Chiral Fermion on Curved Domain-wall

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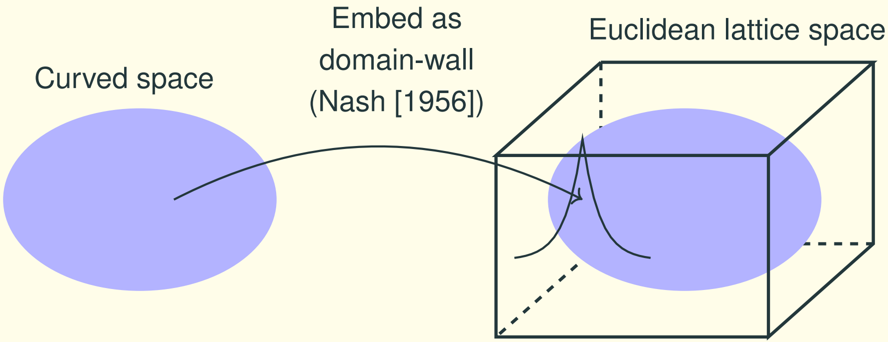
Motivation

We propose a lattice formulation of fermion in a curved space, embedding a curved domain-wall into a flat square lattice.



Cf. Brower et al. [2017] and Ambjørn et al. [2001] studied on triangle lattices.

Embedding a curved space

For any n-dim. Riemann space (M^n, g), there is an embedding $f : M^n \to \mathbb{R}^m$ $(m \gg n)$ such that M^n is identified as

$$x^\mu = f^\mu(\tilde{x}^1, \cdots, \tilde{x}^n) \; (\mu = 1, \cdots, m) \tag{1}$$

$$\left(\begin{array}{ll} x^\mu & \text{: Cartesian coordinates of } \mathbb{R}^m \\ \tilde{x}^i & \text{: coordinates of } M^n \end{array} \right.$$

and the metric g is induced as

$$g_{ij} = \sum_{\mu\nu} \frac{\partial f^\mu}{\partial \tilde{x}^i} \delta_{\mu\nu} \frac{\partial f^\nu}{\partial \tilde{x}^j}. \tag{2}$$

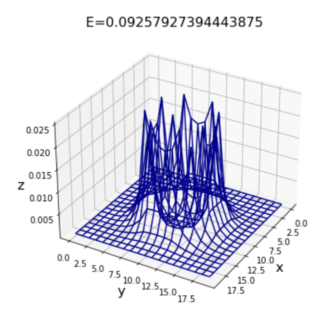
Therefore (M^n, g) can be identified as a submanifold of $\mathbb{R}^m$.

cf. Nash [1956].

Our Work

We find that

- Edge states are localized at the curved domain-wall (S^1 or S^2 in this work),
- They feel gravity (through induced spin connection).



Cf. Similar studies in condensed matter physics [Imura et al. [2012], Parente et al. [2011]].

Contents

Plan of this section

We embed an S^1 domain-wall into $\mathbb{R}^2$ and study

- Spectrum of Dirac operator,
- Edge states,
- Their effective Dirac op.

both in the continuum and on the lattice.

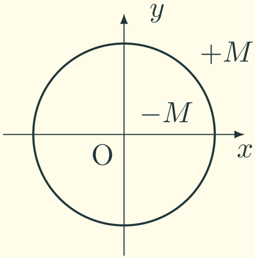


Fig 1: Continuum Case

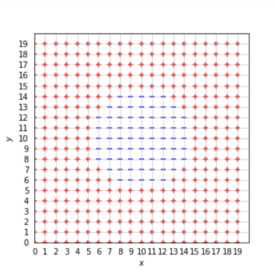
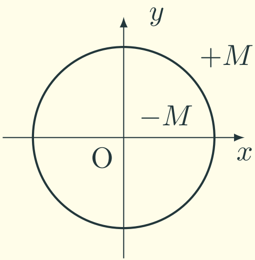


