New universality classes of the non-Hermitian Dirac operator in QCD-like theories

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Outline

Introduction

RMT

- random matrix theory (RMT) describes universal features of eigenvalue spectra
  - determined by (global) symmetries
  - independent of details of the dynamics
- useful in many applications in physics and beyond
  - also in (lattice) QCD, e.g., to determine LECs or spectral sum rules
- physical systems fall into distinct symmetry classes
  - Hermitian RMT: 10 classes (Dyson 1962; Altland-Zirnbauer arXiv:cond-mat/9602137)
  - non-Hermitian RMT: 38 classes (Kawabata-Shiozaki-Ueda-Sato arXiv:1812.09133, Zhou-Lee arXiv:1812.10490)
- for short-range correlations in the bulk of the spectrum: only 3 universality classes (i.e., different symmetry classes can give same bulk correlations)
  - Hermitian RMT: Wigner-Dyson ensembles (GOE, GUE, GSE)
  - non-Hermitian RMT: Ginibre, $AI^{\dagger}$, $AII^{\dagger}$

Universality classes in non-Hermitian RMT

- transposition symmetries of the non-Hermitian matrix X

Class	Matrix
Ginibre	X
AI†	$X^T = X$
AII†	$X^T = \sigma_2 X \sigma_2$

- for a long time, only the Ginibre universality class was known
  - in LQCD, e.g., Markum, Pullirsch, TW arXiv:hep-lat/9906020
- recently, it was found that AI† and AII† are different universality classes (Jaiswal-Prakash-Pandey arXiv:1904.12484, Hamazaki-Kawabata-Kura-Ueda arXiv:1904.13082)
- this talk:
  - Do AI† and AII† play a role in high-energy physics?
  - Can we observe the non-Ginibre behavior in Dirac spectra?

Non-Hermitian Dirac operators: continuum and lattice symmetries

Continuum Dirac operator

- couple Euclidean Dirac operator to a chiral U(1) field B

$$D = \gamma_\nu(\partial_\nu - iA_\nu^a\tau_a - i\gamma_5 B_\nu)$$

  - for $B_\nu \to \delta_{\nu 4} i\mu_5$ with $\mu_5 \in i\mathbb{R}$, last term becomes $\mu_5\gamma_5\gamma_4$ (imaginary chiral chemical potential)
  - physical significance: source term for spatially inhomogeneous chiral condensate
- consider gauge groups with pseudoreal and real representations
  - e.g., SU(2) fundamental (pseudoreal) and adjoint (real)
- in the following: K = complex conjugation
  $C = i\gamma_4\gamma_2$ = charge conjugation
- pseudoreal representations
  - without B: antiunitary symmetry $[iD, C\tau_2 K] = 0$
  - broken by B, but still transposition symmetry $D^T = C\tau_2 D C\tau_2 \;\to\; D^T = D$ in suitable basis
- real representations
  - without B: antiunitary symmetry $[iD, CK] = 0$
  - broken by B, but still transposition symmetry $D^T = CDC \;\to\; D^T = \sigma_2 D\sigma_2$ in suitable basis

Transposition symmetries and nonstandard random matrix ensembles

- we always have chiral symmetry $\{D, \gamma_5\} = 0 \quad \rightarrow \quad D$ has block form $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$
- in a suitable basis, D has the following transposition symmetries

Representation	Symmetry of D	Matrix form	Matrix elements	Class
pseudoreal	$D^T = D$	$\begin{pmatrix} 0 & V \\ V^T & 0 \end{pmatrix}$	complex	AI†
real	$D^T = \Sigma_2 D \Sigma_2$	$\begin{pmatrix} 0 & V \\ \sigma_2 V^T \sigma_2 & 0 \end{pmatrix}$	complex quaternion	AII†

here, σ_2 acts on the Dirac indices, and $\Sigma_2 = \sigma_2 \oplus \sigma_2$

- for AII†, every eigenvalue is twofold degenerate (Kramers degeneracy)

Staggered lattice Dirac operator with chiral U(1) field

- remnant chiral symmetry: $\{D, \varepsilon\} = 0$ with $\varepsilon_{xy} = \varepsilon(x)\delta_{xy}$ and $\varepsilon(x) = (-1)^{x_1+x_2+x_3+x_4}$
- staggered Dirac operator with chiral U(1) field $\theta_\mu(x) = \exp(i\varepsilon(x)\phi_\mu(x))$

$$D(\theta)_{xy} = \frac{1}{2}\sum_{\mu=1}^{4} \eta_\mu(x)\left[U_\mu(x)\theta_\mu(x)\delta_{x+\mu,y} - U_\mu(y)^\dagger\theta_\mu(y)\delta_{x,y+\mu}\right]$$

