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The gradient flow at higher orders in perturbation theory

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based on work with
Johannes Artz, Yannick Kluth, Fabian Lange, Tobias Neumann, Mario Prausa

TTK Institute for Theoretical Particle Physics and Cosmology | RWTH AACHEN UNIVERSITY

The gradient flow

flowed gauge field:

$$\frac{\partial}{\partial t} B_\mu(t,x) = \mathcal{D}_\nu G_{\nu\mu}(t,x)$$

$$B_\mu(t=0,x) = A_\mu(x)$$

flowed quark field:

$$\frac{\partial}{\partial t} \chi(t,x) = \mathcal{D}^2 \chi(t,x)$$

$$\chi(t=0,x) = \psi(x)$$

Lüscher ’10, ‘13

The gradient flow

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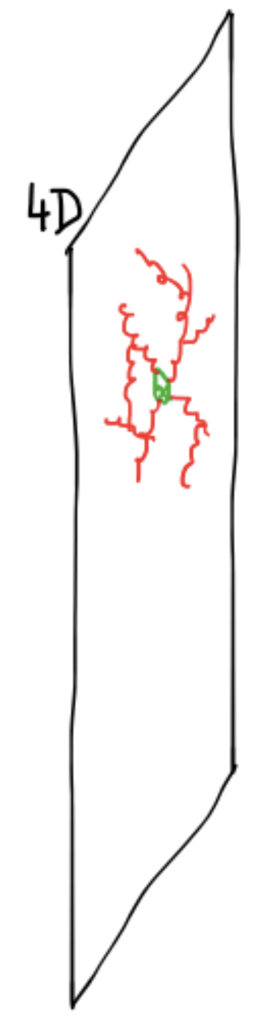

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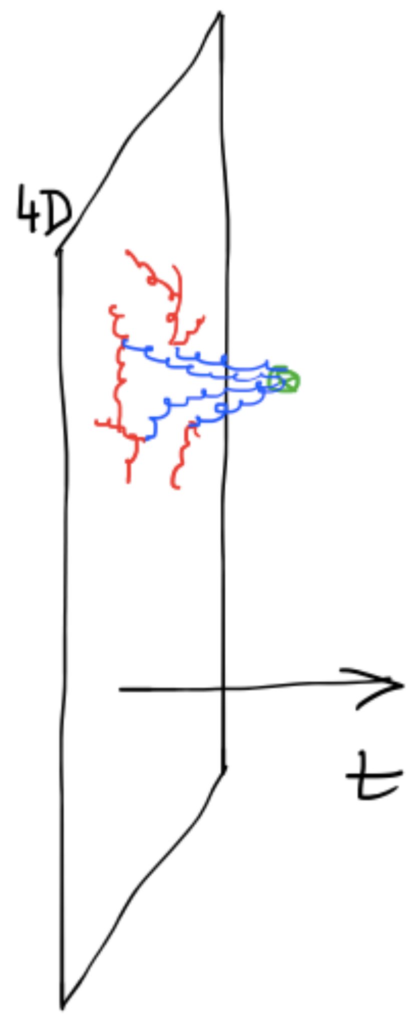

$$B_\mu(t,p) \sim e^{-tp^2} A_\mu(p)$$

A common problem

Observable: $$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

perturbation theory → C_n

lattice → $\langle \mathcal{O}_n \rangle$

match renormalization schemes?

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Instead:

$$R = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

perturbation theory → $\tilde{C}_n(t)$

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gradient flow renormalization

Institute for Theoretical Particle Physics and Cosmology
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gradient flow renormalization

see also talks by A. Hasenfratz, C. Monahan, M. Rizik, …

Small flow-time expansion

Observable: $$R = \sum_n C_n \langle \mathcal{O}_n \rangle = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

small flow-time expansion: $$\tilde{\mathcal{O}}_n(t) \overset{t\to 0}{\to} \sum_m \zeta_{nm}(t)\, \mathcal{O}_m$$

Lüscher, Weisz '11

$$\tilde{C}_n(t) \overset{t\to 0}{\to} \sum_m C_m\, \zeta^{-1}_{mn}(t)$$

⇒ need $\zeta_{nm}(t)$ for small t ⇒ perturbation theory

Method of projectors

$$\tilde{\mathcal{O}}_n(t) \stackrel{t\to 0}{\to} \sum_m \zeta_{nm}(t)\, \mathcal{O}_m$$

define projector: $P_k[\mathcal{O}_m] \equiv \delta_{km}$ to all orders in perturbation theory

$$P_k[\tilde{\mathcal{O}}_n(t)] \stackrel{t\to 0}{\to} \sum_m \zeta_{nm}(t)\, P_k[\mathcal{O}_m] \quad \Rightarrow \quad \zeta_{nk}(t) = P_k[\tilde{\mathcal{O}}_n(t)]$$

$$P_k[\,\cdot\,] \sim H(\partial/\partial p_i, \partial/\partial m)\, \langle 0 |\cdot| p_1, \ldots\rangle \Big|_{p_i=m=0}$$

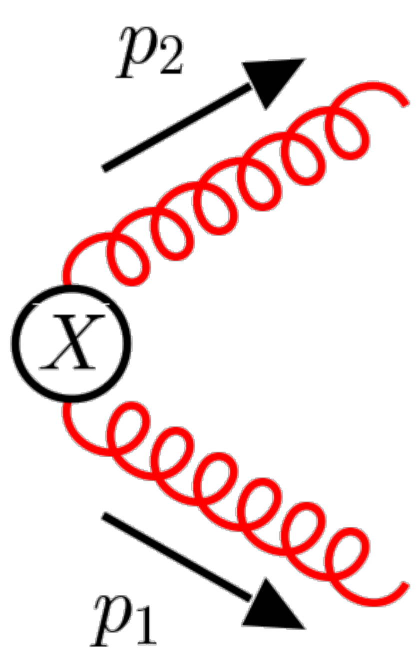
Gorishny, Larin, Tkachov '83

Method of projectors

Example: dim-4 operators of QCD:

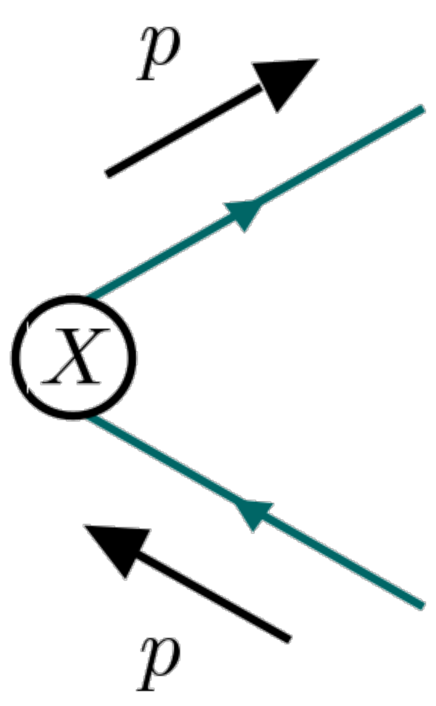
$$\mathcal{O}_1 \sim F^a_{\mu\nu} F^a_{\mu\nu} \qquad\qquad \mathcal{O}_2 \sim m\bar{\psi}\psi$$

$$P_1[X] \sim \frac{\partial}{\partial p_1} \cdot \frac{\partial}{\partial p_2}$$



$$\Bigg|_{p_1 = p_2 = m = 0}$$

$$P_2[X] \sim \frac{\partial}{\partial m}$$



$$\Bigg|_{p = m = 0}$$

note: scale-less integrals = 0 in dimensional regularization!

