Fields on 2D hyperbolic geometry

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Motivation

- Maldacena's conjecture: compactifications of field theory on anti- de-Sitter spacetime is dual to conformal field theory on its boundary (arXiv:hep-th/9711200, hep-th/9802150,hep-th/9802109) .
- Mathematically the conjecture can be expressed as,

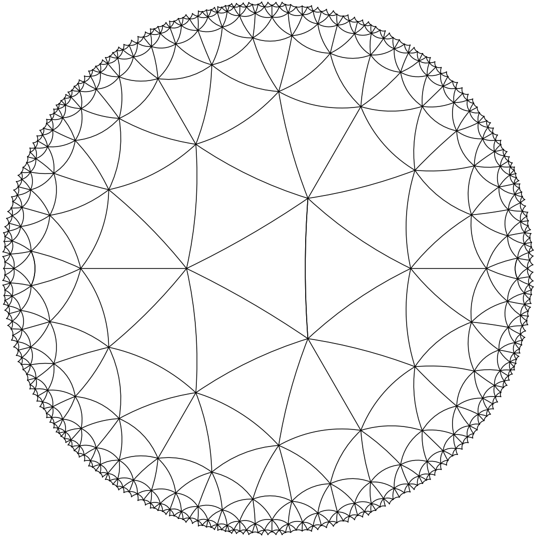
$$Z_{\mathrm{ADS}}\left[\phi(\vec{x},z)|_{z\to 0}=\phi_0(x)\right]\equiv Z_{\mathrm{CFT}}[\phi_0]$$

$$\int_{\phi_0}\mathcal{D}\phi\exp(-S(\phi))\equiv\langle\exp\left(\int_{\partial\Omega}d^dx\mathcal{O}\phi_0\right)\rangle$$

- Explore holography numerically in 2D and 3D hyperbolic space. To develop a lattice analysis so that the idea of **holography** can be explored with **backreaction** of the matter fields in a straightforward manner.
- Boundary correlators demonstrate **power law dependence** with the boundary distance: $C(r)\sim r^{-\Delta}$. Δ depends on the dimension of the boundary d and the radius of curvature L.

$$\boxed{\Delta_{\pm}=\frac{d}{2}\pm\sqrt{(\frac{d}{2})^2+m^2L^2}}$$

$\{3,7\}$ tessellated disk



- Poincare disk representation of a 2D hyperbolic geometry is shown here.
- Infinite ways to tessellate the hyperbolic geometry with p-sided polygons with q connectivity of each lattice point: $(p-2)(q-2) > 4$.
- Two length scales are related: lattice spacing a and length associated with the intrinsic curvature L. In general, for a regular tessellated disk $a \propto L$.

Discretized lattice action

- Discretized action can be written as

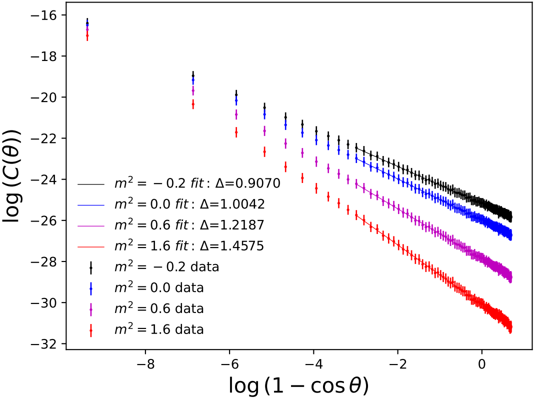
$$S_{\mathrm{scalar}} = \frac{1}{2}\sum_{\langle xy\rangle} q_{xy} V_e \left(\phi_x - \phi_y\right)^2 + \frac{1}{2}\sum_{x} q_x m_0^2 V_v \phi_x^2 = \sum_{x,y} \phi_x L_{xy} \phi_y .$$

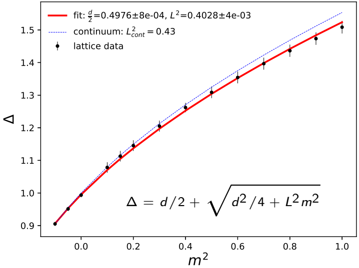
  Here V_e denotes the dual volume of the lattice associated with an edge, V_v denotes the dual volume associated with a vertex, a denotes the lattice spacing, q_{xy} denotes the coordination number of the link xy and q_x denotes the coordination number of the lattice site x.
- Correlators can be obtained by computing the inverse of the discrete Lattice Laplacian L.
- In the continuum, two-point boundary boundary correlation

$$C(\delta\theta) \propto \frac{1}{|1 - \cos(\delta\theta)|^{\Delta}}$$

  where $\delta\theta$ is the angular distance between two points along the boundary in the Poincare disk and Δ is the scaling dimension of the boundary operator.

