A physicist-friendly reformulation of the mod-two Atiyah-Patodi-Singer index

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Lattice 2021

Message

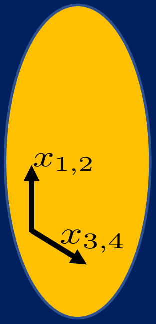
An SU(2) global anomaly inflow can be reformulated w/o unphysical boundary condition

This talk

- Continuum theory
- But our approach is useful in Lattice theory

Anomaly inflow [Callan and Harvey 1985]

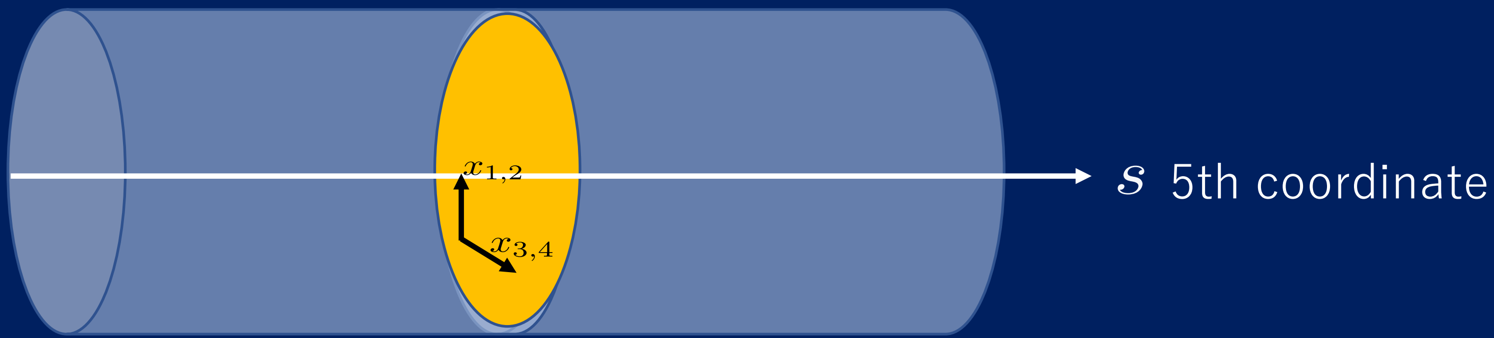
A **chiral** gauge theory on 4-dim manifold.



A perturbative gauge anomaly exists

Anomaly inflow [Callan and Harvey 1985]

A **chiral** gauge theory on 4-dim manifold.



A perturbative gauge anomaly exists

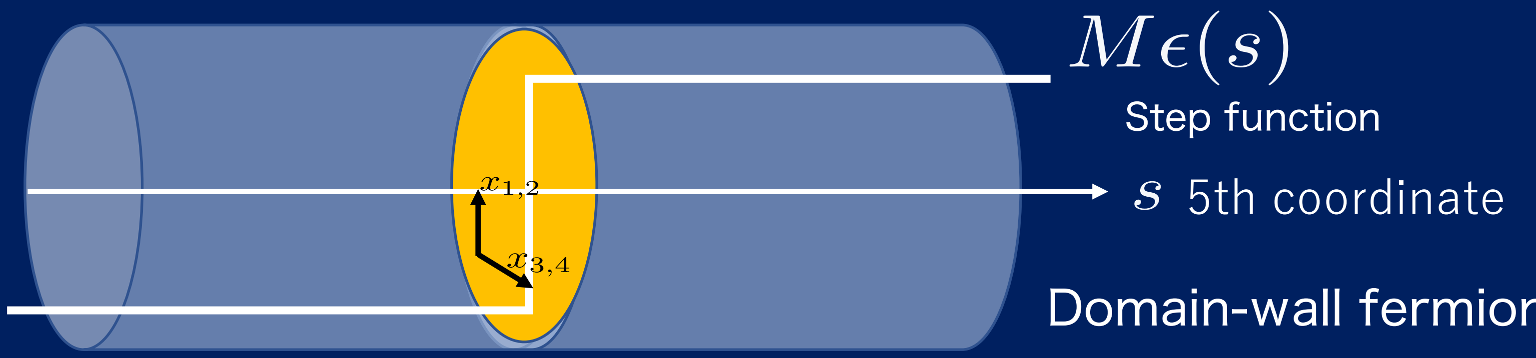
Cancellation mechanism

Embedded into the 5-dimensional theory

The total system is anomaly-free

Anomaly inflow [Callan and Harvey 1985]

A **chiral** gauge theory on 4-dim manifold.