$\Rightarrow \quad P_k[\mathcal{O}_m] \equiv \delta_{km}$ to all orders in perturbation theory

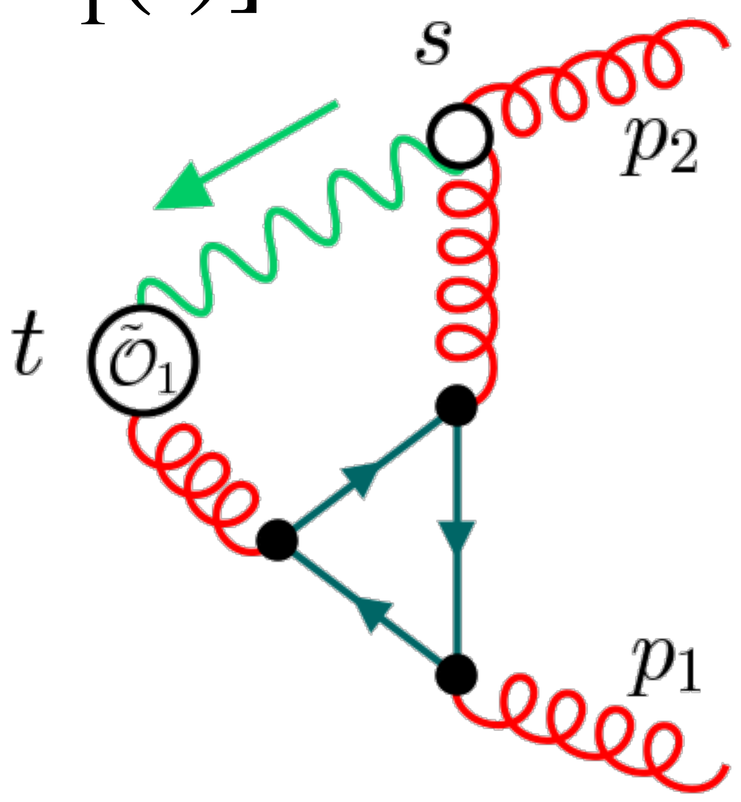
but: $P_k[\tilde{\mathcal{O}}_m(t)]$ is *not* scale-less!

Method of projectors

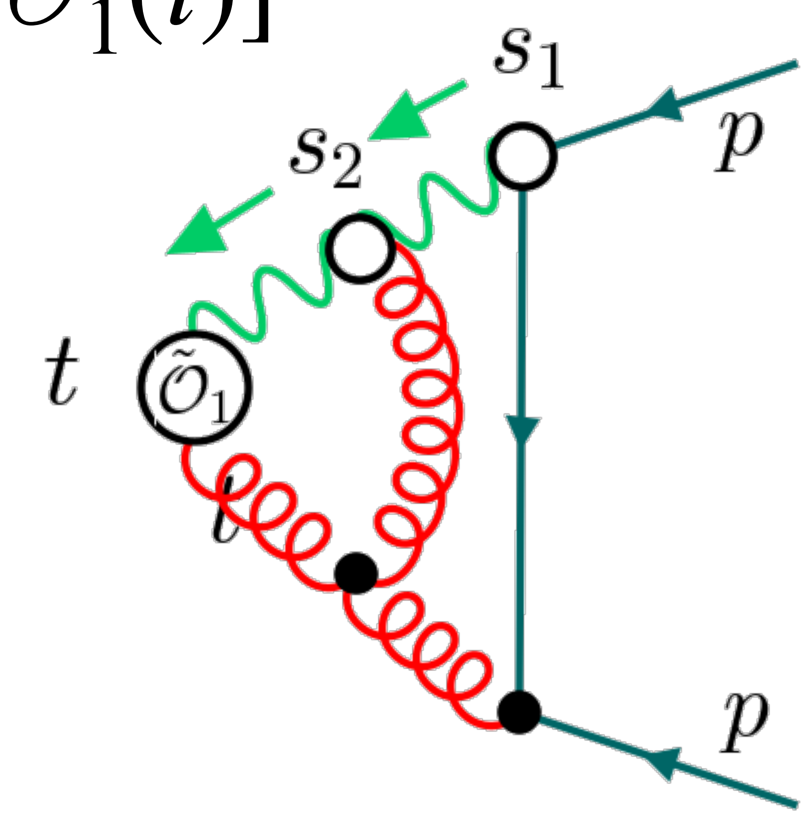
$$\tilde{\mathcal{O}}_n(t) \overset{t\to 0}{\to} \sum_m \zeta_{nm}(t)\, \mathcal{O}_m \qquad\qquad P_k[\tilde{\mathcal{O}}_n(t)] \overset{t\to 0}{\to} \sum_m \zeta_{nm}(t)\, P_k[\mathcal{O}_m]$$

Use 5-dimensional QFT formulation: Lüscher, Weisz '11

$$\zeta_{11}(t) = P_1[\tilde{\mathcal{O}}_1(t)]$$



$$\zeta_{21}(t) = P_2[\tilde{\mathcal{O}}_1(t)]$$



produced with FeynGame

Method of projectors

$$\tilde{\mathcal{O}}_n(t) \overset{t\to 0}{\to} \sum_m \zeta_{nm}(t)\,\mathcal{O}_m \qquad P_k[\tilde{\mathcal{O}}_n(t)] \overset{t\to 0}{\to} \sum_m \zeta_{nm}(t) P_k[\mathcal{O}_m]$$

δ_{km}

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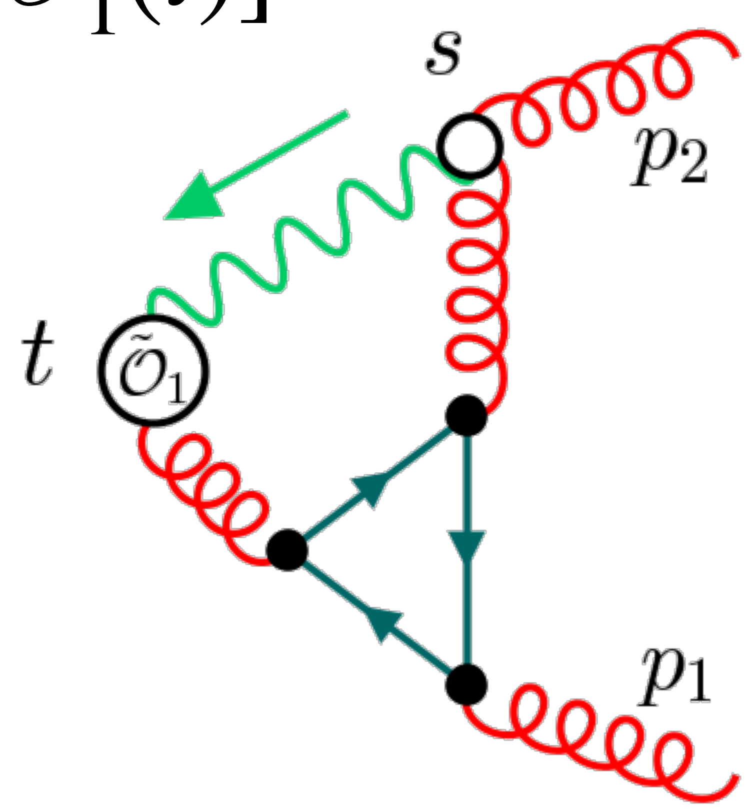

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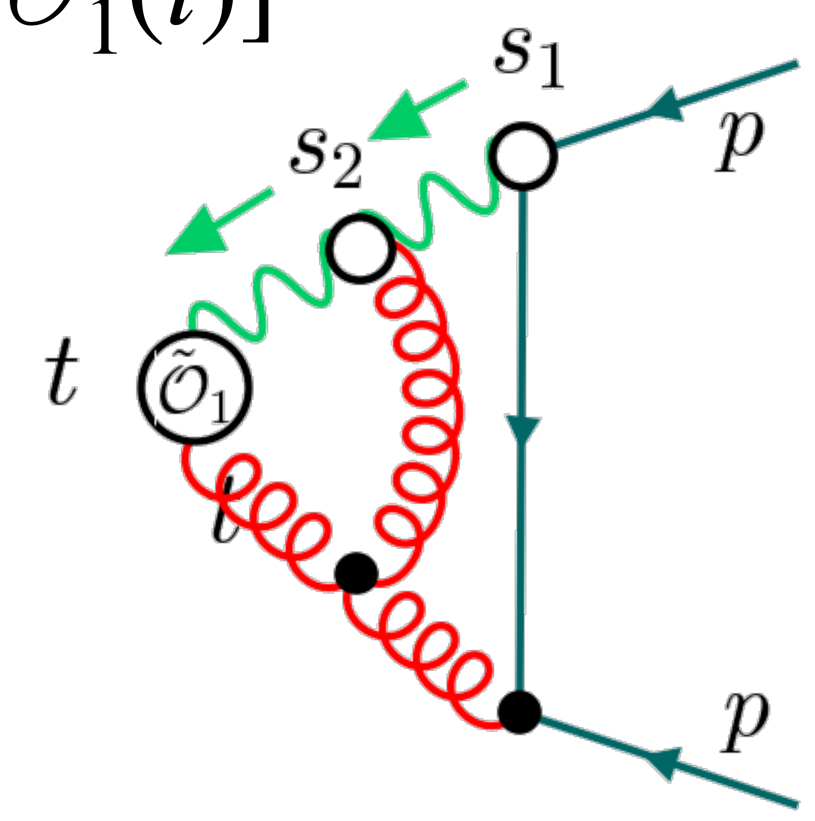