Fig 2: Lattice case

Dirac op in Continuum case

The domain-wall is given by

$$\epsilon_A(r) = \text{sign}(r - r_0) = \begin{cases} -1 & (r < r_0) \\ 1 & (r \geq r_0) \end{cases},$$



and the Dirac op. is

$$\begin{aligned} H &= \sigma_3 \left(\sum_{i=1,2} \left(\sigma_i \frac{\partial}{\partial x^i} \right) + M\epsilon \right) \\ &= \begin{pmatrix} M\epsilon & e^{-i\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \theta}\right) \\ -e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right) & -M\epsilon \end{pmatrix}. \end{aligned} \qquad (3)$$

Edge states

When M is large enough, edge states are

$$\psi^{E,j}_{\text{edge}} \simeq \sqrt{\frac{M}{4\pi r}} e^{-M|r-r_0|} \begin{pmatrix} e^{i(j-\frac{1}{2})\theta} \\ e^{i(j+\frac{1}{2})\theta} \end{pmatrix}.$$

They are "chiral" eigenstates of

$$\begin{aligned} \gamma_{\text{normal}} :=& \sigma_1 \cos\theta + \sigma_2 \sin\theta \\ =& \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}, \end{aligned}$$

with eigenvalue $+1$.

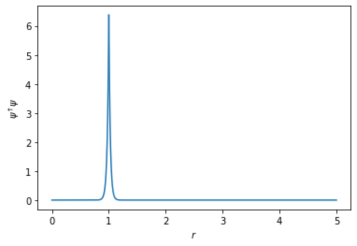


Fig 3: Edge state when $M = 5, r_0 = 1$

Effective Dirac op

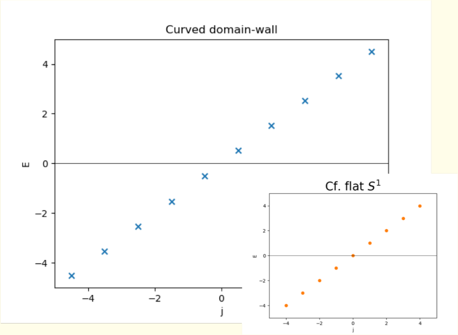


Fig 4: Eigenvalue of edge states at $M = 5, r_0 = 1$

Effective Dirac op on the chiral edge states is

$$H_{S^1} = \frac{1}{r_0}\left(-i\frac{\partial}{\partial\theta} + \frac{1}{2}\right) \tag{4}$$

$$E = \frac{j}{r_0}\left(j = \pm\frac{1}{2}, \pm\frac{3}{2}, \cdots\right). \tag{5}$$

The $\frac{1}{2}$ is identified as the induced spin connection.

Gravity appear as the gap of the spectrum

Lattice domain-wall fermion

Let $(\mathbb{Z}/n\mathbb{Z})^2$ is a 2-dim. lattice. The domain-wall is given by

$$\epsilon(x) = \begin{cases} -1 & (|x| < r_0) \\ 1 & (|x| \geq r_0) \end{cases},$$

and the (Wilson) Dirac op is

$$H = \sigma_3 \left(\sum_{i=1,2} \left[\sigma_i \frac{\nabla_i^f + \nabla_i^b}{2} - \frac{r}{2} \nabla_i^f \nabla_i^b \right] + \epsilon M \right),$$

$$(\nabla_i^f \psi)_x = \psi_{x+\hat{i}} - \psi_x, \ (\nabla_i^b \psi)_x = \psi_x - \psi_{x-\hat{i}}$$

where periodic boundary condition is imposed in the x and y direction.

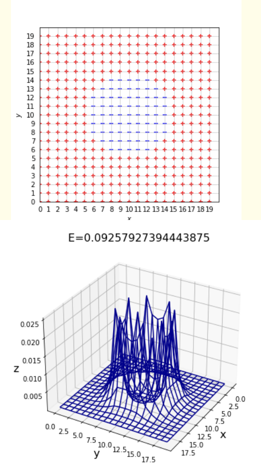


Fig 5: Edge state

Cf. Kaplan [1992] studied a flat domain-wall in $\mathbb{R}^{2m+1}$

Continuum Limit

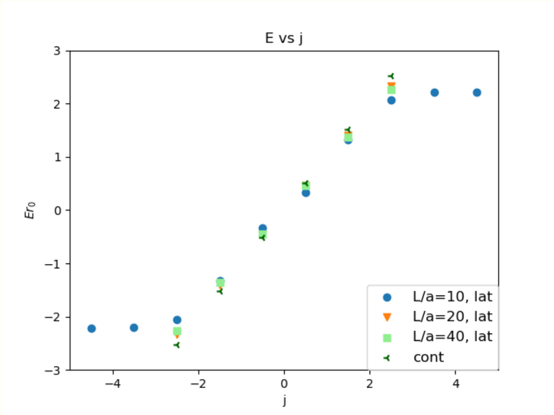


Fig 6: the Dirac eigenvalue spectrum normalized by the circle radius when $Ma = 0.7, r_0 = L/4$.

