q|$ would-be zero-modes in physical branch (unchanged in 4D), additive mass shift
- L/R-eigenmode sandwich $\langle L|.|R\rangle$ for non-chiral D, see [arXiv:1302.0773, PRD]

Proposal: use as overlap-kernel, already close to shifted-unitary [arXiv:1701.00726].

Eigenvalues and topology with naive fermions







naive: not sensitive to topology with γ_5

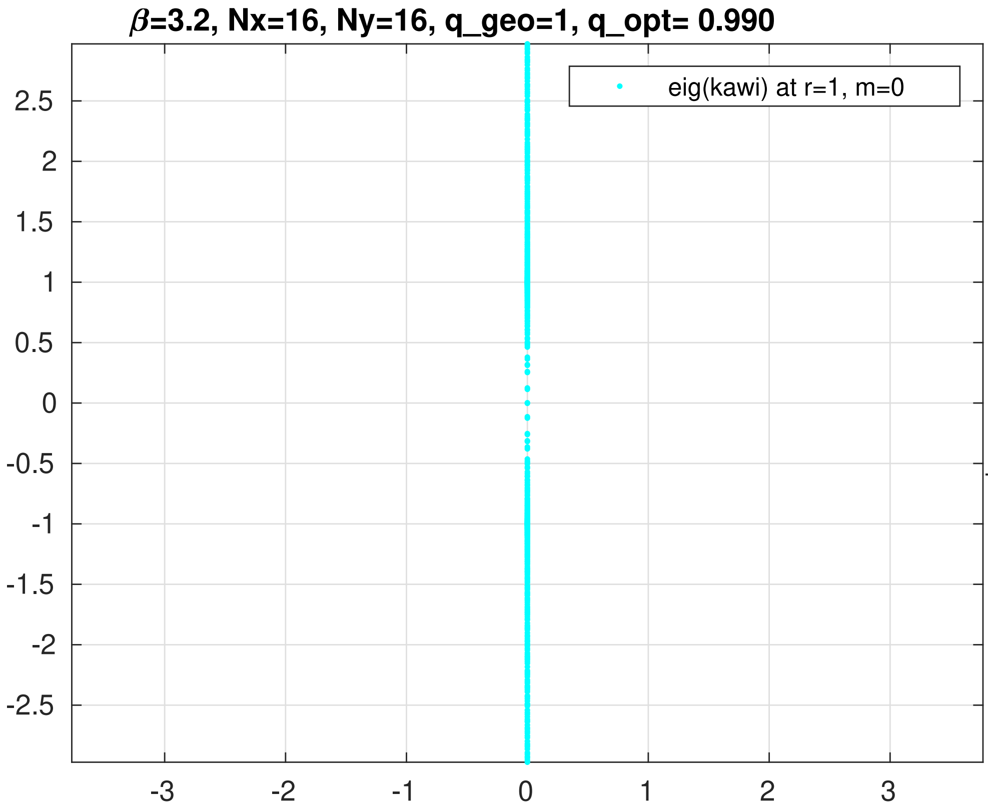
$D_\text{modi} = \sum_\mu \gamma_\mu \nabla_\mu + \frac{r}{2}(C_1 + C_2)^2 \otimes I$

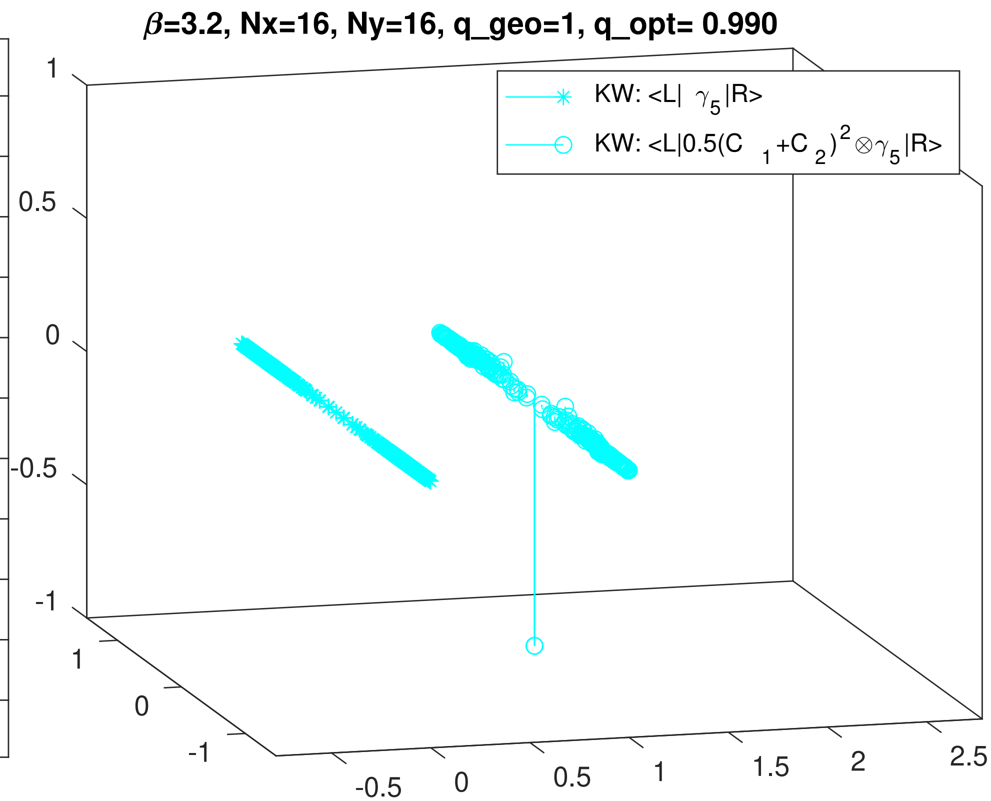
$2|q|$ right-chirality modes in physical branch