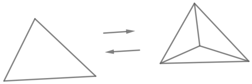
Conformality from correlators: $\{3,8\}$ disk

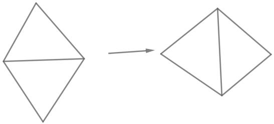




Asaduzzaman *et. el.* **(PRD 102, 034511)**

- Dynamical triangulation of 2D geometry at different topology $\Rightarrow$ discrete quantum gravity. Ambjorn et. el. Classical and quantum gravity, vol 12, no 9. .
- Building block of the simplicial geometry: d-dimensional simplex with edge length 1. Coordination number in $d-2$ dimensional subsimplex contain the information of the curvature of the geometry. Scott et. el. arXiv:hep-lat/0105002





$$Z = \int \frac{\mathcal{D}g_{ab}(\xi)}{\text{Vol (diff)}} \mathrm{e}^{-S_{\text{EH}}[g;\mu,G]} \int \mathcal{D}_g\phi(\xi)\mathrm{e}^{-S_{\mathrm{m}}[\phi,g;\lambda_i]}, \; S_{\text{EH}} = \int \mathrm{d}^2\xi\sqrt{g}\left[\mu - \frac{1}{4\pi G}R\right]$$

$$\int \frac{\mathcal{D}g_{ab}(\xi)}{\text{Vol (diff)}} \sim \sum_{T\in\mathcal{T}} \frac{1}{C_T}$$

$$\mathrm{Einstein-Hilbert} \sim \mathrm{Einstein-Regge\ action}$$

$$\phi(\xi) \sim \phi_i$$

- Background geometry $\Rightarrow$ $\{3,7\}$ tessellated disk.
- Action

$$S = \underbrace{-\kappa_0 N_0 + \kappa_2 N_2}_{\text{Einstein-Regge action}} + \overbrace{\beta \sum_{i \in \text{bulk}} (q_i - 7)^2}^{\text{modified } R^2 \text{ term}} .$$

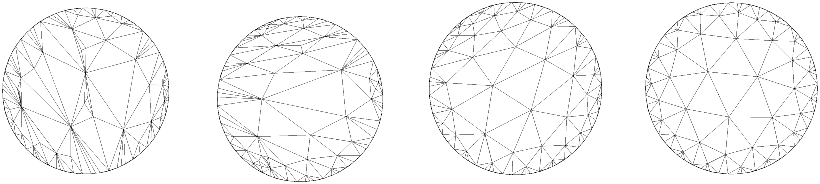
where, N_0=number of nodes, N_2=number of triangles, q_i denotes the coordination number of the i$^{\text{th}}$ bulk node, then the action for the scalar field can be written as

- Canonical partition function,

$$Z = \sum_{\mathcal{T}} \exp(-S).$$

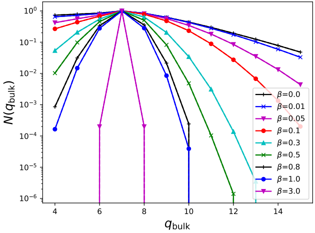
We move at different points in the triangulation space with the local Pachnar moves. Pachnar moves introduce fluctuations in the local curvature of the geometry in the Regge sense.

Snapshot of generated disks at different β

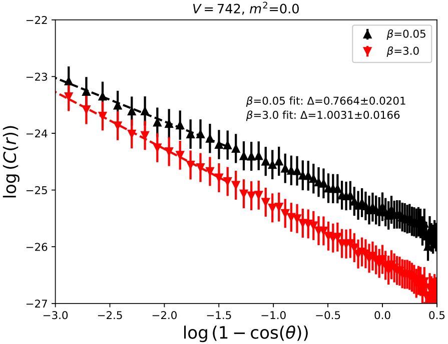


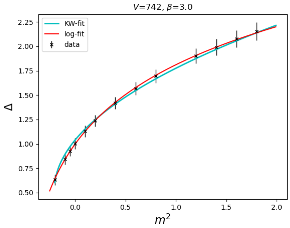
$\beta = 0.0$ $\beta = 0.1$ $\beta = 1.0$ $\beta = 3.0$

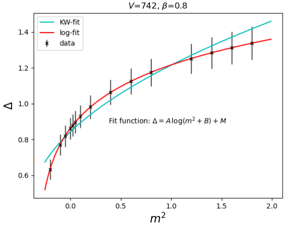
As $\beta \to \infty$ pick out regular (3,7) tessellation of H^2



Boundary-boundary correlators







Previous construction employed an R^2 action put in by hand.