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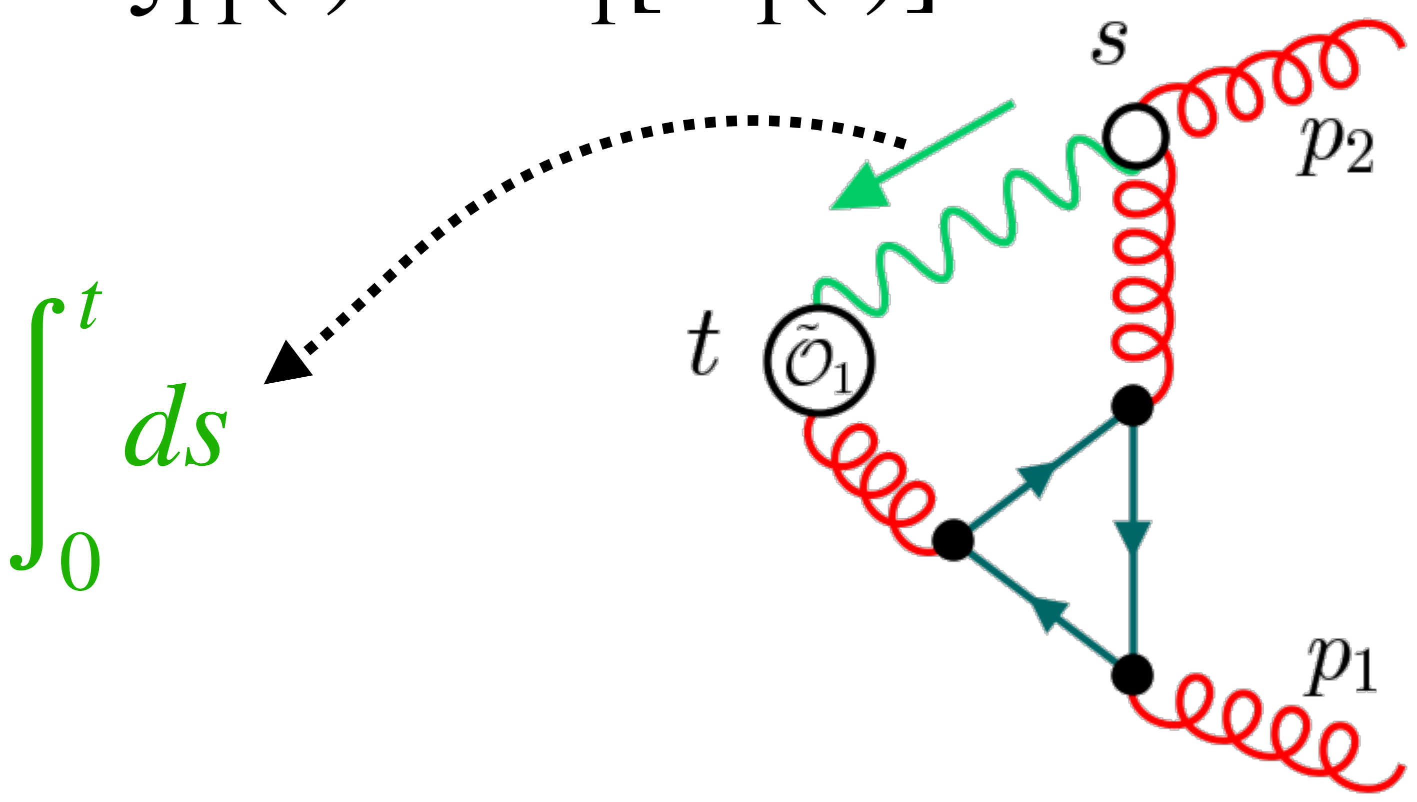

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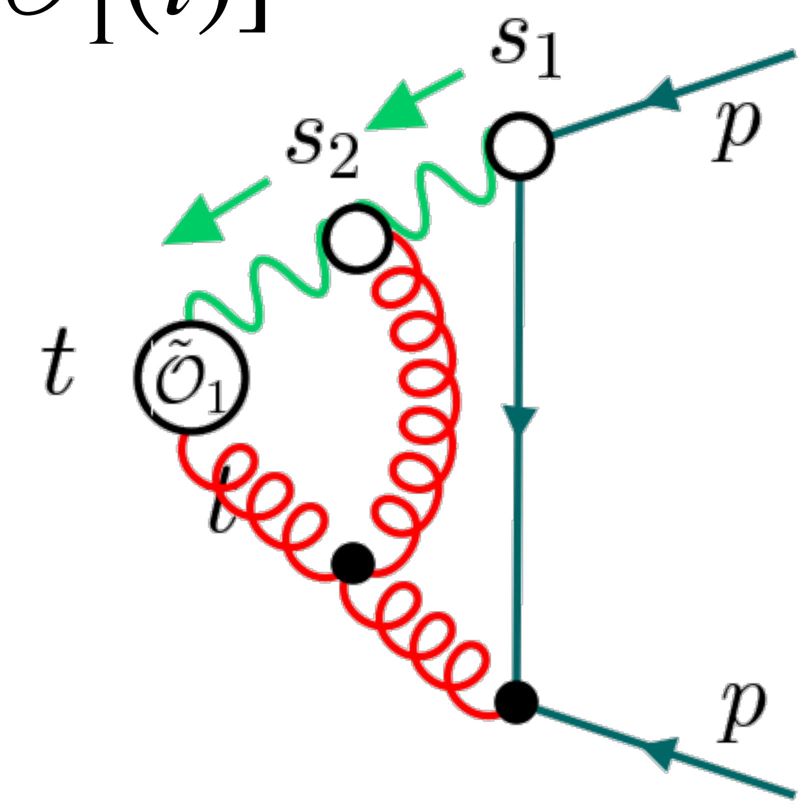

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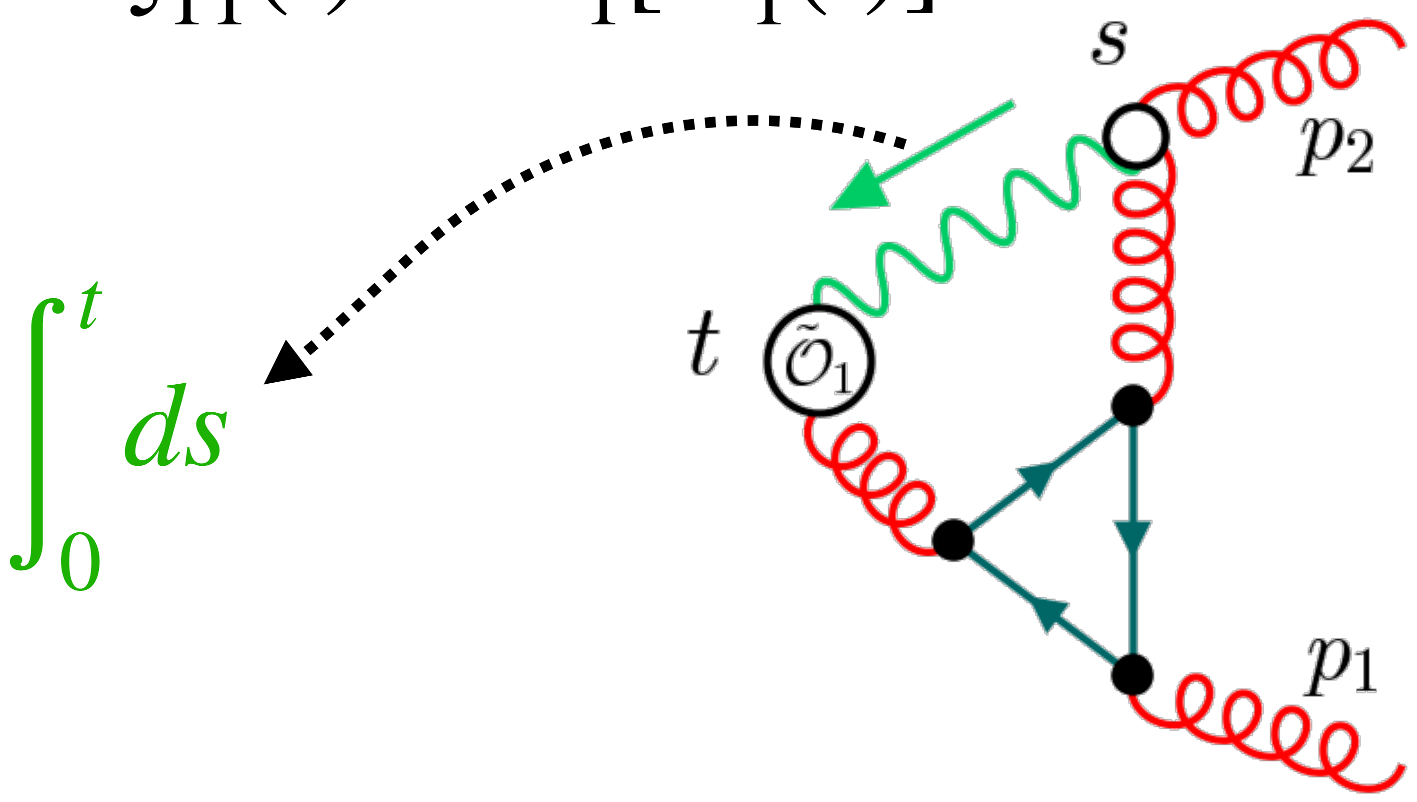