$2|q|$ wrong-chirality modes in doubler branch

$C_\mu \equiv \triangle_\mu + 2$ borrowed from Adams/Hölbling

Eigenvalues and topology with Karsten-Wilczek fermions

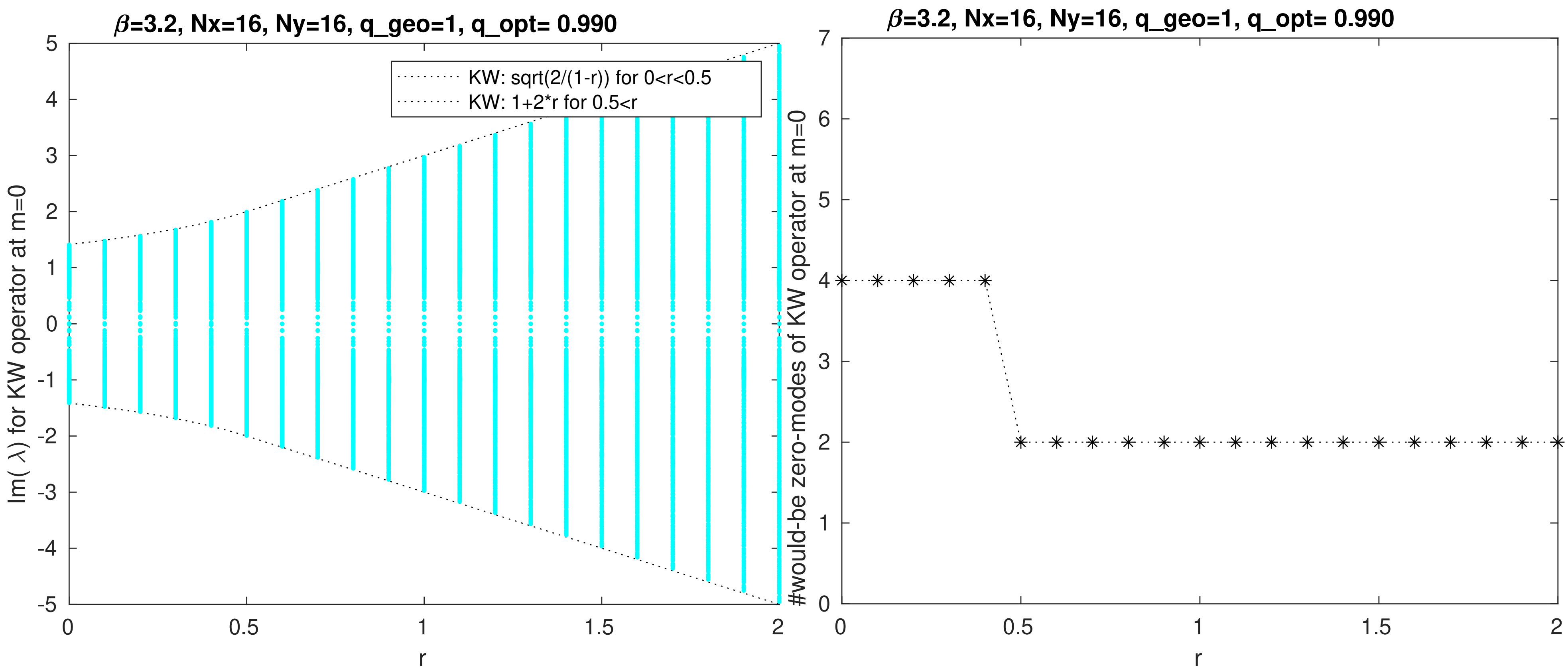




- $2|q|$ would-be zero-modes at $r=1$ (unchanged in 4D), remnant chiral symmetry
- pertinent L/R-eigenmodes of D_{KW} not sensitive to γ_5

Other operators X can be crafted to have $\langle L|X|R\rangle \neq 0$ with L/R-eigenmodes of D_{KW}; current favorite is $X=\frac{1}{2}(C_1+C_2)^2 \otimes \gamma_5$ (data shifted by $+1$).

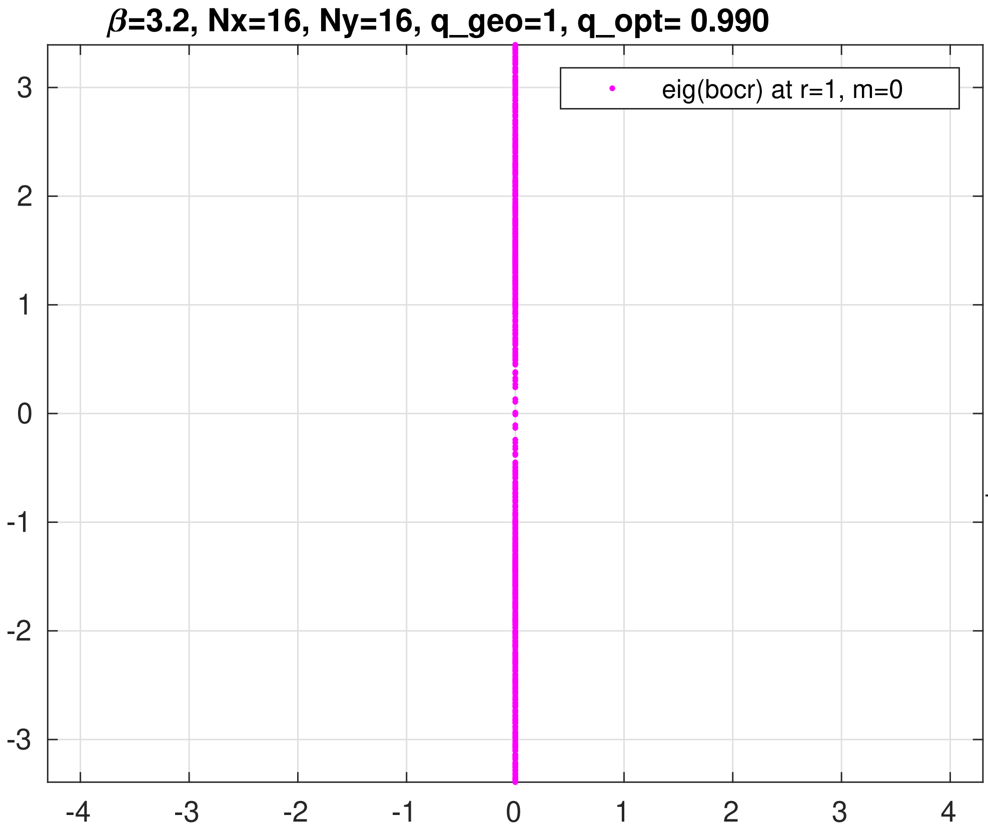
• Transition $D_{\rm naive} \to D_{\rm KW}$ as a function of r

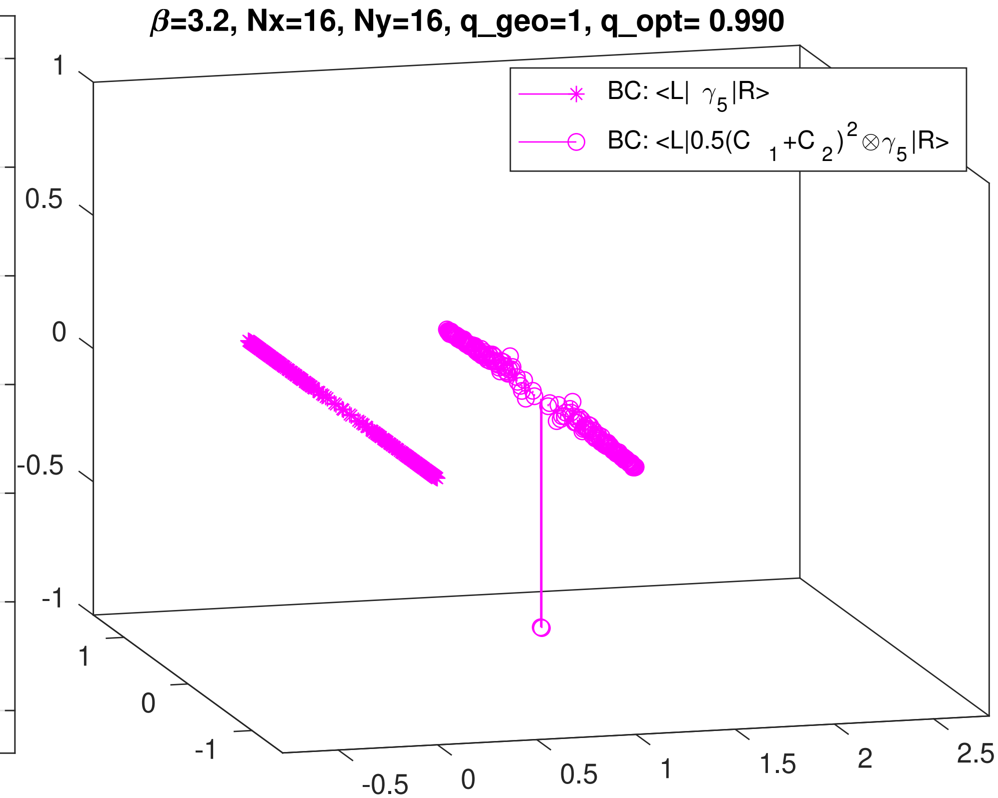


$\mathrm{Im}(\lambda_{\rm KW}(r))$ near-saturate free-field bound for KW-fermions [arXiv:2003.10803, PRD]

$D_{\rm KW}(r=0)$ is naive $\Longrightarrow$ number of would-be zero-modes evolves from $4|q|$ to $2|q|$