- consider SU(2) gauge field $U_\mu(x)$ in fundamental and adjoint representation, where

$$U_\mu^A(x)_{ab} = \frac{1}{2}\,\mathrm{tr}(\tau_a U_\mu^F(x)\tau_b U_\mu^F(x)^\dagger)$$

- fundamental representation: $D^F(\theta)^T = -\tau_2 D^F(\theta)\tau_2$ → class AII†
- adjoint representation: $D^A(\theta)^T = -D^A(\theta)$ → class AI†

→ staggered symmetries are reversed compared to continuum (same as in Hermitian case)

Staggered lattice Dirac operator with imaginary chiral chemical potential

- following Braguta et al. arXiv:1503.06670, μ_5 is introduced as follows

$$D(\mu_5)_{xy} = D(0)_{xy} + \frac{1}{2}\mu_5 s(x)\left[\bar{U}_\delta(x)\delta_{x+\delta,y} + \bar{U}_\delta^\dagger(y)\delta_{x,y+\delta}\right]$$

$$s(x) = (-1)^{x_2}\,, \quad \delta = (1,1,1,0)\,, \quad \bar{U}_\delta(x) = \frac{1}{6}\sum_{i,j,k=\mathrm{perm}(1,2,3)} U_i(x)U_j(x+\hat{i})U_k(x+\hat{i}+\hat{j})$$

- same symmetries as $D(\theta)$
- continuum limit of μ_5 term is $\mu_5\gamma_5\gamma_4$ as required
- for $\mu_5 \notin \mathbb{R}$, eigenvalues move into complex plane (we use $\mu_5 \in i\mathbb{R}$)

Spectral sum rules

- exact spectral sum rules are useful check for correct computation of eigenvalues
- well known for standard staggered Dirac operator (with V = lattice volume)

$$\operatorname{tr} D^2 = \sum_n \lambda_n^2 = -2N_{\text{rep}}V$$

$N_{\text{rep}} = 2$ for SU(2) fundamental, $N_{\text{rep}} = 3$ for SU(2) adjoint

- generalization to chiral U(1) field and chiral chemical potential

$$\operatorname{tr} D^2 = \sum_n \lambda_n^2 = -V\Big[2N_{\text{rep}}\langle\theta_\mu^2(x)\rangle_{x\mu} + \frac{1}{2}\mu_5^2\big\langle \operatorname{tr} \bar{U}_\delta(x)\bar{U}_\delta^\dagger(x)\big\rangle_x\Big]$$

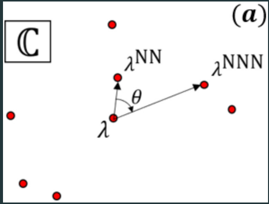
where $\langle\cdots\rangle_{x\mu}$ and $\langle\cdots\rangle_x$ means average over links and sites

Complex spacing ratios

Complex spacing ratios

- Dirac eigenvalues in complex plane: unfolding difficult
  - Markum, Pullirsch, TW arXiv:hep-lat/9906020
  - Akemann, Mielke, Kieburg, Prosen arXiv:1910.03520
- avoid the problem by taking spacing ratios: Sa, Ribeiro, Prosen arXiv:1910.12784

$$z_k = \frac{\lambda_k^{\mathrm{NN}} - \lambda_k}{\lambda_k^{\mathrm{NNN}} - \lambda_k}$$



→ distribution $P(z)$ is universal and described by RMT

- $z = re^{i\theta} \to |r| \leq 1$
- we also compute $P(r)$, $P(\theta)$ and some moments
- RMT results not known analytically yet → generated numerically