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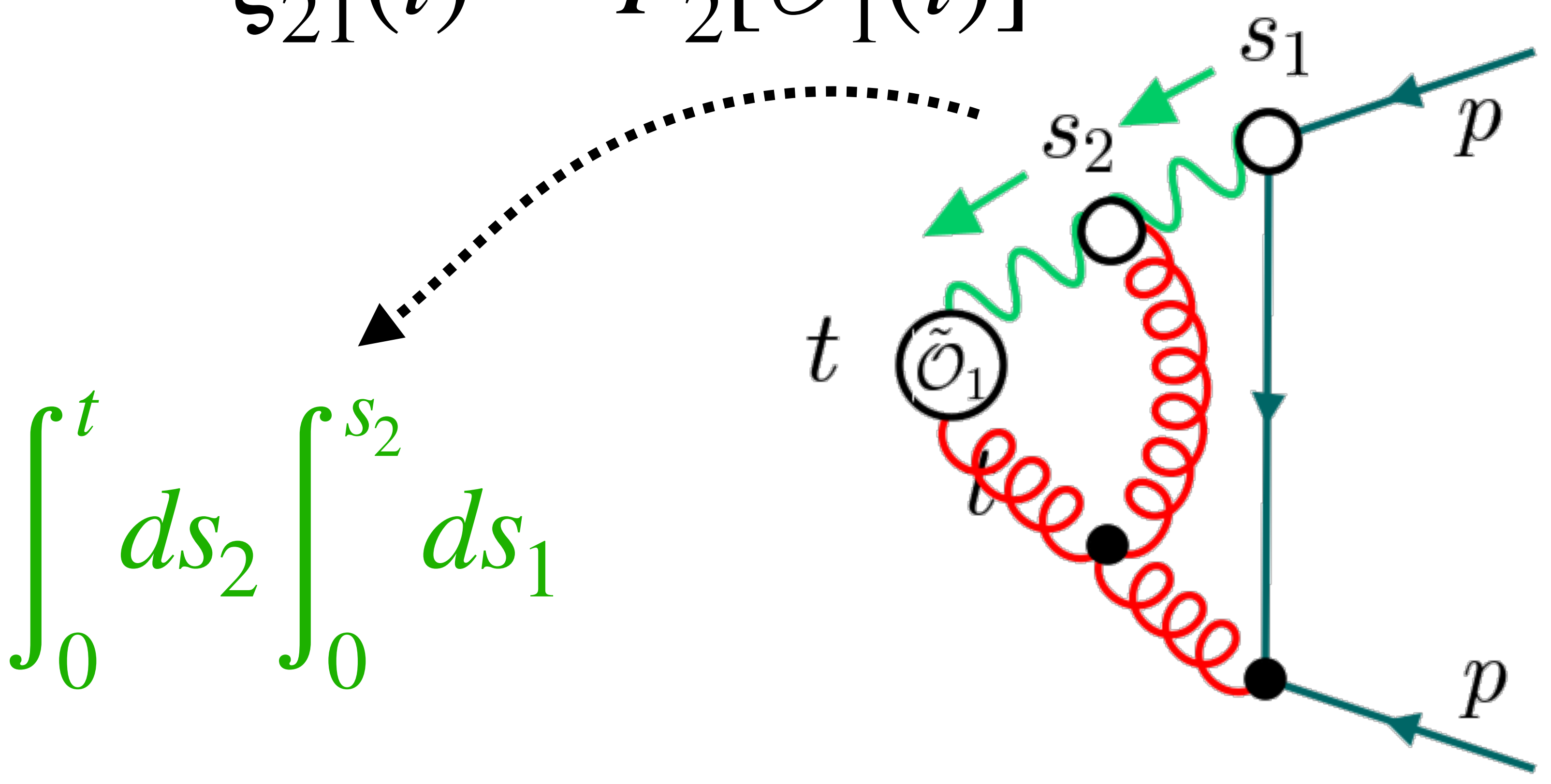

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The perturbative toolbox

[For details, see: Artz, RH, Lange, Neumann, Prausa]

Diagram generation:	`qgraf`	Nogueira
Diagram analyzation:	`q2e/exp`	RH, Seidensticker, Steinhauser
Algebraic manipulations:	`FORM`	Vermaseren

Reduction to masters:

Chetyrkin, Tkachov
Laporta

`Kira` ⊗ `FireFly`

Usovitsch, Uwer, Maierhöfer ⊗ Klappert, Klein, Lange

Sector Decomposition:

Binoth, Heinrich

$$\int d^Dk \int d^Dp \int_0^t ds \frac{e^{-tp^2 - s(k-p)^2}}{k^2 p^2 (k - p^2)} = \frac{A}{\epsilon^2} + \frac{B}{\epsilon} + C + \ldots$$

Ex. 1: QCD energy-momentum tensor Suzuki, Makino '13, '14

$$T_{\mu\nu} = \sum_n C_n \mathcal{O}_{n,\mu\nu}$$

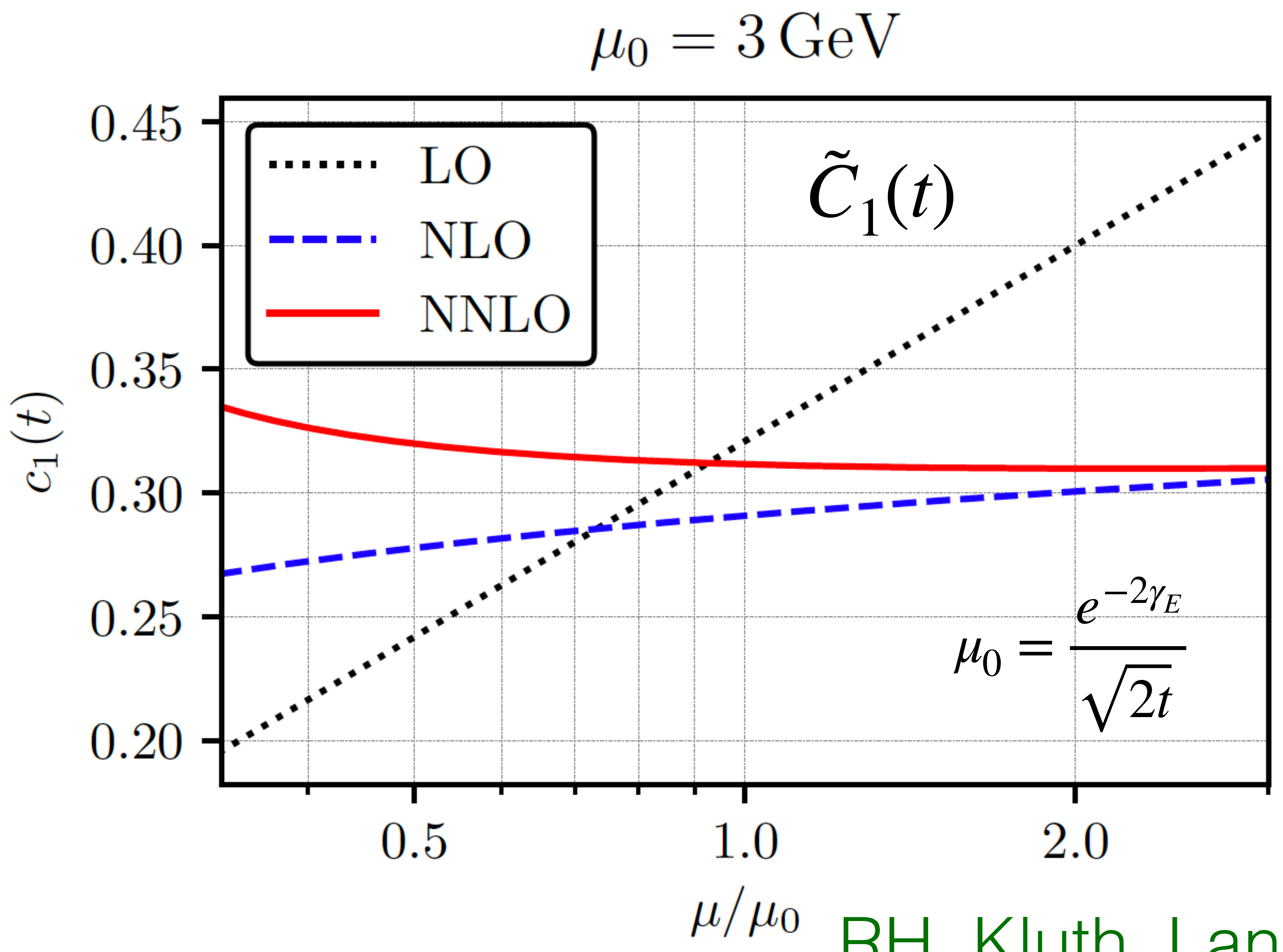
$$T_{\mu\nu} = \sum_n \tilde{C}_n(t)\tilde{\mathcal{O}}_{n,\mu\nu}(t)$$