Eigenvalues and topology with Borici-Creutz fermions

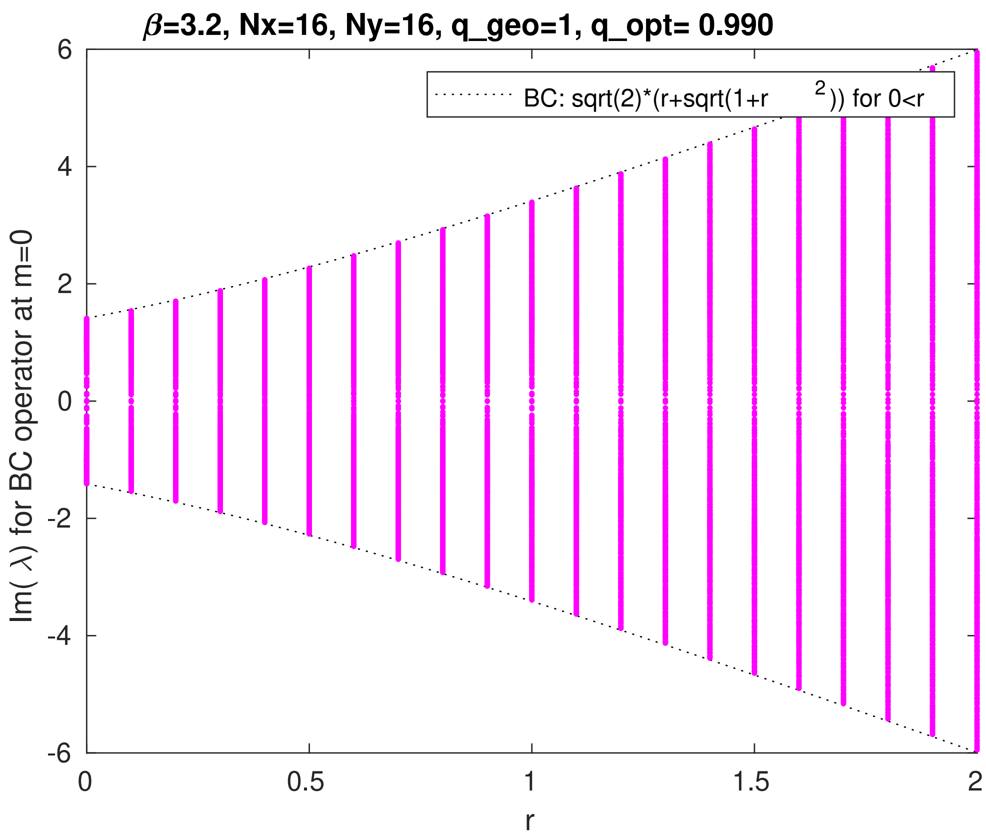


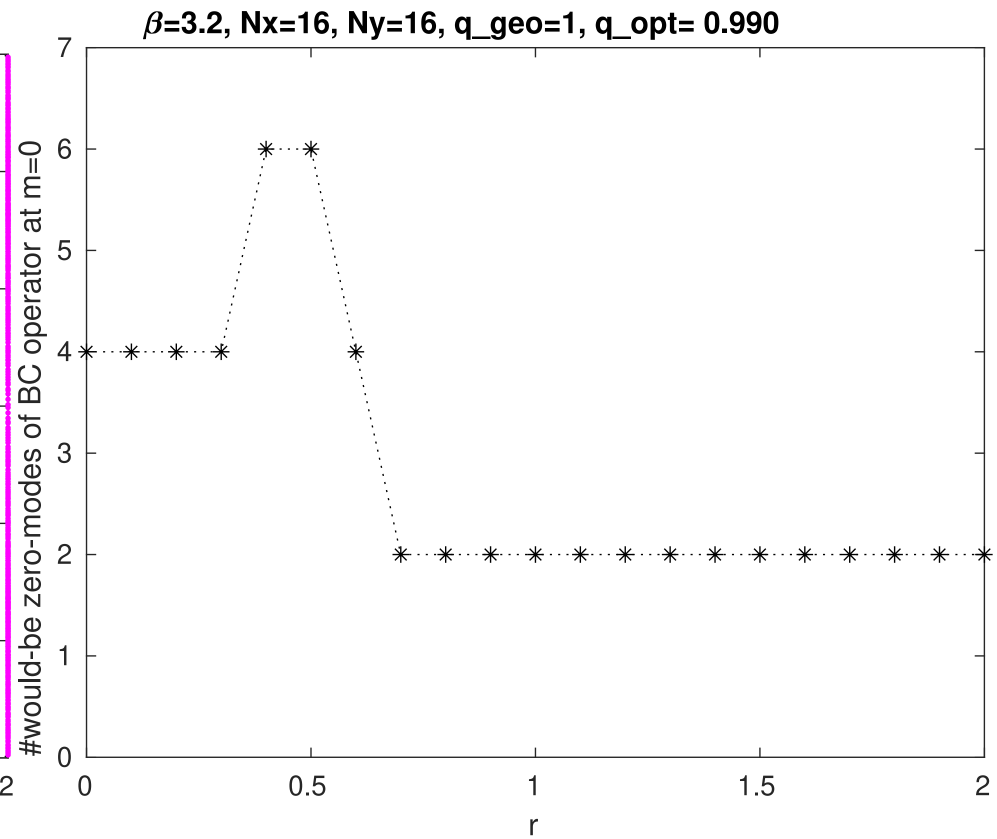


- $2|q|$ would-be zero-modes at $r = 1$ (unchanged in 4D), remnant chiral symmetry
- pertinent L/R-eigenmodes of D_{BC} not sensitive to γ_5

Other operators X can be crafted to have $\langle L|X|R\rangle \neq 0$ with L/R-eigenmodes of D_{BC}; current favorite is $X = \frac{1}{2}(C_1 + C_2)^2 \otimes \gamma_5$ (data shifted by $+1$).

- **Transition $D_{\text{naive}} \to D_{\text{BC}}$ as a function of r**





$\text{Im}(\lambda_{\text{BC}}(r))$ near-saturate free-field bound for BC-fermions [arXiv:2003.10803, PRD]

$D_{\text{BC}}(r=0)$ is naive $\Longrightarrow$ number of would-be zero-modes evolves from $4|q|$ to $2|q|$

Summary

On one gauge configuration with $|q|=1$ in 2D we find:

- 2 would-be zero-modes of D_{stag} with *opposite chiralities* (invisible to ϵ)
- 1 would-be zero-mode of D_{Adams} with *correct chirality*
- 1 would-be zero-mode of D_{wils} with *correct chirality*
- 1 would-be zero-mode of D_{bril} with *correct chirality*
- 4 would-be zero-modes of D_{naiv} with *opposite chiralities* (invisible to γ_5)
- 2 would-be zero-modes of D_{KW} with *opposite chiralities* (invisible to γ_5)
- 2 would-be zero-modes of D_{BC} with *opposite chiralities* (invisible to γ_5)
- 2 would-be zero-modes of D_{modi} with *correct chiralities* (Adams-like constr.)

KW and BC fermions have matrix size like Wilson fermions ($N_c 4 N_x N_y N_z N_t$ in 4D).

KW and BC fermions have exact chiral symmetry (eigenvalues on imaginary axis).

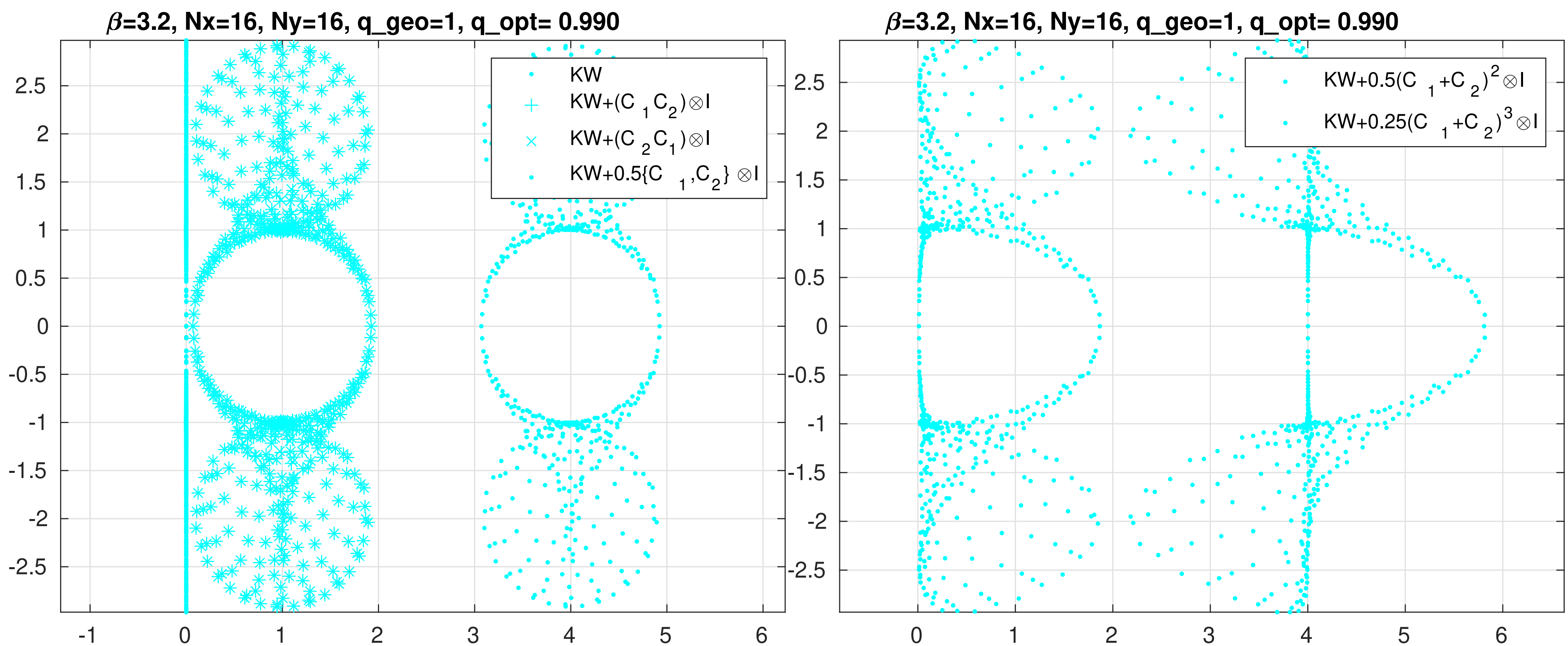
KW and BC fermions have condition numbers less favorable than staggered fermions.