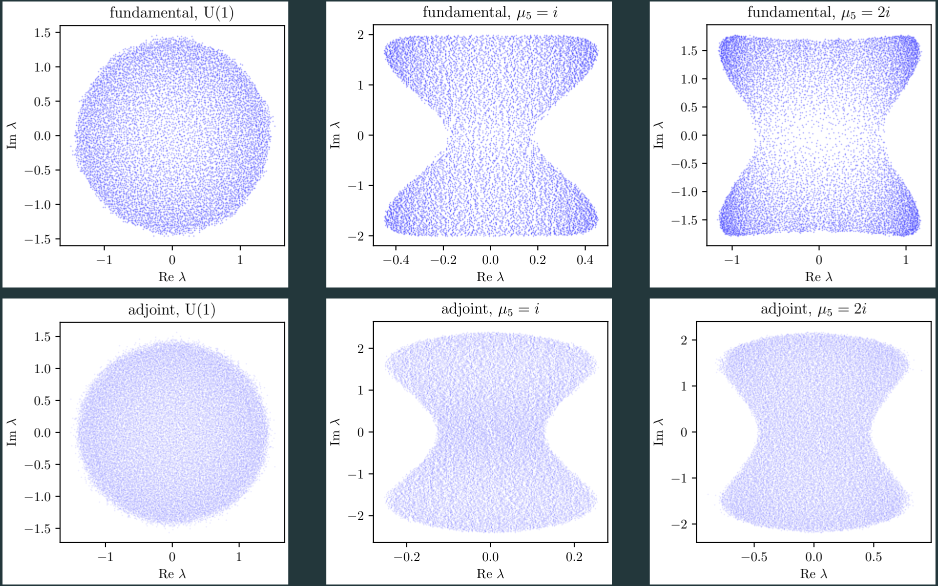
Numerical results

Simulation details

- quenched simulations (dynamical fermions would yield same bulk correlations)
- $\beta_{SU(2)} = 2.0$, $\beta_{U(1)} = 0.9 \rightarrow$ confined phase
- lattice volume $8^3 \times 16 \rightarrow$ can easily compute all eigenvalues
- generation of SU(2) fields: Creutz–Kennedy-Pendleton heatbath algorithm
- generation of U(1) fields: Hattori-Nakajima heatbath algorithm
- k-d tree algorithm for nearest-neighbor search
- number of configurations

Representation	U(1)	$\mu_5 = i$	$\mu_5 = 2i$
SU(2) fundamental	271	415	415
SU(2) adjoint	267	267	266

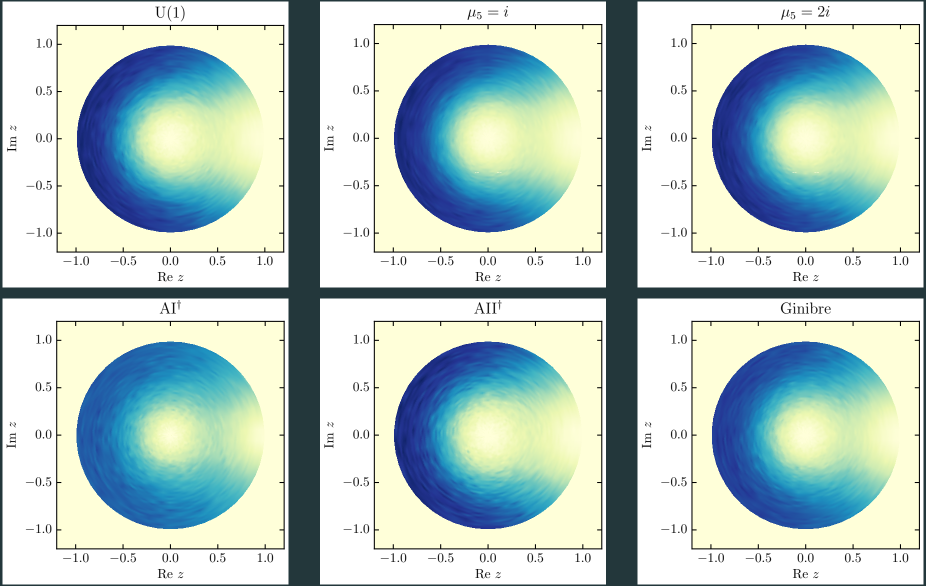
Scatter plots of eigenvalues (nonuniversal)