$$\mathcal{O}_{1,\mu\nu} = \frac{1}{g_0^2} F^a_{\mu\rho}F^a_{\nu\rho} \qquad C_1 \equiv 1$$

$$\mathcal{O}_{2,\mu\nu} = \frac{\delta_{\mu\nu}}{g_0^2} F^a_{\rho\sigma}F^a_{\rho\sigma} \qquad C_2 \equiv -\frac{1}{4}$$

$$\mathcal{O}_{3,\mu\nu} = \bar{\psi}\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\psi \qquad C_3 \equiv \frac{1}{4}$$

$$\mathcal{O}_{4,\mu\nu} = \delta_{\mu\nu}\bar{\psi}\overleftrightarrow{\not{D}}\psi \qquad C_4 \equiv 0$$



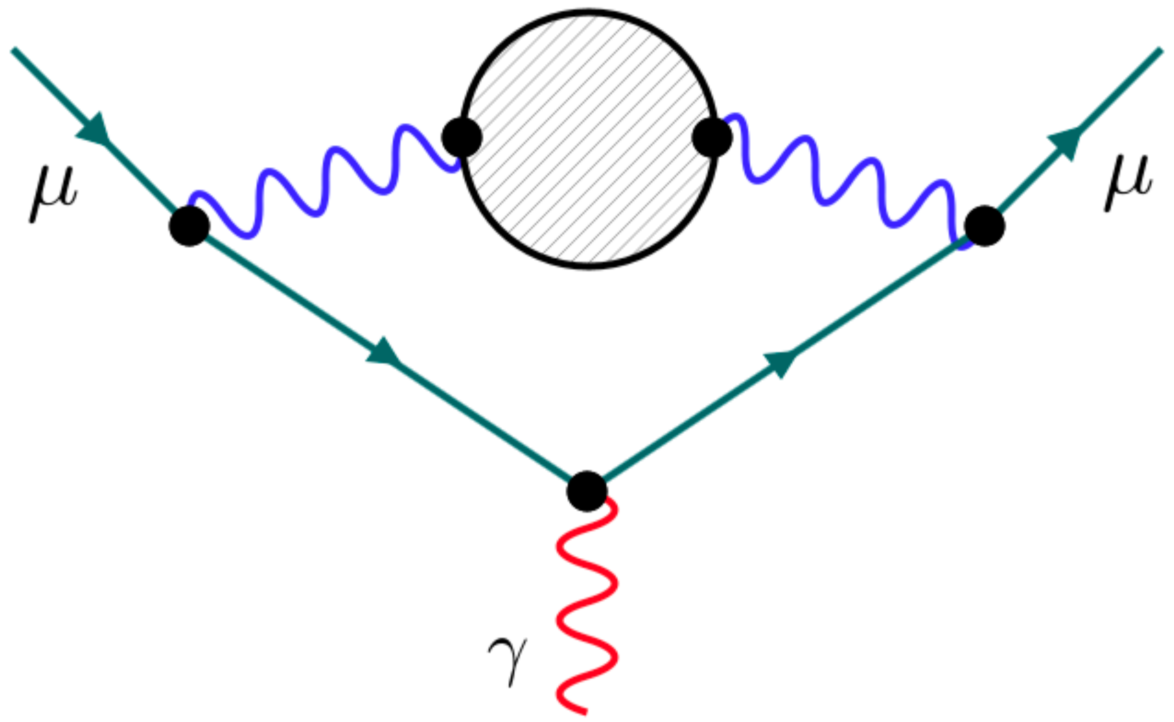
RH, Kluth, Lange '18

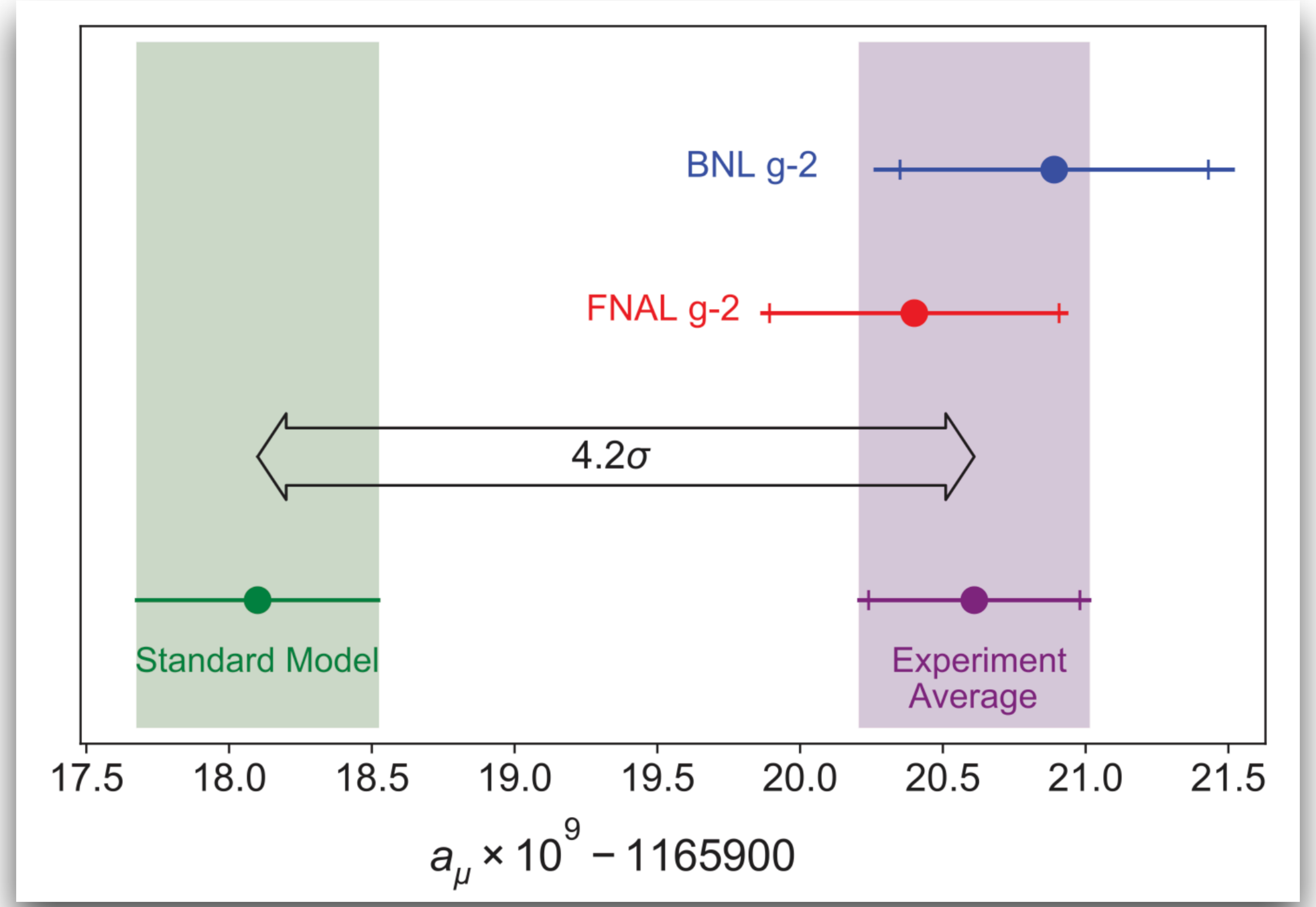
application: see WHOT collaboration

Ex. 2: Hadronic vacuum polarization

$$\int d^4x\, e^{iQx}\langle T\, j(x)\, j(0)\rangle$$

contribution to $(g-2)_\mu$





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$$\int d^4x\, e^{iQx} \langle T j(x) j(0) \rangle$$