L:Lattice size
a:lattice spacing

Relative Error

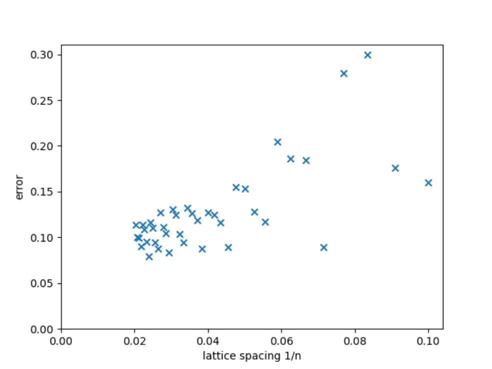


Fig 7: error $= \left|E^{\text{con}}_{\frac{1}{2}} - E^{\text{lat}}_{\frac{1}{2}}\right| / E^{\text{con}}_{\frac{1}{2}}$ is a relative error of $E_{1/2}$ between continuum and lattice when $r_0 = \frac{L}{4}$. a is lattice distance and $n \to \infty$ means continuum limit.

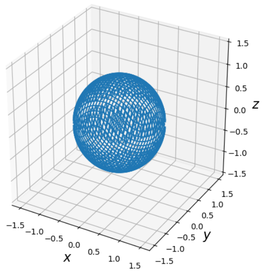
Finite a scaling is not monotonic but decreasing.

Contents

S^2 domain-wall

The domain-wall is given by

$$\epsilon(r) = \text{sign}(r - r_0) = \begin{cases} -1 & (r < r_0) \\ 1 & (r \geq r_0) \end{cases},$$



and the Dirac op is

$$H = \gamma^0 \left(\gamma^j \frac{\partial}{\partial x^j} + M\epsilon \right) = \begin{pmatrix} M\epsilon & \sigma^j \partial_j \\ -\sigma^j \partial_j & -M\epsilon \end{pmatrix} \quad (6)$$

$$\gamma^0 = \sigma_3 \otimes 1, \gamma^j = \sigma_1 \otimes \sigma^j \quad (7)$$

Edge states and Their spectrum

In the large M limit, edge states are

$$\psi_{\text{edge}}^{\pm E,j,j_3} \simeq \sqrt{\frac{M}{2}}\frac{e^{-M|r-r_0|}}{r}\begin{pmatrix}\chi_{j,j_3}^{(\pm)} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{r}\chi_{j,j_3}^{(\pm)}\end{pmatrix},$$

$$E \simeq \frac{j+\frac{1}{2}}{r_0}\ \left(j=\frac{1}{2},\frac{3}{2},\cdots\right)$$

Edge states are "chiral" states of

$$\gamma_{\text{normal}} := \sum_{i=1}^{3}\frac{x^i}{r}\gamma^i = \begin{pmatrix}0 & \frac{\boldsymbol{x}\cdot\boldsymbol{\sigma}}{r} \\ \frac{\boldsymbol{x}\cdot\boldsymbol{\sigma}}{r} & 0\end{pmatrix} \tag{8}$$

with eigenvalue $+1$.

Effective Dirac op is obtained as

$$H_{S^2} = \frac{1}{r_0}(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1), \tag{9}$$

which acts on a two-component spinor χ.