Domain-wall fermion has edge localized modes at the kink

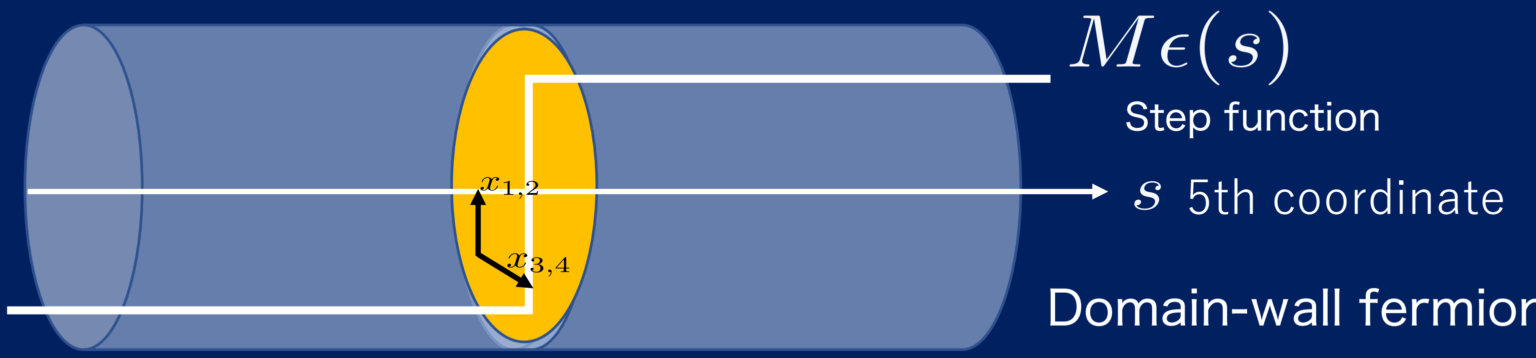
Perturbative anomaly on edge + Bulk CS current = Anomaly-free

$$Z_{\rm edge} \times Z_{\rm bulk} = Z_{\rm total}$$

Anomalous   Anomalous   **Anomaly-free**

Anomaly inflow [Callan and Harvey 1985]

A **chiral** gauge theory on 4-dim manifold.



Domain-wall fermion has edge localized modes at the kink

Perturbative anomaly on edge + Bulk CS current = Anomaly-free

This talk

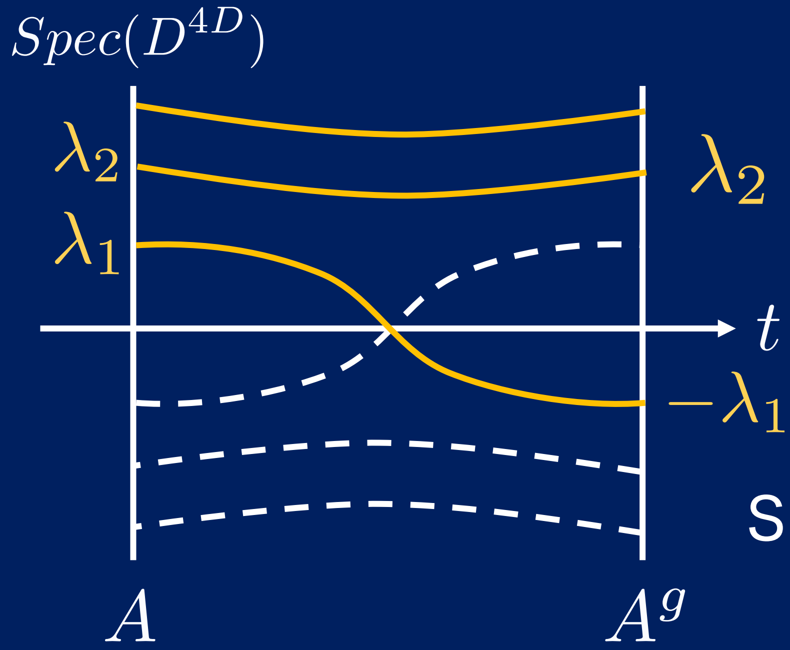
Application of anomaly inflow to **SU(2) global anomaly**