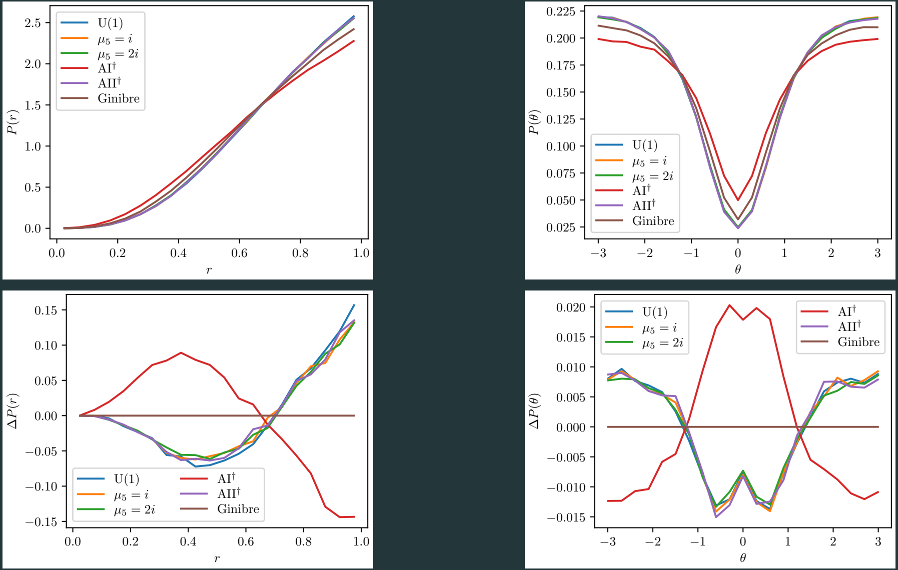
Distribution $P(z)$ of spacing ratios (universal): SU(2) fundamental $\sim$ AII†

Lattice:

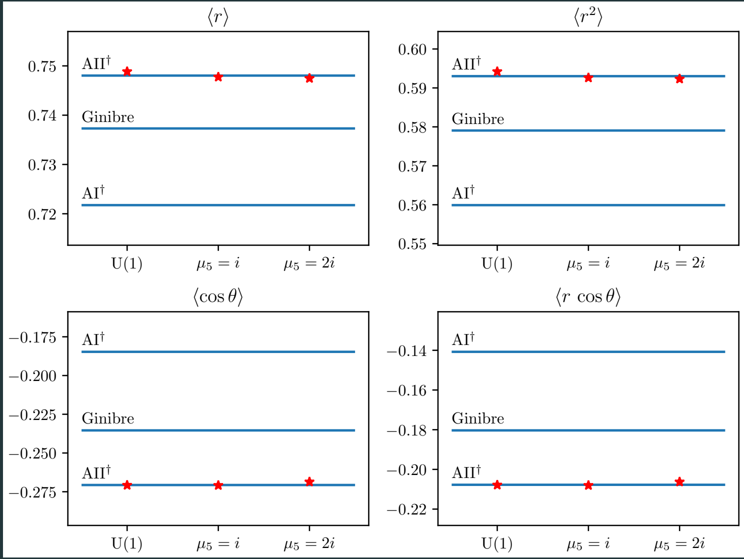
RMT:



