

Funded by



The gradient flow at higher orders in perturbation theory

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RWTH Aachen University

based on work with

Johannes Artz, Yannick Kluth, Fabian Lange, Tobias Neumann, Mario Prausa

The gradient flow

flowed gauge field:

$$\frac{\partial}{\partial t} B_\mu(t, x) = \mathcal{D}_\nu G_{\nu\mu}(t, x)$$
$$B_\mu(t=0, x) = A_\mu(x)$$

flowed quark field:

$$\frac{\partial}{\partial t} \chi(t, x) = \mathcal{D}^2 \chi(t, x)$$
$$\chi(t=0, x) = \psi(x)$$

Lüscher '10, '13

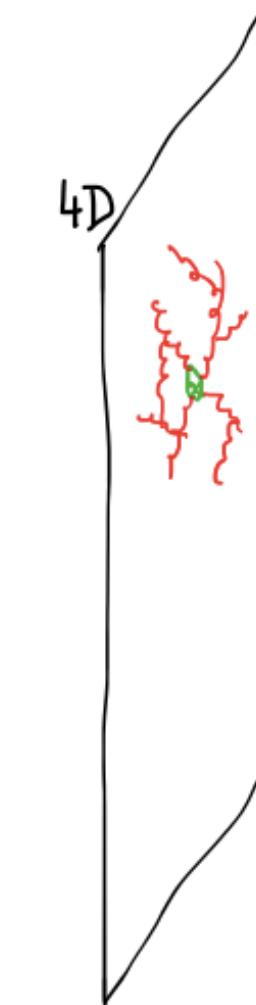
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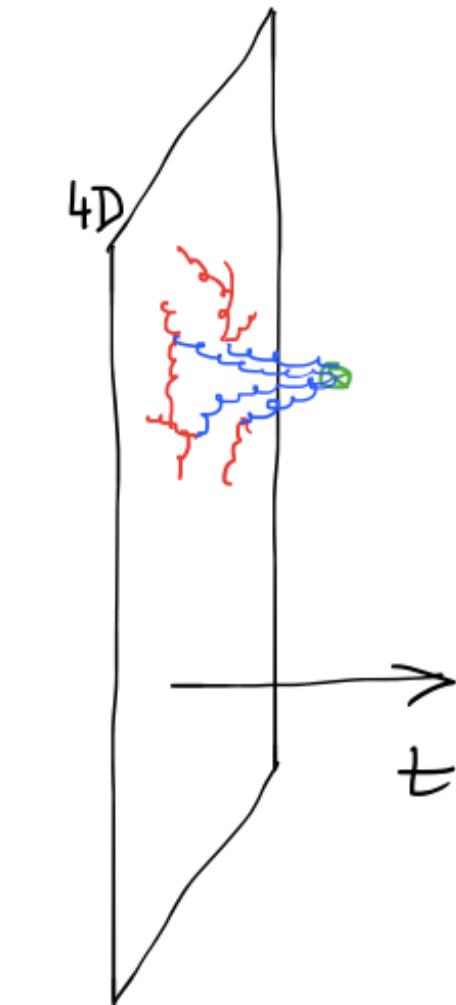
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Lüscher '10, '13



$$B_\mu(t, p) \sim e^{-tp^2} A_\mu(p)$$

A common problem

Observable:

$$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

perturbation theory lattice

match
renormalization
schemes?

A common problem

Observable:

$$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

The diagram illustrates the relationship between perturbation theory and lattice calculations. On the left, the word "perturbation theory" is written in green. To its right is the mathematical expression for the observable R , which is a sum over n of the product of a coefficient C_n and the expectation value of an operator \mathcal{O}_n . Two curved arrows point from the text "perturbation theory" to the term C_n in the equation. Another curved arrow points from the word "lattice" to the term $\langle \mathcal{O}_n \rangle$.

match
renormalization
schemes?

Instead:

$$R = \sum_n \tilde{C}_n(\textcolor{blue}{t}) \langle \tilde{\mathcal{O}}_n(\textcolor{blue}{t}) \rangle$$

This diagram shows an alternative approach to matching perturbation theory and lattice results. It features the same structure as the first diagram, with "perturbation theory" in green and the observable R equation. However, the coefficients C_n are replaced by $\tilde{C}_n(t)$, and the operators \mathcal{O}_n are replaced by $\tilde{\mathcal{O}}_n(t)$. Curved arrows connect "perturbation theory" to $\tilde{C}_n(t)$ and "lattice" to $\tilde{\mathcal{O}}_n(t)$.

gradient flow
renormalization

A common problem

Observable:

$$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

The diagram illustrates the relationship between perturbation theory and lattice calculations. On the left, the word "perturbation theory" is written in green. To its right is the mathematical expression for the observable R . The summation symbol \sum is positioned above a horizontal line that connects the two parts. Two curved arrows point upwards from this line to the terms C_n and $\langle \mathcal{O}_n \rangle$. To the right of the expression, the word "lattice" is written in red.

match
renormalization
schemes?