SU(2) Global anomaly [Witten 1982]

4-dim a single Weyl fermion theory is not consistent

$$Z_{\mathrm{Weyl}} = (\det D^{4D})^{1/2} = \lambda_1 \lambda_2 \cdots$$

g : nontrivial gauge transformation in $\pi_4[SU(2)]$



Spectral flow is odd

$$Z_{\mathrm{Weyl}} \to -Z_{\mathrm{Weyl}}$$

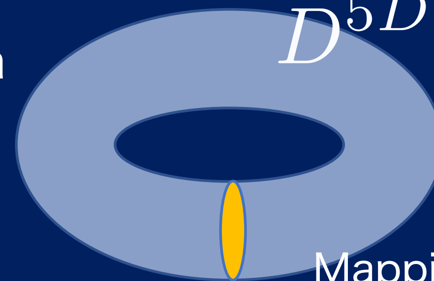
Mod-two APS index

5-dim.

4-dim

A $\quad A^g$

Identification →



D^{5D} Real-antisymmetric

Mapping torus

$$Z_{total} \propto (-1)^{Ind_{AS}(D^{5D})}$$

Gauge invariant

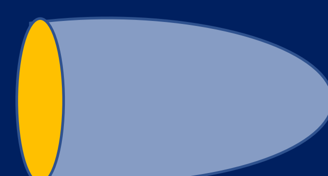
Mod-two AS index

Spec. flow odd = $Ind_{AS}(D^{5D})$ is one

# of zero-modes of $D^{5D}\,(mod\ 2)$

Extension [Witten 2015]

Extend →



APS b.c.

Mod-two APS index

$$Z_{total} \propto (-1)^{Ind_{APS}(D^{5D})}$$

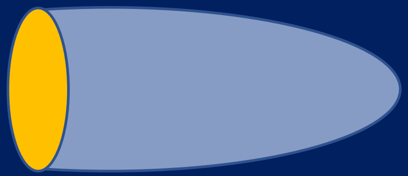
Gauge invariant

Anomaly-free

Expectation

The bulk/edge anomaly inflow description for SU(2) global anomaly

APS b.c. is unphysical



APS b.c.

[Witten 2015, Yonekura 2016]

- Non-local
- It prohibits the edge-localized mode

→The bulk/edge contributions cannot be easily separated

Unphysical

[Witten-Yonekura 2019]

APS b.c. was introduced as a "state" to evaluate the partition function and the unphysical properties are canceled between bra and ket states.

How can we reformulate WITHOUT APS b.c.?

Domain-wall fermion is physical

	APS b.c.	DW fermion
Locality	×	○
Edge-localized mode	×	○
Separation of bulk and edge	×	○

APS index theorem can be physicist-friendlily reformulated with DW fermion

*This APS index is quite different from the mod-two APS index!

c.f. [Fukaya,Onogi,Yamaguchi 2017]
[Fukaya, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita 2019]
[Fukaya, Kawai, YM, Mori, Nakayama, Onogi, Yamaguchi 2019]

Plenary talk (30 Jul, 02:40(EDT)): APS index on a lattice by H. Fukaya

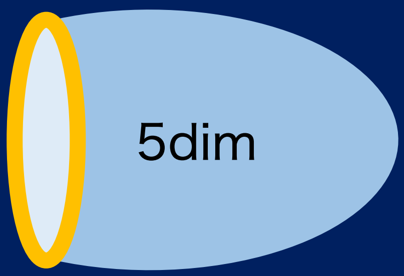
Our work

We reformulate the APS index w/o unphysical b.c. to understand bulk/edge anomaly inflow by using DW fermion