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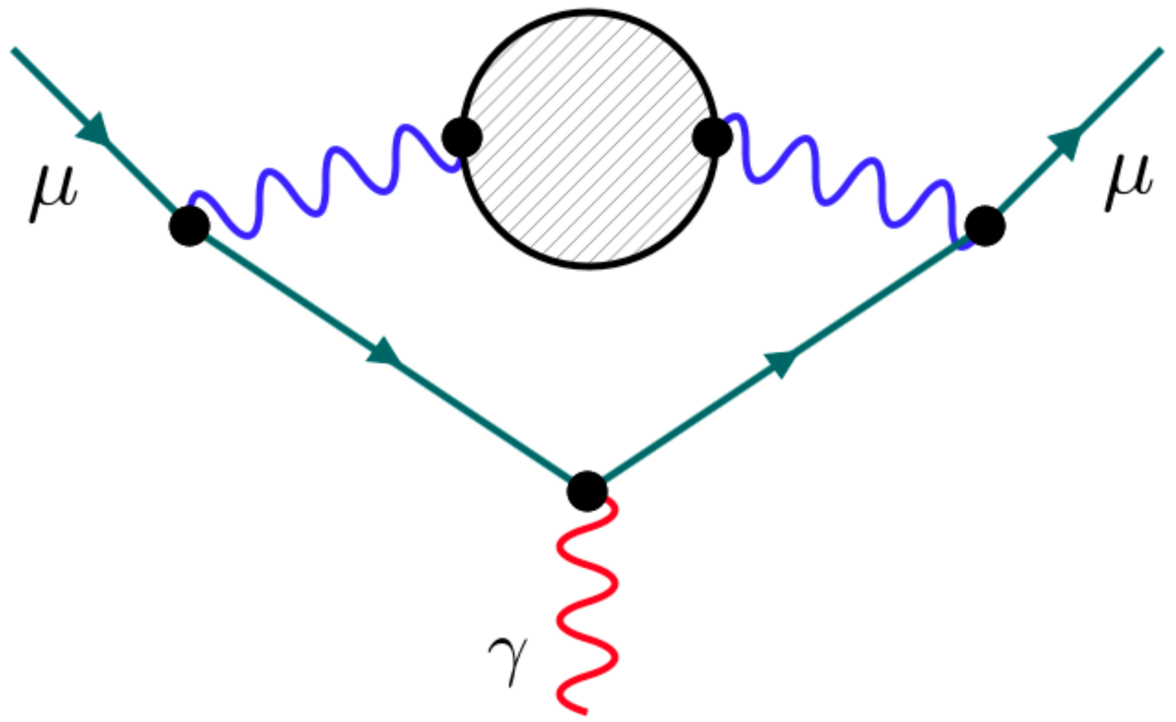

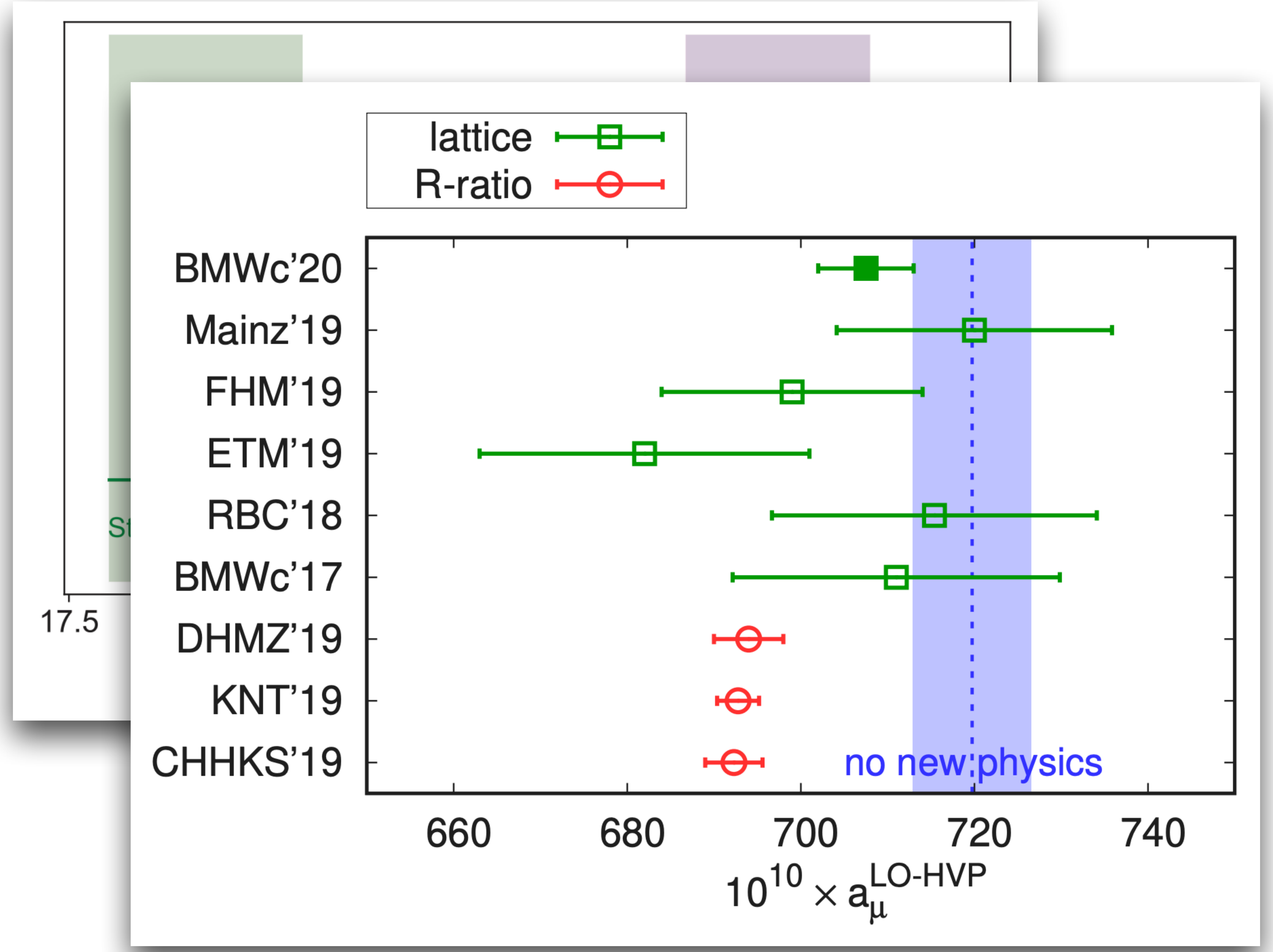

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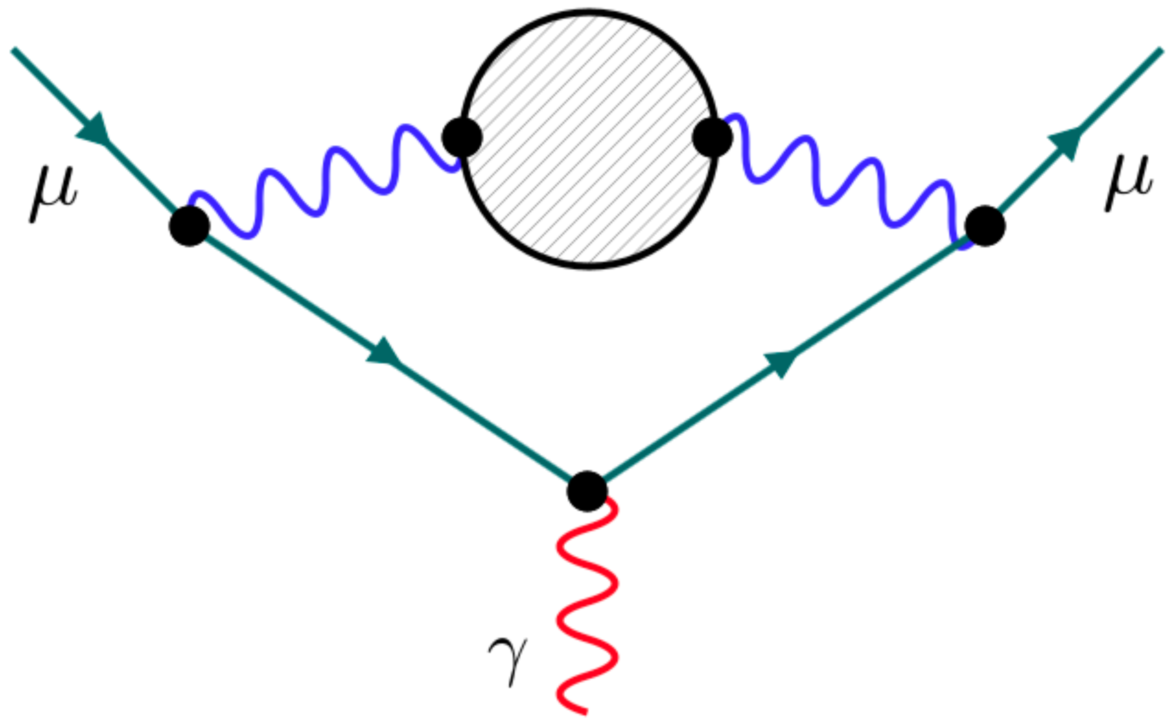