Instead:

$$R = \sum_n \tilde{C}_n(\textcolor{blue}{t}) \langle \tilde{\mathcal{O}}_n(\textcolor{blue}{t}) \rangle$$

This diagram follows the same structure as the one above, but it represents a different approach. The word "perturbation theory" is in green on the left. The mathematical expression for R includes a tilde over the summation symbol and over the coupling constants \tilde{C}_n and operators $\tilde{\mathcal{O}}_n$. The variable t is written in blue. Curved arrows point upwards from the horizontal line to the terms $\tilde{C}_n(t)$ and $\langle \tilde{\mathcal{O}}_n(t) \rangle$. To the right of the expression, the word "lattice" is in red.

gradient flow
renormalization

see also talks by A. Hasenfratz, C. Monahan, M. Rizik, ...

Small flow-time expansion

Observable:

$$R = \sum_n C_n \langle \mathcal{O}_n \rangle = \sum_n \tilde{C}_n(\textcolor{blue}{t}) \langle \tilde{\mathcal{O}}_n(\textcolor{blue}{t}) \rangle$$

small flow-time expansion:

Lüscher, Weisz '11

$$\tilde{\mathcal{O}}_n(\textcolor{blue}{t}) \xrightarrow{t \rightarrow 0} \sum_m \zeta_{nm}(t) \mathcal{O}_m$$

$$\tilde{C}_n(\textcolor{blue}{t}) \xrightarrow{t \rightarrow 0} \sum_m C_m \zeta_{mn}^{-1}(t)$$

\Rightarrow need $\zeta_{nm}(t)$ for small t \Rightarrow perturbation theory

Method of projectors

$$\tilde{\mathcal{O}}_n(\textcolor{blue}{t}) \xrightarrow{t \rightarrow 0} \sum_m \zeta_{nm}(t) \mathcal{O}_m$$

define projector: $\textcolor{red}{P}_k[\mathcal{O}_m] \equiv \delta_{km}$ to all orders in perturbation theory

$$\textcolor{red}{P}_k[\tilde{\mathcal{O}}_n(\textcolor{blue}{t})] \xrightarrow{t \rightarrow 0} \sum_m \zeta_{nm}(t) \textcolor{red}{P}_k[\mathcal{O}_m] \quad \Rightarrow \quad \zeta_{nk}(t) = P_k[\tilde{\mathcal{O}}_n(\textcolor{blue}{t})]$$

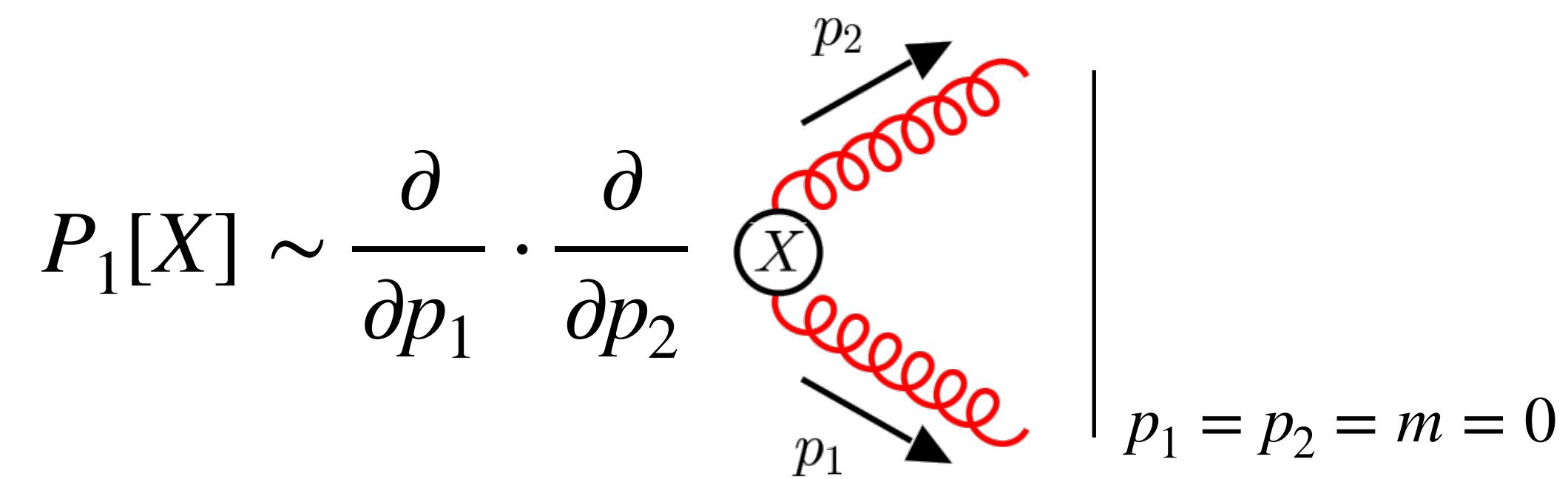
$$P_k[\cdot] \sim H(\partial/\partial p_i, \partial/\partial m) \langle 0 | \cdot | p_1, \dots \rangle \Big|_{p_i=m=0}$$