Effective Dirac op and Dirac op. of S^2

Effective Dirac op is obtained as

$$H_{S^2} = \frac{1}{r_0}(\boldsymbol{\sigma} \cdot \boldsymbol{L} + 1) \tag{10}$$

$\downarrow$ gauge trsf. by $s = \begin{pmatrix} e^{-i\frac{\phi}{2}} \cos\left(\frac{\theta}{2}\right) & -e^{-i\frac{\phi}{2}} \sin\left(\frac{\theta}{2}\right) \\ e^{i\frac{\phi}{2}} \sin\left(\frac{\theta}{2}\right) & e^{i\frac{\phi}{2}} \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$

$$\begin{aligned} s^{-1} H_{S^2} s &= -\frac{\sigma_3}{r_0}\left(\sigma_1 \frac{\partial}{\partial\theta} + \sigma_2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \underbrace{- \frac{\cos\theta}{2\sin\theta}\sigma_1\sigma_2}_{\text{Spin conn. of } S^2}\right)\right) \\ &= -\frac{\sigma_3}{r_0} \mathbb{D}_{S^2}. \end{aligned} \tag{11}$$

Edge states feel gravity of the spherical domain-wall

Cf. [Takane and Imura [2013]].

Euler number of S^2

We get the spin connection

$$\omega_\Delta = -\frac{\cos\theta}{2\sin\theta}\sigma_1\sigma_2\sin\theta d\phi = -\frac{1}{2}i\sigma_3\cos\theta d\phi, \tag{12}$$

So the Levi-Civita connection ω and Riemann curvature R is given by

$$\omega = \begin{pmatrix} 0 & -\cos\theta d\phi \\ \cos\theta d\phi & 0 \end{pmatrix} \tag{13}$$

$$\frac{R}{2\pi} = \frac{d\omega + \omega^2}{2\pi} = \begin{pmatrix} 0 & \frac{\sin\theta}{2\pi}d\theta d\phi \\ -\frac{\sin\theta}{2\pi}d\theta d\phi & 0 \end{pmatrix} \tag{14}$$

Euler class of S^2

The Euler number of S^2 is identified as

$$\chi(S^2) = \int_{S^2}\frac{\sin\theta}{2\pi}d\theta d\phi = 2. \tag{15}$$

Induced gravity makes a gap in the spectrum.

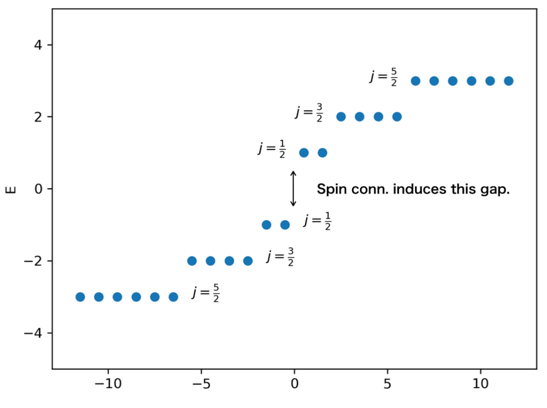


Fig 8: Spectrum of edge states when $M = 5, r_0 = 1$

Eigenvalue

$$E \simeq \pm \frac{j + \frac{1}{2}}{r_0} \qquad (16)$$

Degeneracy

$$2j + 1 \qquad (17)$$

$(\text{Euler number of } S^2) = 2$

Lattice Domain-wall Fermion

Let $(\mathbb{Z}/n\mathbb{Z})^3$ is a 3-dim. lattice. The domain-wall is given by

$$\epsilon(x) = \begin{cases} -1 & (|x| < r_0) \\ 1 & (|x| \geq r_0) \end{cases},$$

and the (Wilson) Dirac op is

$$H = \gamma_3 \left(\sum_{i=1,2} \left[\gamma_i \frac{\nabla_i^f + \nabla_i^b}{2} - \frac{r}{2} \nabla_i^f \nabla_i^b \right] + \epsilon M \right).$$

$$(\nabla_i^f \psi)_x = \psi_{x+\hat{i}} - \psi_x,\ (\nabla_i^b \psi)_x = \psi_x - \psi_{x-\hat{i}}$$