Reformulation

Our main theorem

We formulate the SU(2) global anomaly inflow in terms of the domain-wall fermion



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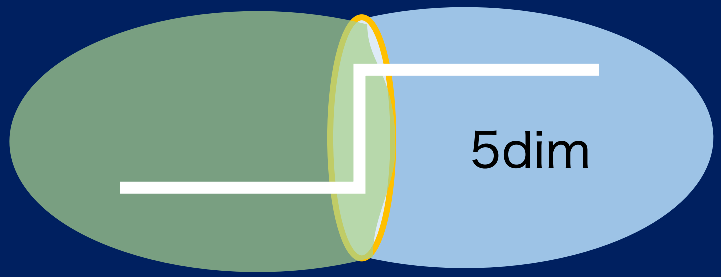


$$\mathrm{sgn}\, Z_{total} = (-1)^{Ind_{\mathrm{APS}}(D^{5D})}$$

$$\mathrm{sgn}\, Z_{DW} = \mathrm{sgn}\,\mathrm{Det}\left(\frac{D^{5D} + M\epsilon(s)}{D^{5D} - M}\right)$$

Pauli-Villars det.

$$(-1)^{Ind_{\mathrm{APS}}(D^{5D})} = \mathrm{sgn}\ \mathrm{Det}\left(\frac{D^{5D} + M\epsilon(s)}{D^{5D} - M}\right)$$

mod 2 APS index

Two steps of the proof

$$(-1)^{Ind_{\mathrm{APS}}(D^{5D})} = \mathrm{sgn}\ \mathrm{Det}\left(\frac{D^{5D} + M\epsilon(s)}{D^{5D} - M}\right)$$

mod 2 APS index $Ind_{APS}(D^{5D})$

STEP1

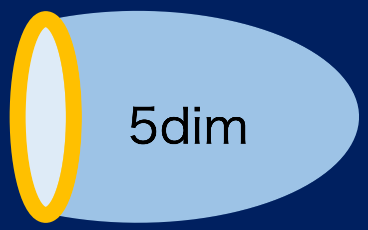
$$Ind_{AS}(D^{6D}) = Ind_{APS}(D^{5D})$$

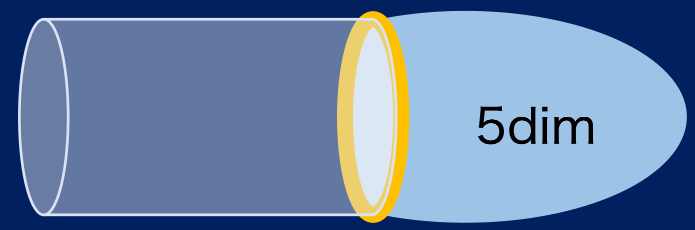
STEP2

$$(-1)^{Ind(D^{6D})} = \mathrm{sgn}\ \mathrm{Det}\left(\frac{D^{5D} + \epsilon(s)M}{D^{5D} - M}\right)$$