They have $2|q|$ would-be zero-modes with *opposite chiralities* (as staggered) in 2D.

This figure remains $2|q|$ in 4D (like Adams, while staggered fermions have $4|q|$ in 4D).

More on Boriçi-Creutz fermions: R. Osmanaj on Fri 05:45 in Hadron Spectroscopy.

• Hitchhiker's guide to D_{KW} plus species-splitting term



$D_{\mathrm{KW}} + C_1 C_2 \otimes I$ lacks γ_5-pairing

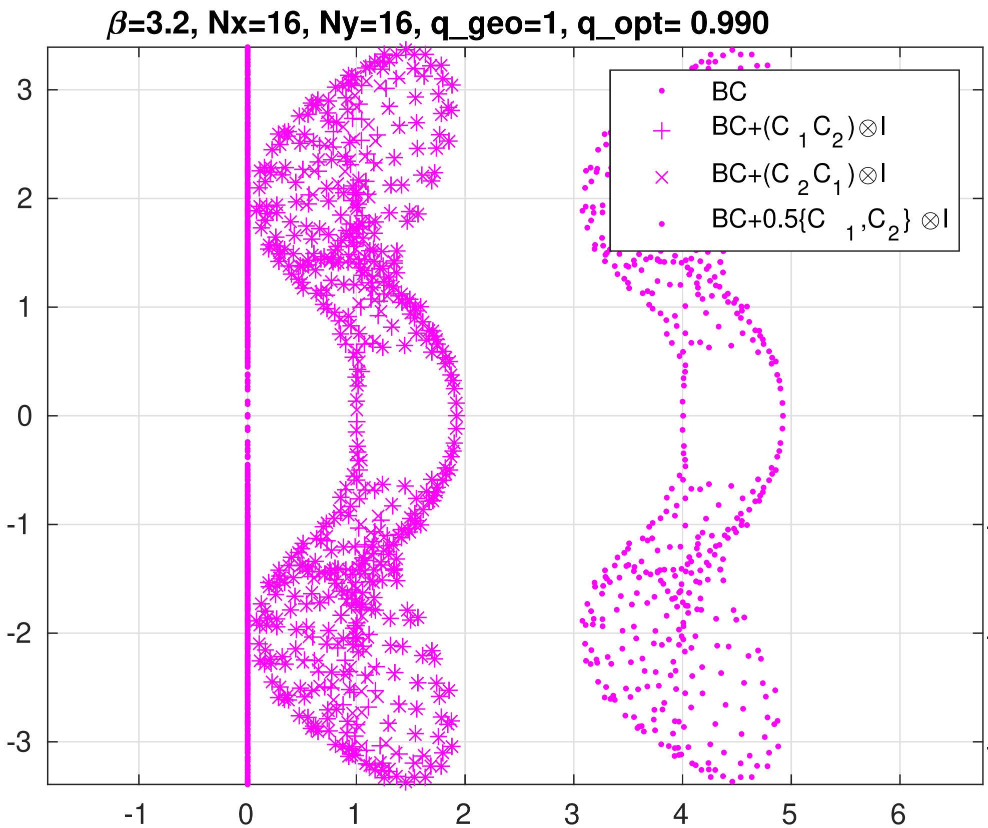
$D_{\mathrm{KW}} + C_2 C_1 \otimes I$ lacks γ_5-pairing (difference invisibly small)

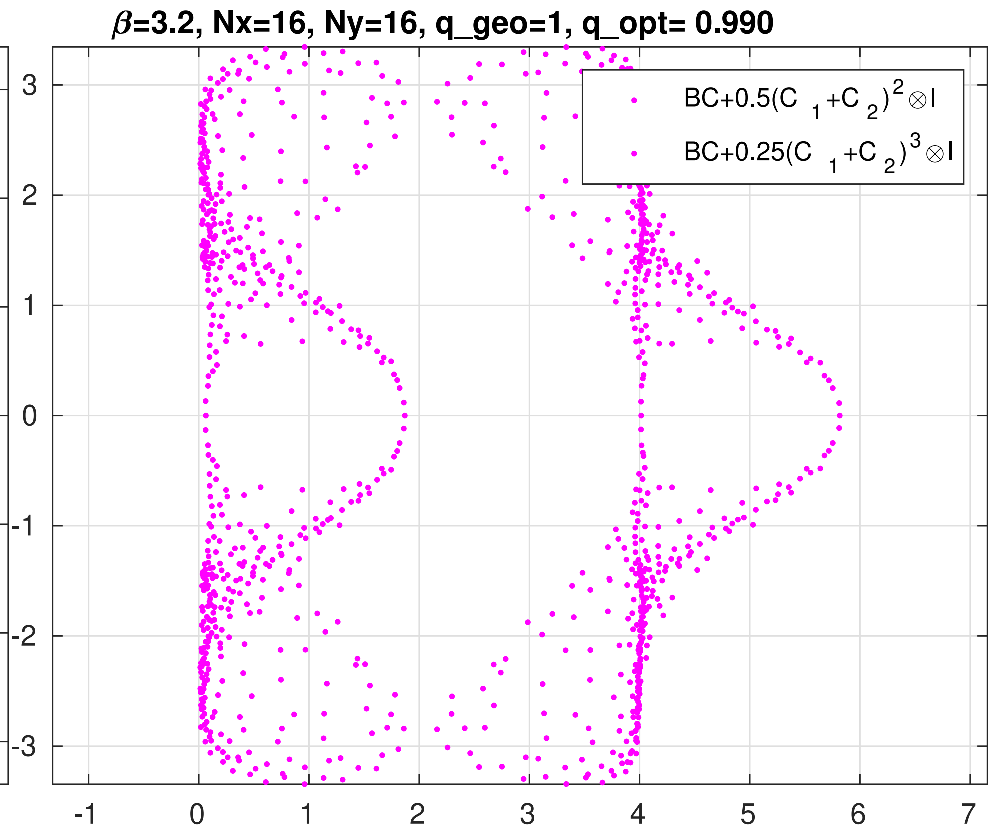
$D_{\mathrm{KW}} + \frac{1}{2}\{C_1, C_2\} \otimes I$ looks fine (shift +3 for clarity)

$D_{\mathrm{KW}} + \frac{1}{2}(C_1 + C_2)^2 \otimes I$ has smaller additive mass shift

$D_{\mathrm{KW}} + \frac{1}{4}(C_1 + C_2)^3 \otimes I$ has tiny additive mass shift (shift +4 for clarity)

• Hitchhiker's guide to $D_{\rm BC}$ plus species-splitting term





$D_{\rm KW} + C_1 C_2 \otimes I$ lacks γ_5-pairing

$D_{\rm KW} + C_2 C_1 \otimes I$ lacks γ_5-pairing (difference visible)