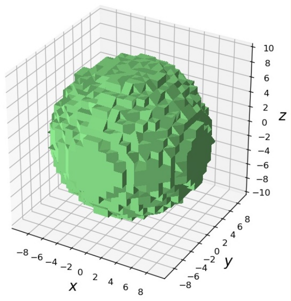


Fig 9: S^2 Domain-wall on lattice

Edge states

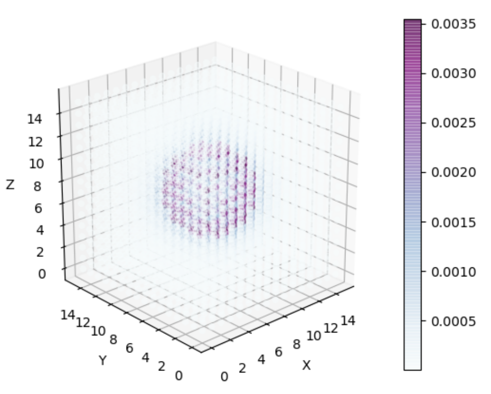


Fig 10: Edge state localized at S^2 when M=0.7 and lattice size $= 16^3$

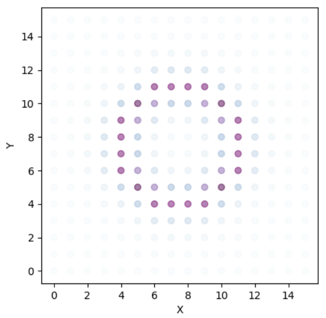


Fig 11: Slice at $z = 7$

Spectrum in Lattice case

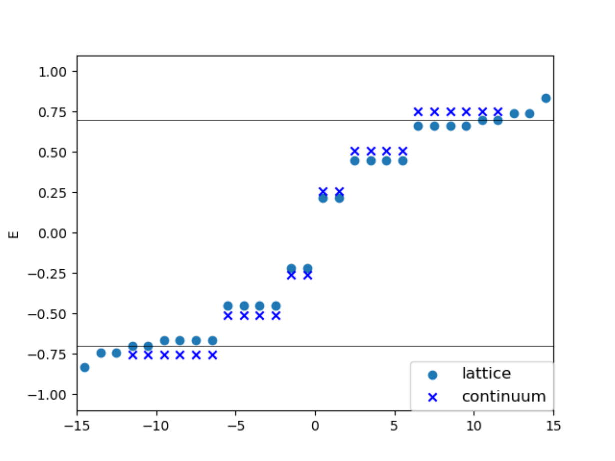


Fig 12: Spectrum of edge states at S^2 when $n = 16, M = 0.7$.

It reproduces the spectrum of continuum!

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Summary

We have considered S^1 and S^2 as a curved domain-wall on square lattice. We have confirmed that

- Chiral edge-localized states appear at the domain-wall.
- They feel gravity through the induced spin connection.

Outlook

- Systematics in the continuum limit
- Gravitational anomaly inflow.
- Index theorem with a nontrivial curvature.

Reference i

Ambjørn, J., Jurkiewicz, J., and Loll, R. (2001). Dynamically triangulating lorentzian quantum gravity. Nuclear Physics B, 610(1):347–382.

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Parente, V., Lucignano, P., Vitale, P., Tagliacozzo, A., and Guinea, F. (2011). Spin connection and boundary states in a topological insulator. Phys. Rev. B, 83:075424.

Takane, Y. and Imura, K.-I. (2013). Unified description of dirac electrons on a curved surface of topological insulators. Journal of the Physical Society of Japan, 82(7):074712.

Contents

Effective Dirac op

We consider a normalized edge state as

$$\psi_{\text{edge}} = \rho(r)\begin{pmatrix}\chi(\theta)\\ e^{i\theta}\chi(\theta)\end{pmatrix},\ \chi(\theta+2\pi)=\chi(\theta) \tag{18}$$

$$\int_0^\infty dr 2r\rho^2 = 1,\ \int_0^{2\pi} d\theta\chi^\dagger\chi = 1 \tag{19}$$

and let $2r\rho^2 \to \delta(r-r_0)$ $(M\to\infty)$. Then we obtain

$$\int dxdy\psi^\dagger_{\text{edge}} H\psi_{\text{edge}} \to \int_0^{2\pi} d\theta\chi^\dagger \underbrace{\frac{1}{r_0}\left(-i\frac{\partial}{\partial\theta}+\frac{1}{2}\right)}_{\text{Effective Dirac op } H_{S^1}\text{ !!}}\chi \tag{20}$$

The factor $\frac{1}{2}$ means induced spin connection.