Ex. 2: Hadronic vacuum polarization

$$\int d^4x\, e^{iQx} \langle T j(x) j(0) \rangle \to \sum_n C_n(Q) \langle \mathcal{O}_n \rangle$$

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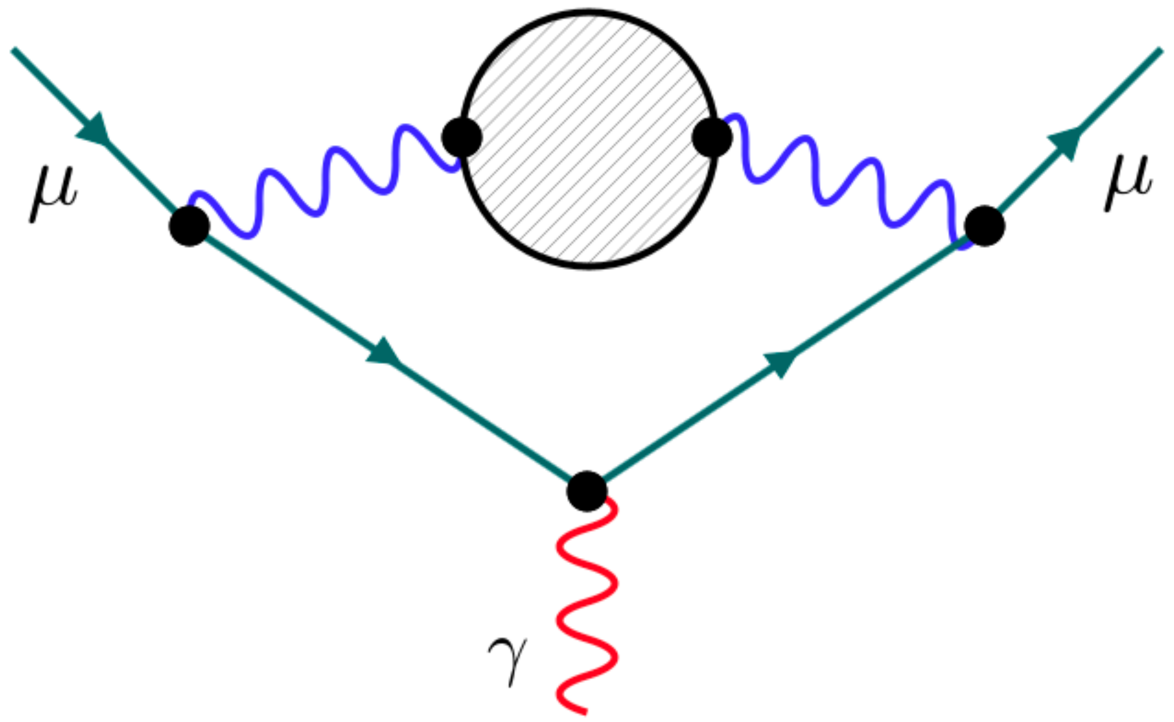

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Known to high orders in perturbation theory

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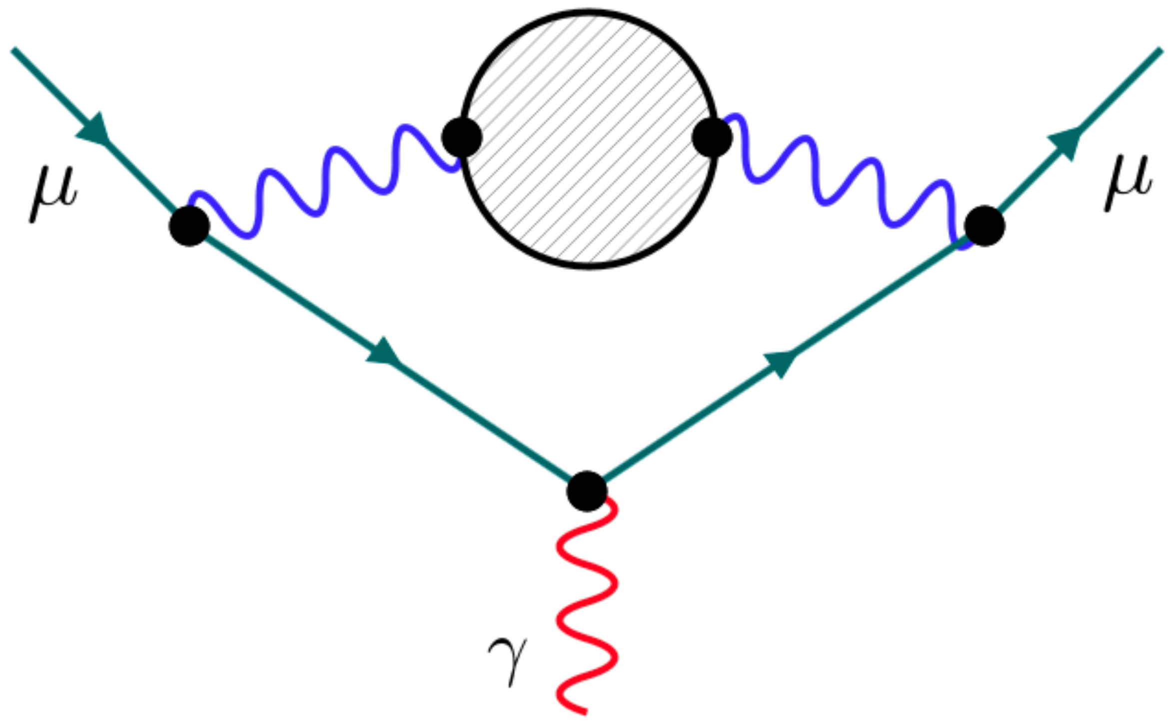
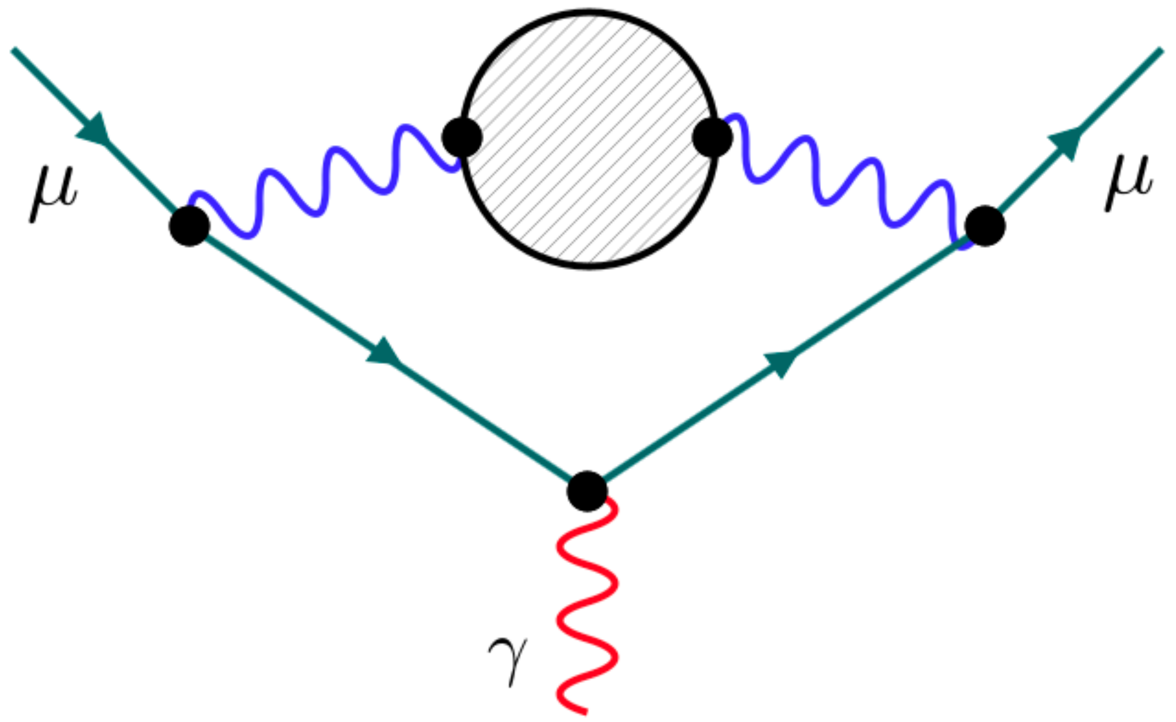