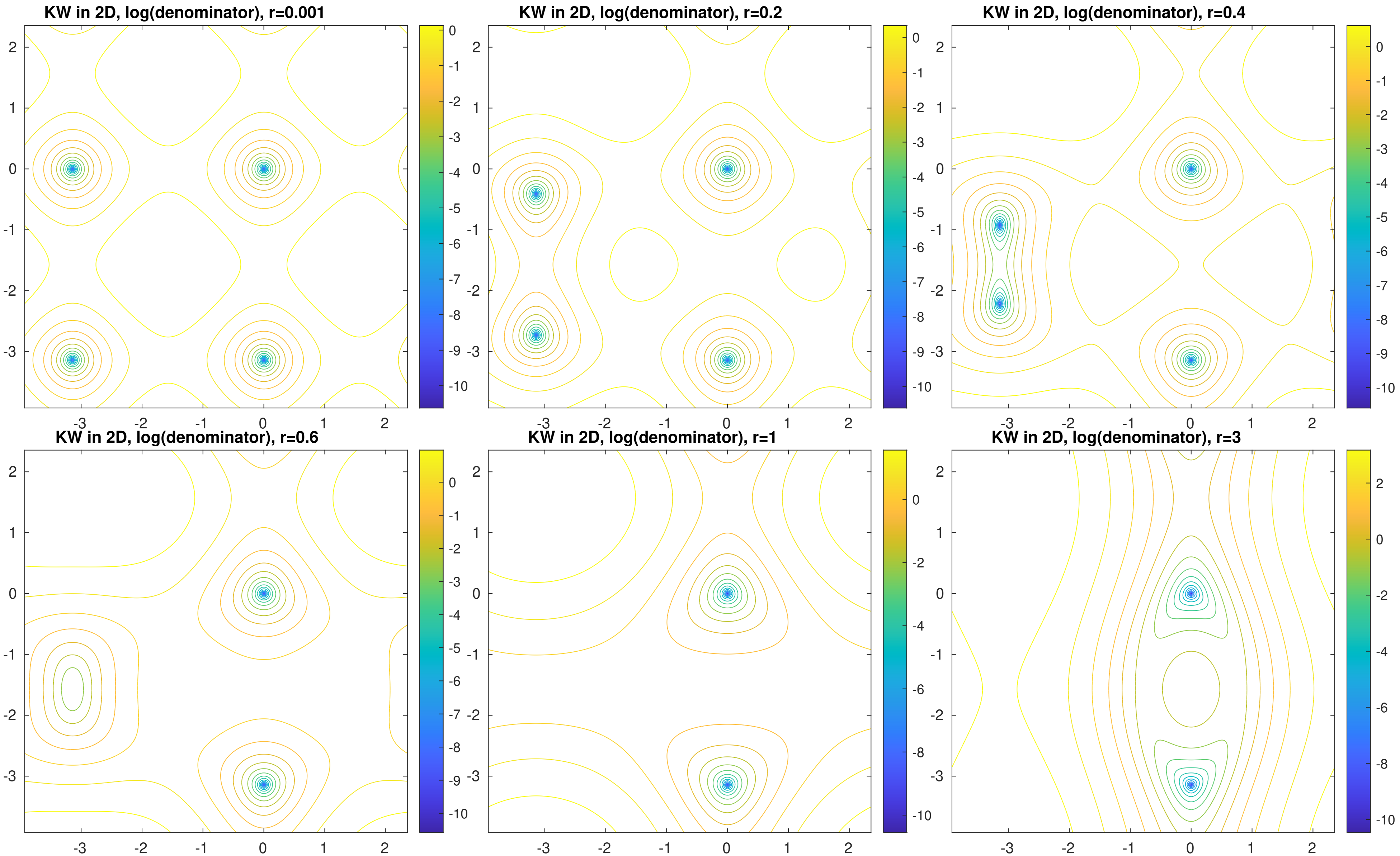
$D_{\rm KW} + \frac{1}{2}\{C_1, C_2\} \otimes I$ looks fine (shift +3 for clarity)

$D_{\rm KW} + \frac{1}{2}(C_1 + C_2)^2 \otimes I$ has smaller additive mass shift

$D_{\rm KW} + \frac{1}{4}(C_1 + C_2)^3 \otimes I$ has tiny additive mass shift (shift +4 for clarity)

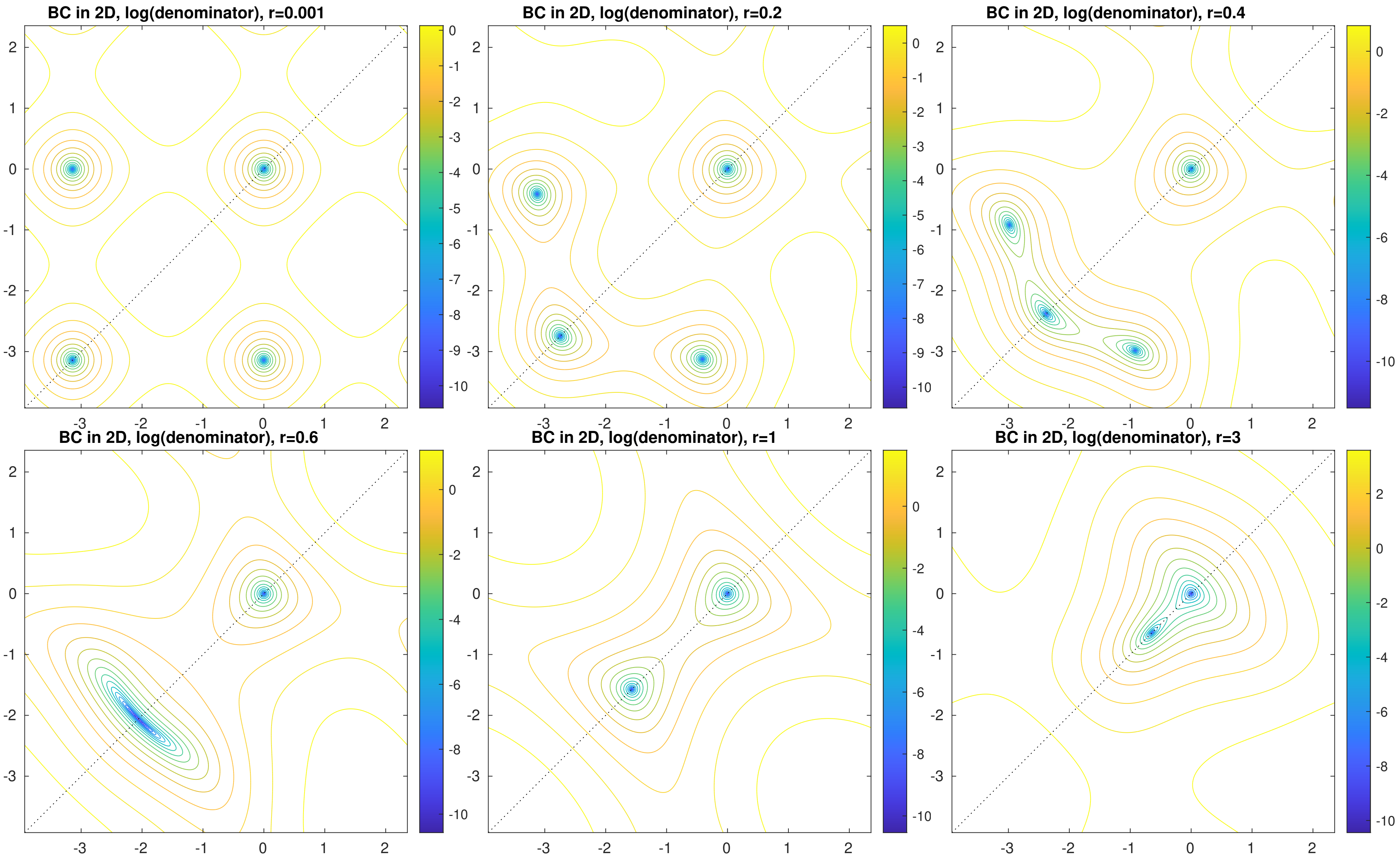
BACKUP PAGES

Pole position drift for KW fermions in 2D (annihilate at r=0.5)



See [arXiv:2003.10803, PRD] for details.