Distributions $P(r)$ and $P(\theta)$: SU(2) fundamental $\sim$ AII†

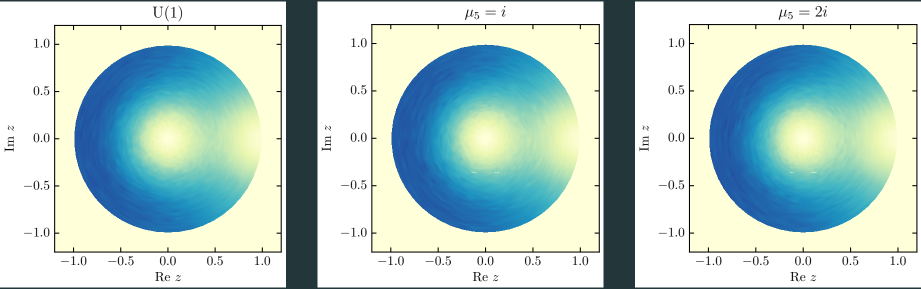


Moments of $P(z)$: SU(2) fundamental $\sim$ AII†

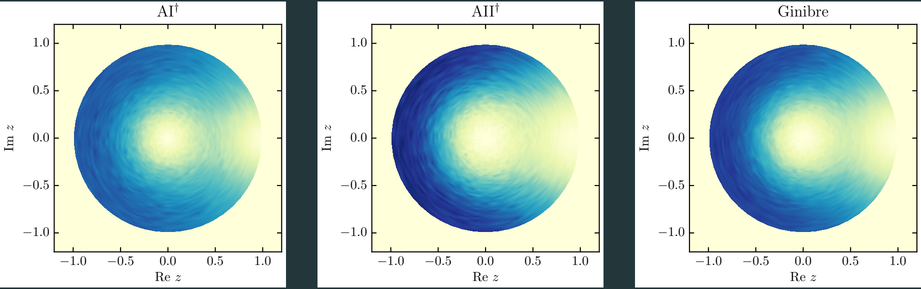


Distribution $P(z)$ of spacing ratios: SU(2) adjoint $\sim$ AI†

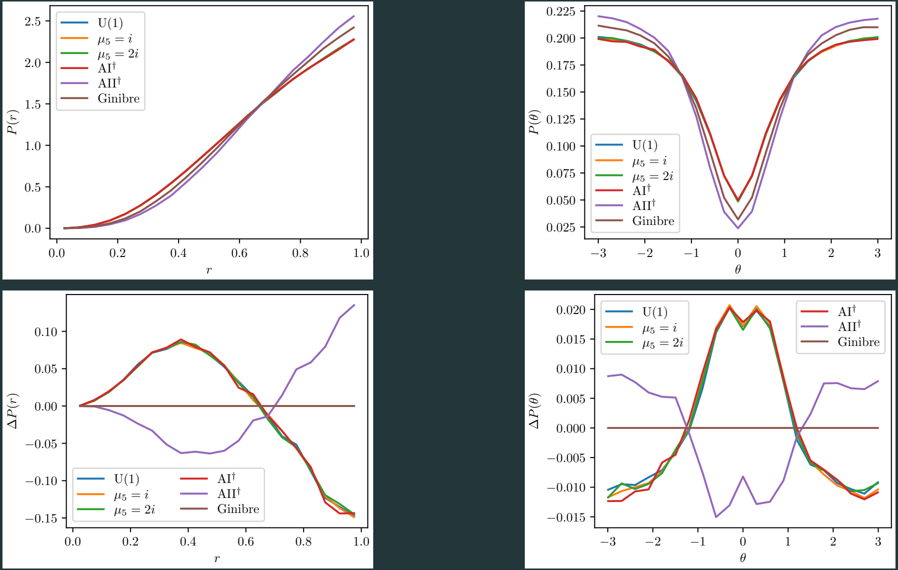
Lattice:



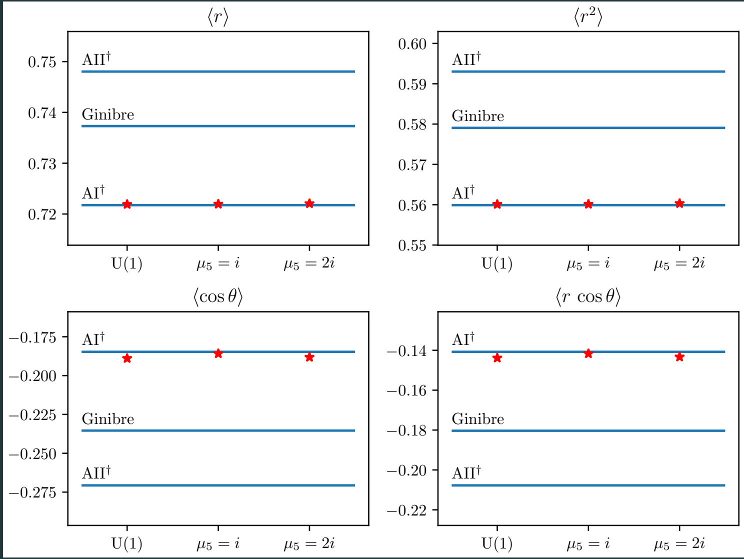
RMT:



Distributions $P(r)$ and $P(\theta)$: SU(2) adjoint $\sim$ AI†



Moments of $P(z)$: SU(2) adjoint $\sim$ AI†



Summary and outlook

- nonstandard universality classes $AI^\dagger$ and $AII^\dagger$ of non-Hermitian RMT are realized in the bulk spectral correlations of the Dirac operator coupled to a chiral U(1) gauge field or an imaginary chiral chemical potential
  - continuum: $AI^\dagger$ for pseudoreal representations, $AII^\dagger$ for real ones
  - staggered lattice Dirac operator: symmetries reversed
- checked numerically in lattice simulations for complex spacing ratios
- derived novel spectral sum rules (satisfied for all configurations)
- future work
  - deconfined phase
  - spectral correlations near zero
  - derivation of analytical RMT results