Ex. 2: Hadronic vacuum polarization

$$\int d^4x\, e^{iQx}\langle T j(x) j(0)\rangle \to \sum_n C_n(Q)\langle \mathcal{O}_n\rangle = \sum_n \tilde{C}_n(Q, t)\langle \tilde{\mathcal{O}}_n(t)\rangle$$

contribution to $(g-2)_\mu$



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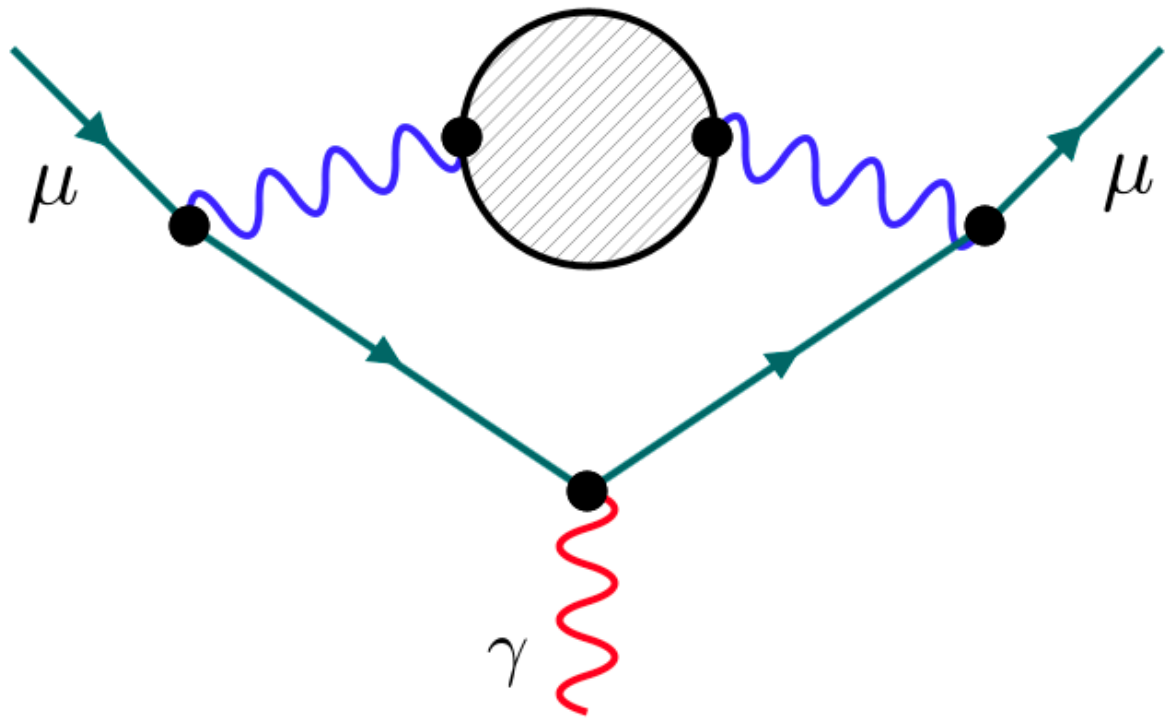

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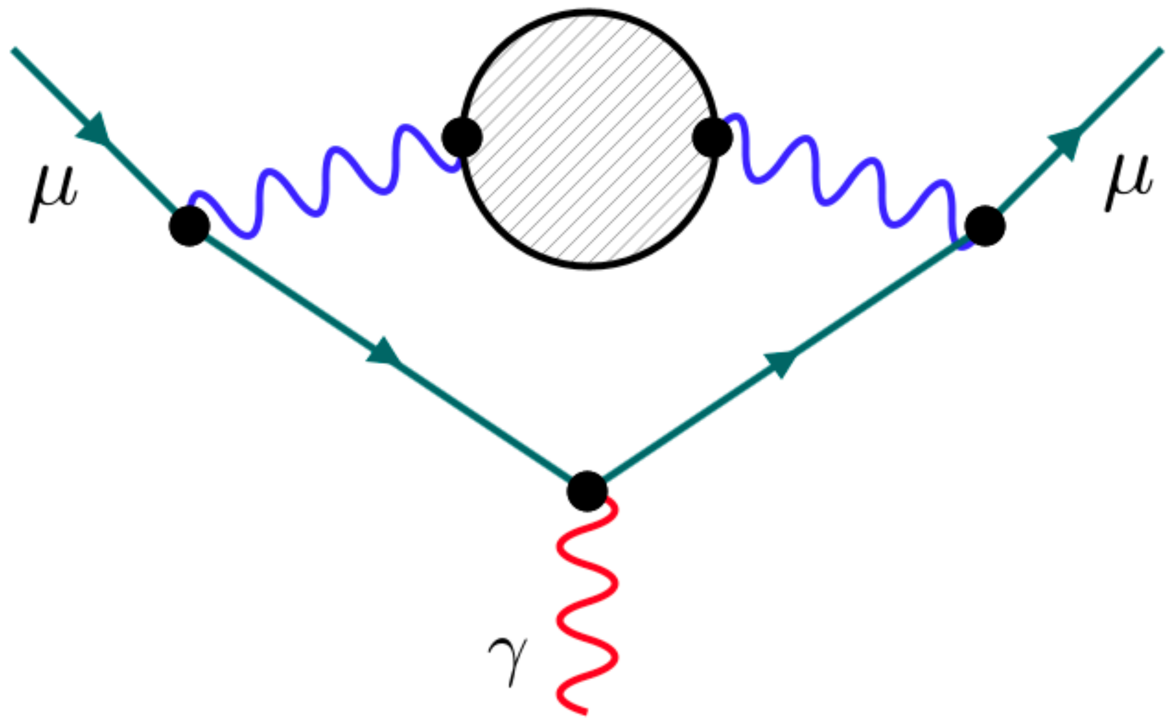

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$\tilde{C}_n(Q,t)$ to NNLO

RH, Lange, Neumann ‘20

Flow-time evolution

$t \to 0:$ $$\tilde{\mathcal{O}}_n(t) = \sum_m \zeta_{nm}(t)\,\mathcal{O}_m$$

matrix notation: $$\tilde{\mathcal{O}}(t) = \zeta(t)\,\mathcal{O}$$

$$t\frac{\partial}{\partial t}\tilde{\mathcal{O}}(t) = \left(t\frac{\partial}{\partial t}\zeta(t)\right)\mathcal{O} = \left(t\frac{\partial}{\partial t}\zeta(t)\right)\zeta^{-1}(t)\,\tilde{\mathcal{O}}(t)$$

$$t\frac{\partial}{\partial t}\tilde{\mathcal{O}}(t) = \tilde{\gamma}(t)\tilde{\mathcal{O}}(t)$$ with $$\tilde{\gamma}(t) = \left(t\frac{\partial}{\partial t}\zeta(t)\right)\zeta^{-1}(t)$$

RH, Lange, Neumann '20

Conclusions and Outlook

$$R = \sum_n C_n \langle \mathcal{O}_n \rangle = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

unique possibilities for lattice/p.t. synergies

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Applications so far:

- energy-momentum tensor (NNLO) — RH, Kluth, Lange '18
- hadronic vacuum polarization (NNLO) — RH, Lange, Neumann '20

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See also:

- ~~CP~~ operators (NLO) — SymLat '20
- Bag parameter (NLO) — A. Suzuki, Taniguchi, H. Suzuki, Kanada '20

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3-loop gradient flow coupling / beta-function — RH, Neumann, Lange '16, '20

see talks by K. Holland, J. Kuti, C. Peterson, …