Effective Dirac op

We consider a normalized edge state as

$$\psi_{\text{edge}} = \rho(r) \begin{pmatrix} \chi(\theta,\phi) \\ \frac{\boldsymbol{x}\cdot\boldsymbol{\sigma}}{r}\chi(\theta,\phi) \end{pmatrix} \tag{21}$$

$$\int_0^\infty drr^2 2\rho^2 = 1, \; \int_{S^2} \chi^\dagger \chi = 1, \tag{22}$$

and we assume $2r^2\rho^2 \to \delta(r - r_0)$ $(M \to \infty)$. Thus

$$\begin{aligned} \int dx^3 \psi^\dagger_{\text{edge}} H \psi_{\text{edge}} &= \int_0^\infty dr 2r^2\rho^2 \int_{S^2} \chi^\dagger \frac{1}{r}(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)\chi \\ &\to \int_{S^2} \chi^\dagger \frac{1}{r_0}(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)\chi \;(M\to\infty), \end{aligned} \tag{23}$$

Effective Dirac op H_{S^2} !!

where $\boldsymbol{L}$ is an orbital angular momentum.

Effective Dirac op and Dirac op. of S^2

The gauge transformation using

$$s = \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) & -e^{-i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) \\ e^{i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) & e^{i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) \end{pmatrix} \tag{24}$$

changes $\chi \to s^{-1}\chi$ and

$$\begin{aligned} H_{S^2} \to & s^{-1} H_{S^2} s \\ = & \frac{1}{r_0}\begin{pmatrix} 0 & -\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} - \frac{1}{2}\frac{\cos\theta}{\sin\theta} \\ \frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} + \frac{1}{2}\frac{\cos\theta}{\sin\theta} & 0 \end{pmatrix} \\ = & -\frac{\sigma_3}{r_0}\left(\sigma_1\frac{\partial}{\partial\theta} + \sigma_2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - \underbrace{\frac{\cos\theta}{2\sin\theta}\sigma_1\sigma_2}_{\text{Spin conn. of } S^2}\right)\right) \\ = & -\frac{\sigma_3}{r_0}\not{D}_{S^2}. \end{aligned} \tag{25}$$

Edge states are affected by the spin connection of the spherical domain-wall [Takane and Imura [2013]].

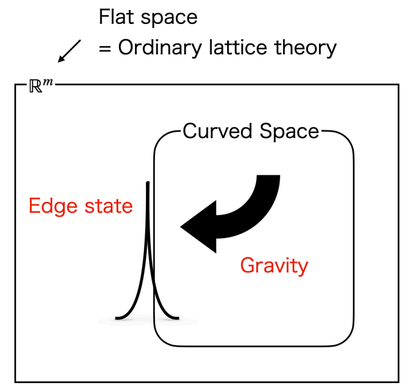
Goal

Embed S^1, S^2 into a square lattice.

Curved domain-wall

↓

- Edge states appear !
- They feel gravity !



Motivation

It is too difficult to consider a lattice theory on a curved space.
If we use

- A square lattice
  $\longrightarrow$ A curved space can NOT be approximated by it.
- Triangulation [Ambjørn et al. [2001]]
  $\longrightarrow$Lattice regularization is different from of lattice gauge theory.

Fig 13: Triangulation of a toy[1]

[1] https://12px.com/blog/2014/02/delaunay/

Main result

- Edge states appear at the curved domain-wall,
- They feel gravity or curvature through the induced spin connection.

Cf. Similar studies in condensed matter physics.[Imura et al. [2012], Parente et al. [2011]].

Domain-wall and edge states

If the sign of mass is flipped as

$$\epsilon(x) = \begin{cases} -1 & (x < 0) \\ 1 & (x > 0) \end{cases},$$

then localized states appear at $x = 0$.

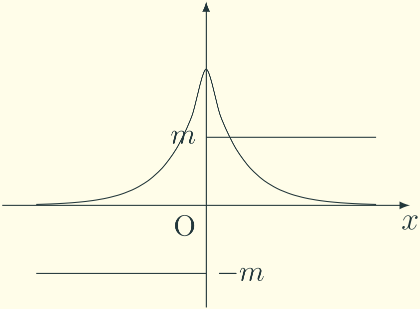


Fig 14: Edge state localized at the domain-wall.