Pole position drift for BC fermions in 2D (merge at r=0.57735)



See [arXiv:2003.10803, PRD] for details.

Review of staggered mass terms

The $(\gamma_\mu \otimes 1)$ and $(\gamma_5 \otimes 1)$ "taste singlet" operators are defined by

$$\Gamma_{\mu 0}(x,y) \equiv \Gamma_\mu(x,y) = \frac{1}{2}\eta_\mu(x)\Big[U_\mu(x)\delta_{x+\hat{\mu},y} + U_\mu^\dagger(x-\hat{\mu})\delta_{x-\hat{\mu},y}\Big]$$

$$\Gamma_{50}(x,y) \equiv \Gamma_5(x,y) = \frac{1}{4!}\sum_{\mathrm{perm}} \epsilon_{\mathrm{perm}} \Gamma_1\Gamma_2\Gamma_3\Gamma_4$$

and the $(1 \otimes \xi_\mu)$ and $(1 \otimes \xi_5)$ "spinor singlet" operators are defined by

$$\Gamma_{0\mu}(x,y) \equiv \Xi_\mu(x,y) = \frac{1}{2}\zeta_\mu(x)\Big[U_\mu(x)\delta_{x+\hat{\mu},y} + U_\mu^\dagger(x-\hat{\mu})\delta_{x-\hat{\mu},y}\Big]$$

$$\Gamma_{05}(x,y) \equiv \Xi_5(x,y) = \frac{1}{4!}\sum_{\mathrm{perm}} \epsilon_{\mathrm{perm}} \Xi_1\Xi_2\Xi_3\Xi_4$$

with the consequence that both Γ_{50} and Γ_{05} are 4-hop operators. Furthermore, the latter two operators relate to each other by a simple Γ_{55} operation (from left or right).

Acceptable mass terms are proportional to $(1 \otimes 1)$ or $(1 \otimes \xi_5)$ or possibly $(1 \otimes \xi_\mu \xi_\nu)$.

Adams species lifting

In practice it is advantageous to introduce the commutators in spinor and taste space

$$\begin{aligned}
\Gamma_{\mu\nu}(x,y) &\equiv \frac{\mathrm{i}}{2}\Big(\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu\Big) &\longleftrightarrow\quad& \gamma_{\mu\nu}\otimes 1 \\
\Xi_{\mu\nu}(x,y) &\equiv \frac{\mathrm{i}}{2}\Big(\Xi_\mu\Xi_\nu - \Xi_\nu\Xi_\mu\Big) &\longleftrightarrow\quad& 1\otimes\xi_{\mu\nu}
\end{aligned}$$

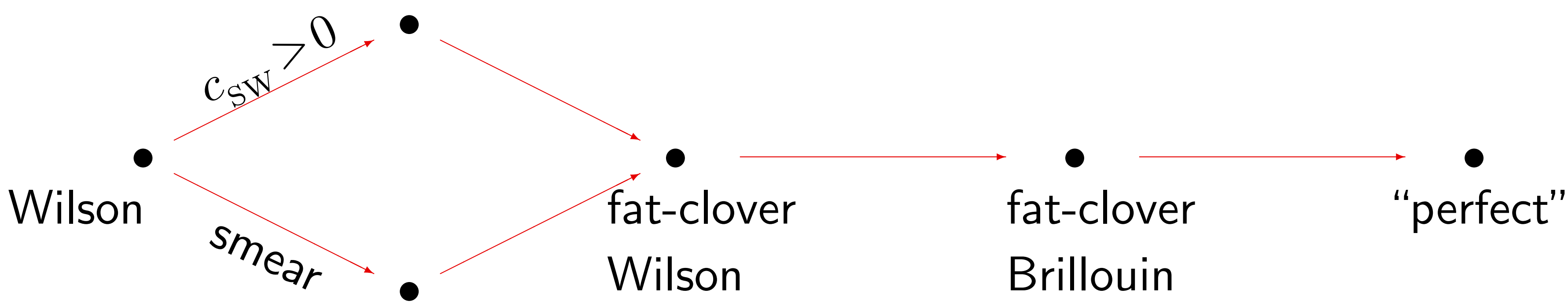
respectively, with $\gamma_{\mu\nu} \equiv \frac{\mathrm{i}}{2}[\gamma_\mu,\gamma_\nu]$ a.k.a. $\sigma_{\mu\nu}$ and $\xi_{\mu\nu} \equiv \frac{\mathrm{i}}{2}[\xi_\mu,\xi_\nu]$, which yields

$$\begin{aligned}
\Gamma_{50}(x,y) &\simeq -\frac{1}{6}\Big(\Gamma_{12}\Gamma_{34} - \Gamma_{13}\Gamma_{24} + \Gamma_{14}\Gamma_{23} + \Gamma_{23}\Gamma_{14} - \Gamma_{24}\Gamma_{13} + \Gamma_{34}\Gamma_{12}\Big) \\
\Gamma_{05}(x,y) &\simeq -\frac{1}{6}\Big(\Xi_{12}\Xi_{34} - \Xi_{13}\Xi_{24} + \Xi_{14}\Xi_{23} + \Xi_{23}\Xi_{14} - \Xi_{24}\Xi_{13} + \Xi_{34}\Xi_{12}\Big)
\end{aligned}$$

Adams: Promote 2 out of the 4 tastes of D_stag to doublers by $\Gamma_{05} = \Xi_5 \simeq (1\otimes\xi_5)$. Key observation is that the remaining 2 physical species share *one chirality* !

Corollary: It makes sense to apply overlap construction to shifted kernel $X = D_\mathrm{A} - \rho$. The resulting operator will be doubled, but the two species will be chiral.

Dirac operator roadmap (pedestrian perspective)



Wilson Dirac operator:

$$D_{\rm W}(x,y)=\sum_\mu \gamma_\mu \nabla_\mu^{\rm std}(x,y)-\frac{a}{2}\triangle^{\rm std}(x,y)+m_0\delta_{x,y}-\frac{c_{\rm SW}}{2}\sum_{\mu<\nu}\sigma_{\mu\nu}F_{\mu\nu}\delta_{x,y}$$

Brillouin Dirac operator:

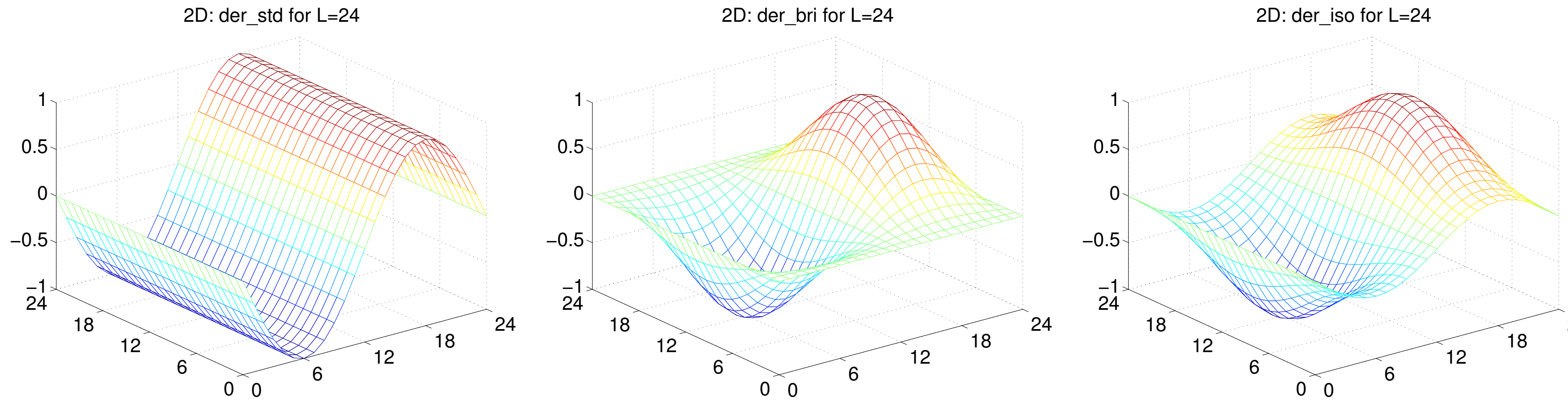
$$D_{\rm B}(x,y)=\sum_\mu \gamma_\mu \nabla_\mu^{\rm iso}(x,y)-\frac{a}{2}\triangle^{\rm bri}(x,y)+m_0\delta_{x,y}-\frac{c_{\rm SW}}{2}\sum_{\mu<\nu}\sigma_{\mu\nu}F_{\mu\nu}\delta_{x,y}$$