Can we dynamically generate curvature fluctuation to the hyperbolic manifolds?

Consider N complex Kahler-Dirac fermions

$$Z = \sum_{\mathcal{T}} \int D\phi \, e^{-S_{EH}(T)} e^{-\phi^a\left(-\Box + m^2\right)\phi^a} \quad a = 1 \ldots N \tag{1}$$

$$= \sum_{\mathcal{T}} \det{}^{\frac{N}{2}}(-\Box(T) + m^2) e^{-S_{EH}(T)} \tag{2}$$

Factorize:

$$\left(-\Box(T) + m^2\right) = \tilde{\Box} = DAD \quad \text{where } D = \mathrm{diag}(\sqrt{q_i} \ldots) \text{ and}$$

$$A_{ii} = 1 + m^2 \tag{3}$$

$$A_{ij} = -\frac{1}{\sqrt{q_i q_j}} C_{ij} \tag{4}$$

where C is the incidence matrix of T

$$\det\tilde{\Box} \approx (\det D)^2 = \prod_i q_i$$

For $m \to \infty$ only D contributes and one finds a local effective action

$$S_{\text{eff}} = -\frac{N}{2}\sum_{i\in\text{bulk}} \ln(q_i) + \lambda(\langle q\rangle - 7) \qquad \boxed{\ln(q_i) \sim -\frac{1}{2\pi^2}R_i^2 + \dots}$$

Consider $N \to \infty$ and use steepest descent:

$$q_i = 7 \qquad \to \text{regular } (3,7) \text{ tessellation}$$

Thus formally retrieve classical gravity for $N \to \infty$

Mapping to the modified R^2 simulation

$$\beta \sim N$$

Vary N and explore transition from holographic to non-holographic regime

Kahler Dirac operator(K) is natural square root of Laplacian $\rightarrow$ Fermionic action
Fundamental operators d and $d^\dagger$: converts a p-form to a $p \pm 1$ form.

$$\Box = K^2 = (d - d^\dagger)^2 = -dd^\dagger - d^\dagger d$$

If Ω is a form field, EOM for KD fermion, $\boxed{(d - d^\dagger + m)\Omega = 0}$

$$\Omega = (\omega_0, \omega_\mu, \ldots, \omega_{\mu_1 \cdots \mu_D})$$

Flat space, form matrix can be written in the γ matrix basis, and each KD is equivalent to 2 copies of Dirac fermion in 2D.

Lattice formulation

- Orient the lattice: edges at neighboring lattice need to have opposite parity
- p-form in continuum manifold maps to p-cochain: real or complex function on p-simplices
- d and $d^\dagger$ $\rightarrow$ boundary and coboundary operators $\delta, \overline{\delta}$ where $\delta(a_1, \ldots, a_p) = \sum_i^p (-1)^j (a_1, \hat{a}_j, \ldots, a_p)$

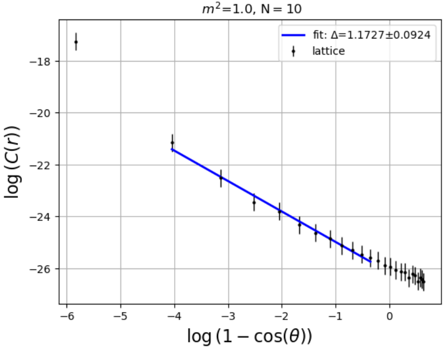
Kähler-Dirac fermions - lattice

Key property: Kähler-Dirac equation can be discretized on arbitrary triangulation without fermion doubling

- Lattice KD equation: $(\delta - \overline{\delta} + m)\Omega_L = 0$

Integrating out pairs of KD fermions yields

$$\det(\Box_0 + m^2)\det(\Box_1 + m^2)\det(\Box_2 + m^2)$$



- It's possible to explore **holography in regular and fluctuating hyperbolic manifolds** numerically.
- Introducing **KD fermions**, **backreaction** of the matter field can be taken into account. We recover classical **H**-space at $N \to \infty$.
- Leading behavior for large m is just the R^2 term for which we showed numerical results.
- Discrete rep allows us to look for a possible **phase transition** to non-holographic regime as we vary the number of fermions N.
- Can be generalized for higher dimension $D > 2$.
- Incorporating other kind of matter fields: Ising spins.
- Explore the **effective boundary theory** from holography: some progress has been made for the Ising model in $\mathbf{H}_2$.