Two known theorems we use

Thm. I





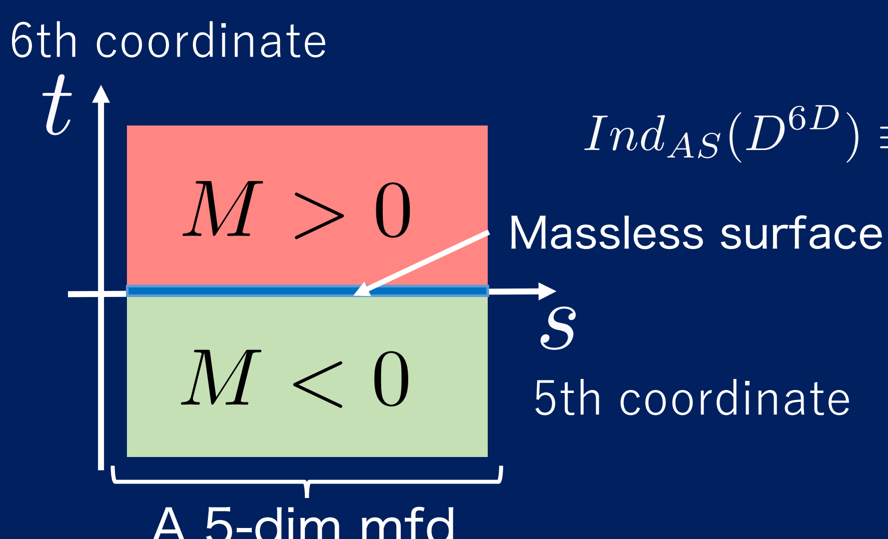
Semi-infinite cylinder

$$Ind_{APS}(D^{5D}) = Ind_{AS}(D^{5D})$$

*On the cylinder, the gauge field is constant in the extra direction

* Even if we bend the cylinder, index is unchanged

Thm. II



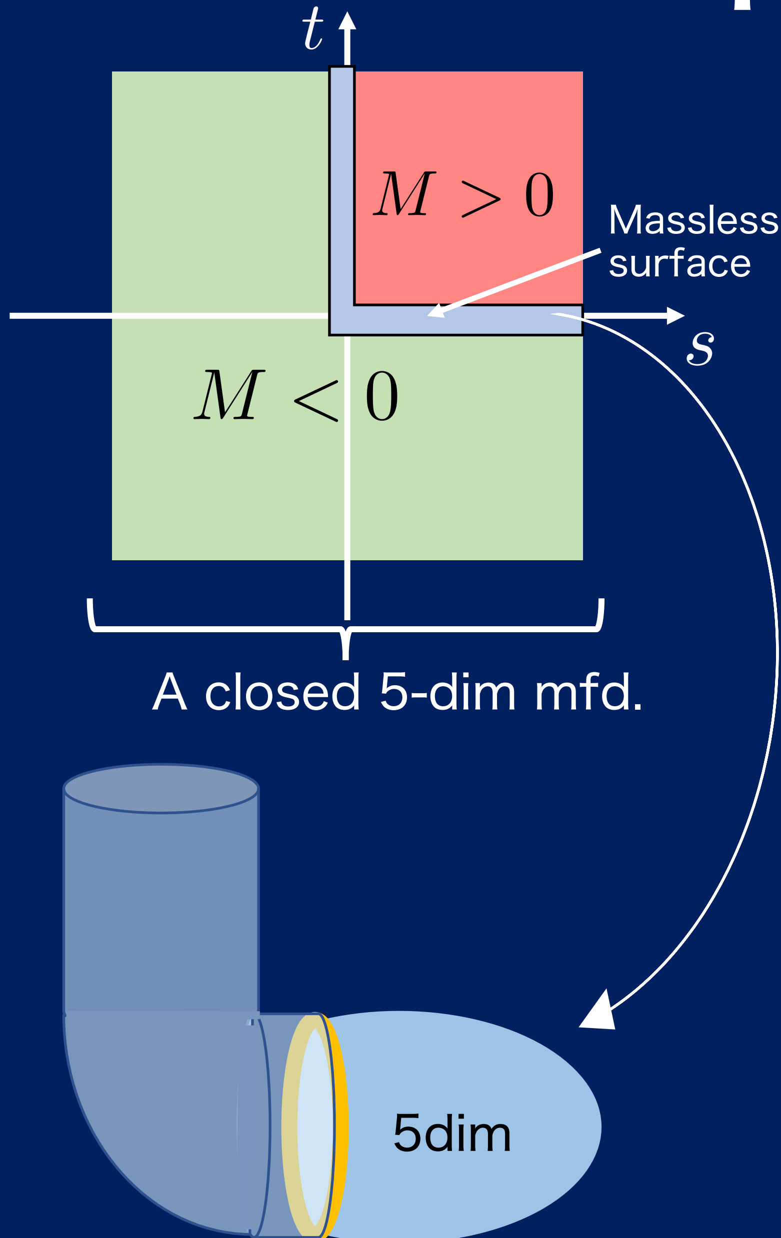
$$Ind_{AS}(D^{6D}) \equiv Ind \begin{pmatrix} \partial_t & D^{5D} + M\epsilon(t) \\ D^{5D} - M\epsilon(t) & -\partial_t \end{pmatrix}$$

$$= Ind(D^{5D}) \times Ind \begin{pmatrix} \partial_t & M\epsilon(t) \\ -M\epsilon(t) & -\partial_t \end{pmatrix}$$

$$\equiv Ind_{AS}(D^{5D}) \cdot Ind_{AS}(D^{1D})$$

Mathematical proof

STEP1 $Ind(D^{6D}) = Ind_{APS}(D^{5D})$



$$D^{6D} = \begin{pmatrix} \partial_t & D^{5D} + M\kappa \\ D^{5D} - M\kappa & -\partial_t \end{pmatrix} \qquad \kappa = \begin{cases} \epsilon(s) & (t > 0) \\ -1 & (t < 0) \end{cases}$$

$$= \begin{pmatrix} 0 & D^{5D} \\ D^{5D} & 0 \end{pmatrix} + \underbrace{\begin{pmatrix} \partial_t & M\kappa \\ -M\kappa & -\partial_t \end{pmatrix}}_{D^{1D}}$$