$\sigma_{\mu\nu}=\frac{\rm i}{2}[\gamma_\mu,\gamma_\nu]$, $F_{\mu\nu}$ the hermitean clover-leaf field-strength tensor, separate $m_0, c_{\rm SW}$

Brillouin operator details

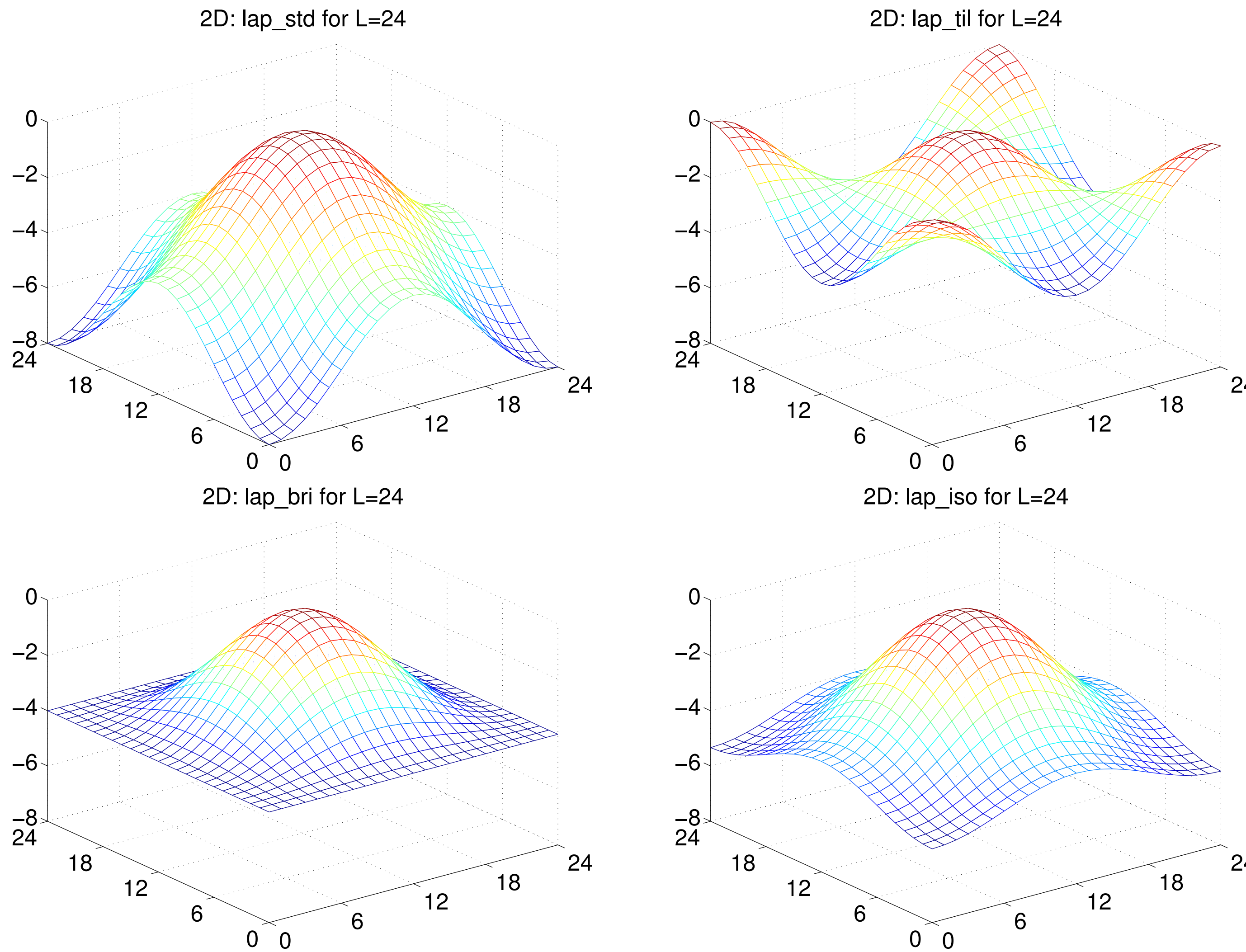
- **3 options for (covariant) Nabla**

Standard Derivative: $\hat{\nabla}_x = \mathrm{i}\sin(k_1)$
Brillouin Derivative: $\hat{\nabla}_x = \mathrm{i}\sin(k_1)[\cos(k_2)+1][\cos(k_3)+1][\cos(k_4)+1]/8$
Isotropic Derivative: $\hat{\nabla}_x = \mathrm{i}\sin(k_1)[\cos(k_2)+2][\cos(k_3)+2][\cos(k_4)+2]/27$



- **4 options for (covariant) Laplacian**

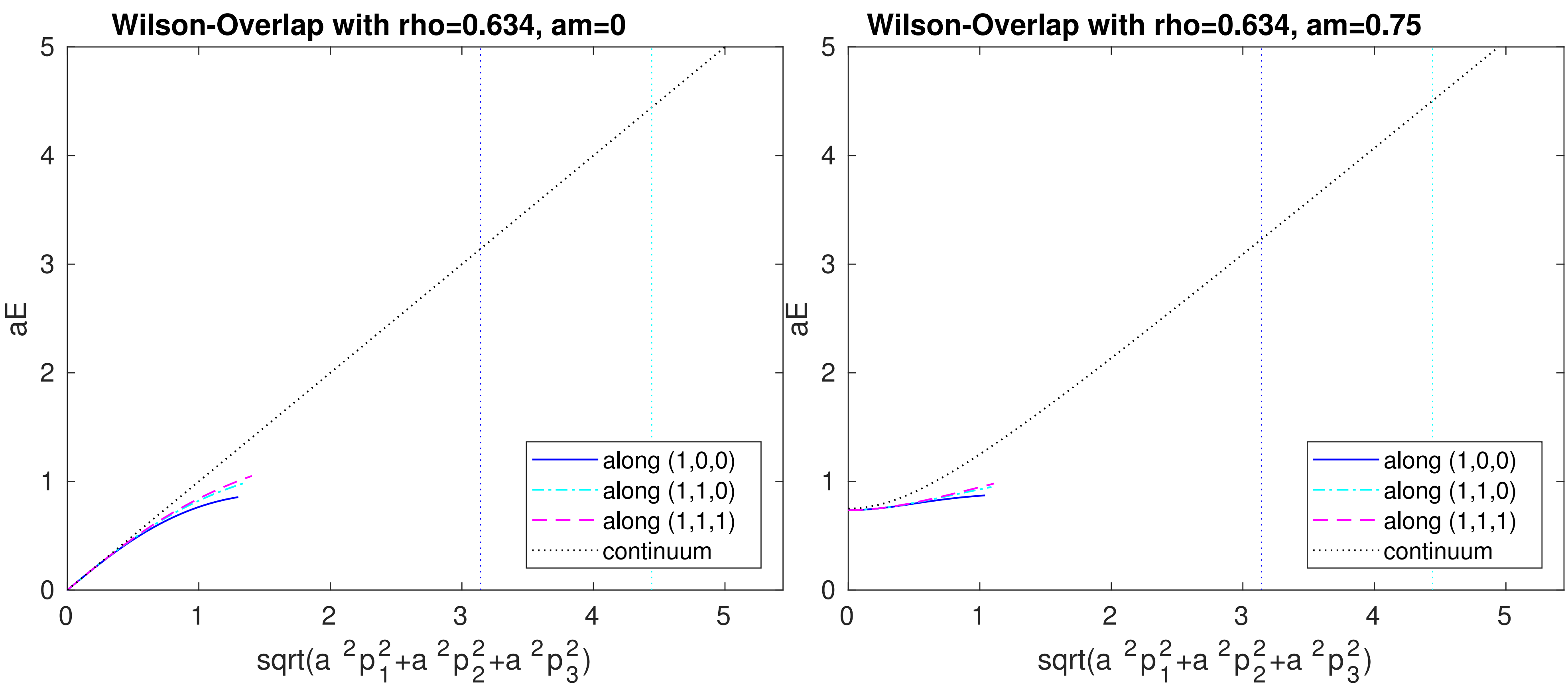
Standard Laplacian: $\hat{\triangle} = 2\cos(k_1)+2\cos(k_2)+2\cos(k_3)+2\cos(k_4)-8$
Tilted Laplacian: $\hat{\triangle} = 2\cos(k_1)\cos(k_2)\cos(k_3)\cos(k_4)-2$
Brillouin Laplacian: $\hat{\triangle} = 4\cos^2(k_1/2)\cos^2(k_2/2)\cos^2(k_3/2)\cos^2(k_4/2)-4$
Isotropic Laplacian: $\hat{\triangle} = [2c_1c_2c_3c_4+7c_1c_2c_3+\ldots+20c_1c_2+\ldots+25c_1+\ldots-250]/54$



- **Selection Procedure**

All 12 options live on $[-1:1]^4$ hypercube (81 sites in 4D, ultralocal). Test them systematically (eigenvalues, dispersion relation). Combination $(\triangle^{\text{bri}}, \nabla^{\text{iso}})$ wins.

• Dispersion relation for Wilson-kernel overlap-descendent



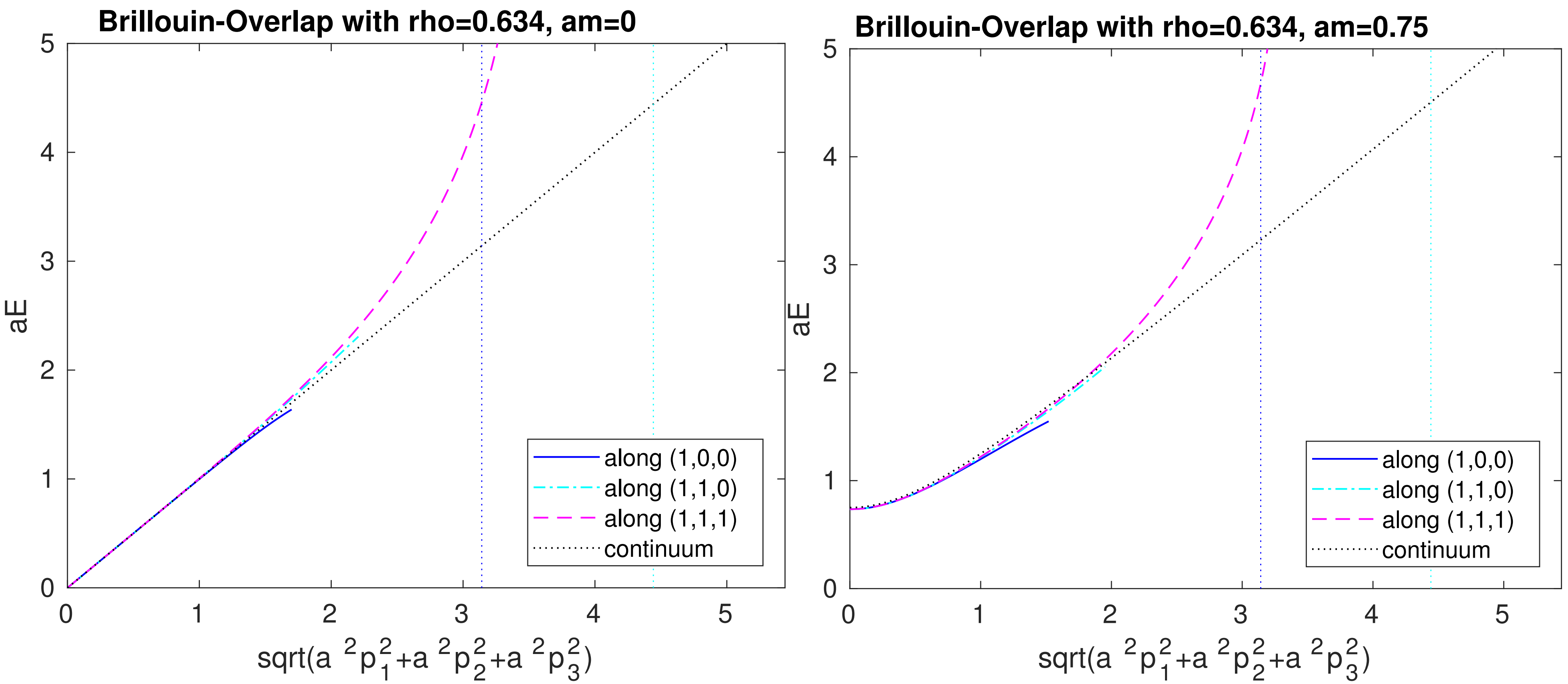
⊖ strong deviation from continuum for any $a|\mathbf{p}| > 1$

⊖ strong rotational symmetry breaking for any $a|\mathbf{p}| > 1$

⊕ mild effect of $am \not\ll 1$ for "magic value" $\rho = 0.634$ (neg. mass of D_W)

See [arXiv:1701.00726] for details.

- **Dispersion relation for Brillouin-kernel overlap-descendent**



⊕ mild deviation from continuum up to $a|\mathbf{p}| \simeq 1.5$

⊕ mild rotational symmetry breaking up to $a|\mathbf{p}| \simeq 1.5$

⊕ mild effect of $am \not\ll 1$ for "magic value" $\rho = 0.634$ (neg. mass of D_W)

See [arXiv:1701.00726] for details.

- **Cut-off effects for Wilson and Brillouin kernel-operators**

Wilson operator:

$$
\begin{aligned}
(aE)^2-(a\mathbf{p})^2 &= \Big[(am)^2-(am)^3+\frac{11}{12}(am)^4-\frac{5}{6}(am)^5\Big] \\
&+ \Big[-\frac{2}{3}(am)^2+\frac{7}{6}(am)^3\Big](a\mathbf{p})^2 \\
&+ \Big[-\frac{2}{3}+\frac{am}{2}\Big]\Big(\sum_{i<j}a^4p_i^2p_j^2+\sum_i(ap_i)^4\Big)+O(a^6)
\end{aligned}
$$

Brillouin operator:

$$
\begin{aligned}
(aE)^2-(a\mathbf{p})^2 &= \Big[(am)^2-(am)^3+\frac{11}{12}(am)^4-\frac{5}{6}(am)^5\Big] \\
&+ \Big[0+\frac{1}{12}(am)^3\Big](a\mathbf{p})^2 \\
&+ \Big[0+\frac{am}{12}\Big]\Big(\sum_{i<j}a^4p_i^2p_j^2+\sum_i(ap_i)^4\Big)+O(a^6)
\end{aligned}
$$

Fermilab reinterpretation manages to get rid of $-(am)^3+\ldots$ part at $\mathbf{p}=0$.

• Cut-off effects for Wilson and Brillouin overlap-descendents

Overlap operator with Wilson kernel:

$$
\begin{aligned}
(aE)^2-(a\mathbf{p})^2 \;&=\; \Big[(am)^2-\frac{2\rho^2-6\rho+3}{6\rho^2}(am)^4\Big] \\
&+\; \Big[-\frac{2}{3}(am)^2+0\Big](a\mathbf{p})^2 \\
&+\; \Big[-\frac{2}{3}+0\Big]\Big(\sum_{i<j}a^4p_i^2p_j^2+\sum_i(ap_i)^4\Big)+O(a^6)
\end{aligned}
$$

Overlap operator with Brillouin kernel:

$$
\begin{aligned}
(aE)^2-(a\mathbf{p})^2 \;&=\; \Big[(am)^2-\frac{2\rho^2-6\rho+3}{6\rho^2}(am)^4\Big] \\
&+\; \Big[0+0\Big](a\mathbf{p})^2 \\
&+\; \Big[0+0\Big]\Big(\sum_{i<j}a^4p_i^2p_j^2+\sum_i(ap_i)^4\Big)+O(a^6)
\end{aligned}
$$

See [arXiv:1701.00726] and references therein for details.