Thm. II

$$Ind_{AS}(D^{6D}) = Ind_{AS}(D^{5D}) \cdot \underbrace{Ind_{AS}(D^{1D})}_{=1}$$

D^{1D} has a single chiral zero-mode

Thm. I

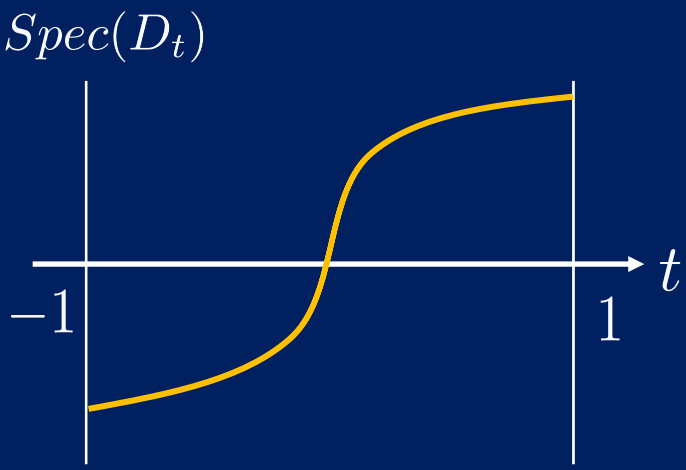
$$Ind_{AS}(D^{5D}) = Ind_{APS}(D^{5D})$$

$$\boxed{Ind_{AS}(D^{6D}) = Ind_{APS}(D^{5D})}$$

Mathematical proof STEP2 $(-1)^{Ind(D^{6D})} = \text{sgn Det}\left(\frac{D^{5D}+\epsilon(s)M}{D^{5D}-M}\right)$

$$D^{6D} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left[\begin{pmatrix} \partial_t & \\ & \partial_t \end{pmatrix} + \underbrace{\begin{pmatrix} & D^{5D}+M\kappa \\ -D^{5D}+M\kappa & \end{pmatrix}}_{D_t} \right]$$

D_t Hermite



[Carey et al 2016]

$$Ind_{AS}(D^{6D}) = \text{Spec. flow of } D_t$$

$$= \frac{1 - \text{sgn Det}A}{2}$$

s.t. $D_1 = A^\dagger D_{-1} A$

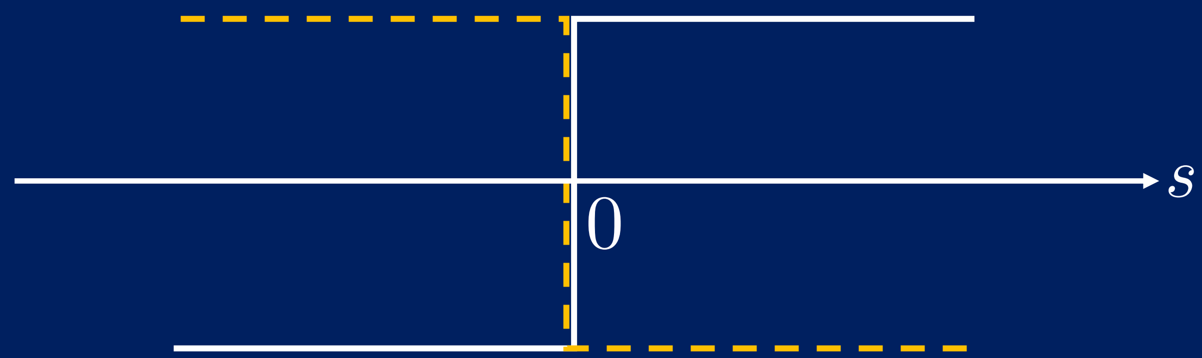
$(\Leftrightarrow \ (-1)^{Ind(D^{6D})} = \text{sgn Det}A \ )$

Evaluating det A, $\text{Det}A = \text{Det}\left(\frac{D^{5D}+\epsilon M}{D^{5D}-M}\right)$

$$(-1)^{Ind(D^{6D})} = \text{sgn Det}\left(\frac{D^{5D}+\epsilon(s)M}{D^{5D}-M}\right)$$

Separation of bulk/edge

Setup



ψ $SU(2)$ doublet ϕ $SU(2)$ singlet

D^{5D} real anti-symmetric operator $O(5)\times SU(2)$

$$\frac{\mathrm{Det}\begin{pmatrix} D^{5D}+M\epsilon(s) & 0 \\ 0 & \not{\partial}-M\epsilon(s)\end{pmatrix}}{\mathrm{Det}\begin{pmatrix} D^{5D}-M & 0 \\ 0 & \not{\partial}-M\end{pmatrix}} \propto (-1)^{Ind_{APS}}$$

Gauge-invariant

Separation of Bulk and Edge

$$\frac{\mathrm{Det}\begin{pmatrix} D^{5D}+M\epsilon(s) & 0 \\ 0 & \not\partial - M\epsilon(s) \end{pmatrix}}{\mathrm{Det}\begin{pmatrix} D^{5D}-M & 0 \\ 0 & \not\partial - M \end{pmatrix}}$$

Gauge breaking term

$$= \frac{\mathrm{Det}\begin{pmatrix} D^{5D}+M\epsilon(s) & 0 \\ 0 & \not\partial - M\epsilon(s) \end{pmatrix}}{\mathrm{Det}\begin{pmatrix} D^{5D}+M\epsilon(s) & \mu \\ \mu & \not\partial - M\epsilon(s) \end{pmatrix}} \cdot \frac{\mathrm{Det}\begin{pmatrix} D^{5D}+M\epsilon(s) & \mu \\ \mu & \not\partial - M\epsilon(s) \end{pmatrix}}{\mathrm{Det}\begin{pmatrix} D^{5D}-M & 0 \\ 0 & \not\partial - M \end{pmatrix}}$$

Cut-off for edge-mode

$$0 \ll \mu \ll M$$

Real **Real**

$$\to \frac{\mathrm{Det}\begin{pmatrix} D^{4D}P_L & 0 \\ 0 & \not\partial P_R \end{pmatrix}}{\mathrm{Det}\begin{pmatrix} D^{4D}P_L & \mu \\ \mu & \not\partial P_R \end{pmatrix}} \cdot \frac{\mathrm{Det}\begin{pmatrix} D^{5D}+M\epsilon(s) & \mu \\ \mu & \not\partial - M\epsilon(s) \end{pmatrix}}{\mathrm{Det}\begin{pmatrix} D^{5D}-M & 0 \\ 0 & \not\partial - M \end{pmatrix}}$$

$$\propto (-1)^{I_{edge}} \cdot (-1)^{I_{bulk}} = (-1)^{Ind_{APS}}$$

Summary and Outlook

We have reformulated the mod-two APS index in a physicist-friendly way, where NO unphysical boundary condition is imposed.

The bulk/edge anomaly inflow is manifest:

$$Ind_{APS}(D^{5D}) = I_{\mathrm{edge}} + I_{\mathrm{bulk}}$$

$$I_{\mathrm{edge}} = \frac{1}{2}\left(1 - \mathrm{sgn}\frac{\mathrm{Det}\begin{pmatrix} D^{4D}P_L & \\ & \not{\partial}P_R \end{pmatrix}}{\mathrm{Det}\begin{pmatrix} D^{4D}P_L & \mu \\ \mu & \not{\partial}P_R \end{pmatrix}}\right) \quad I_{\mathrm{bulk}} = \frac{1}{2}\left(1 - \mathrm{sgn}\frac{\mathrm{Det}\begin{pmatrix} D^{5D} + M\epsilon(s) & \mu \\ \mu & \not{\partial} - M\epsilon(s) \end{pmatrix}}{\mathrm{Det}\begin{pmatrix} D^{5D} - M & \mu \\ \mu & \not{\partial} - M \end{pmatrix}}\right)$$

- **Application to the lattice gauge theory**

$D^{lat}_{DW} = D_W + m\epsilon(s)$ is also real

Wilson-Dirac operator

$$D_W = \gamma_\mu \frac{\nabla_\mu + \nabla^*_\mu}{2} - \frac{\nabla_\mu \nabla^*_\mu}{2}$$