Gorishny, Larin, Tkachov '83

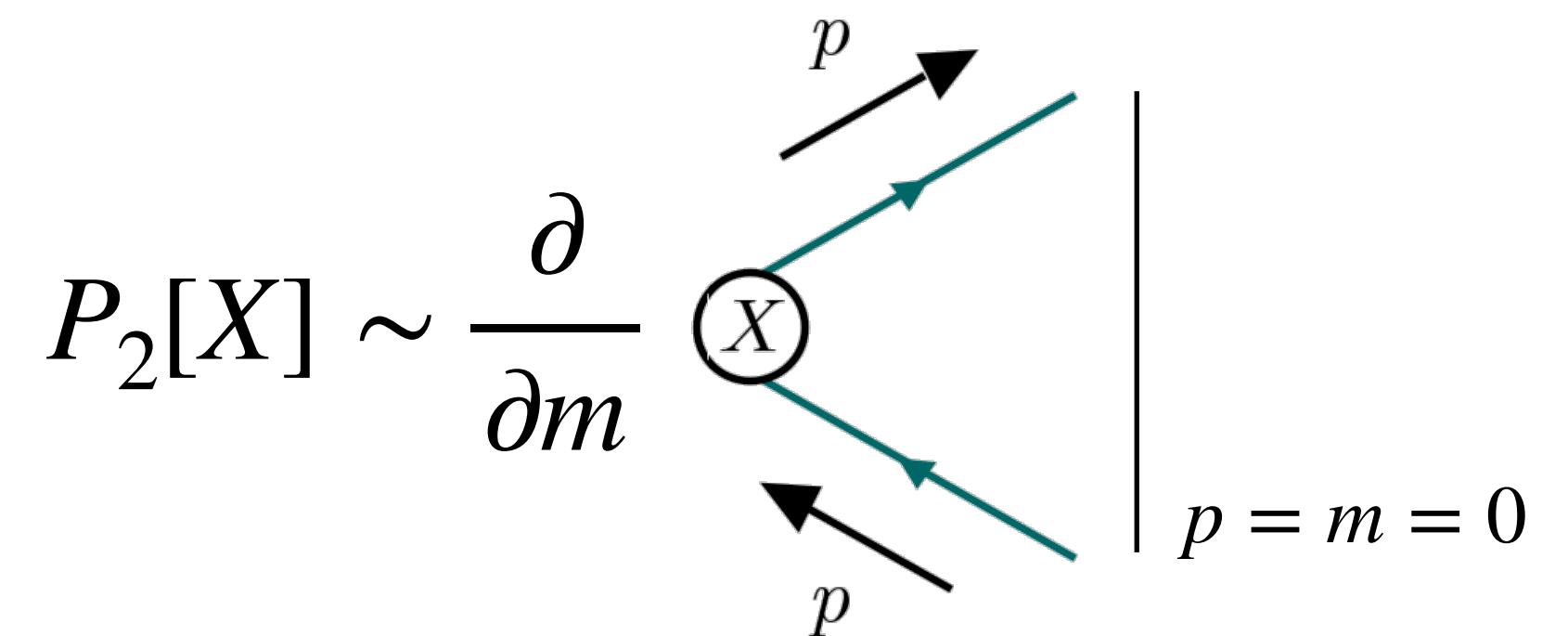
Method of projectors

Example: dim-4 operators of QCD:

$$\mathcal{O}_1 \sim F_{\mu\nu}^a F_{\mu\nu}^a$$



$$\mathcal{O}_2 \sim m \bar{\psi} \psi$$



note: scale-less integrals = 0 in dimensional regularization!

$\Rightarrow P_k[\mathcal{O}_m] \equiv \delta_{km}$ to all orders in perturbation theory

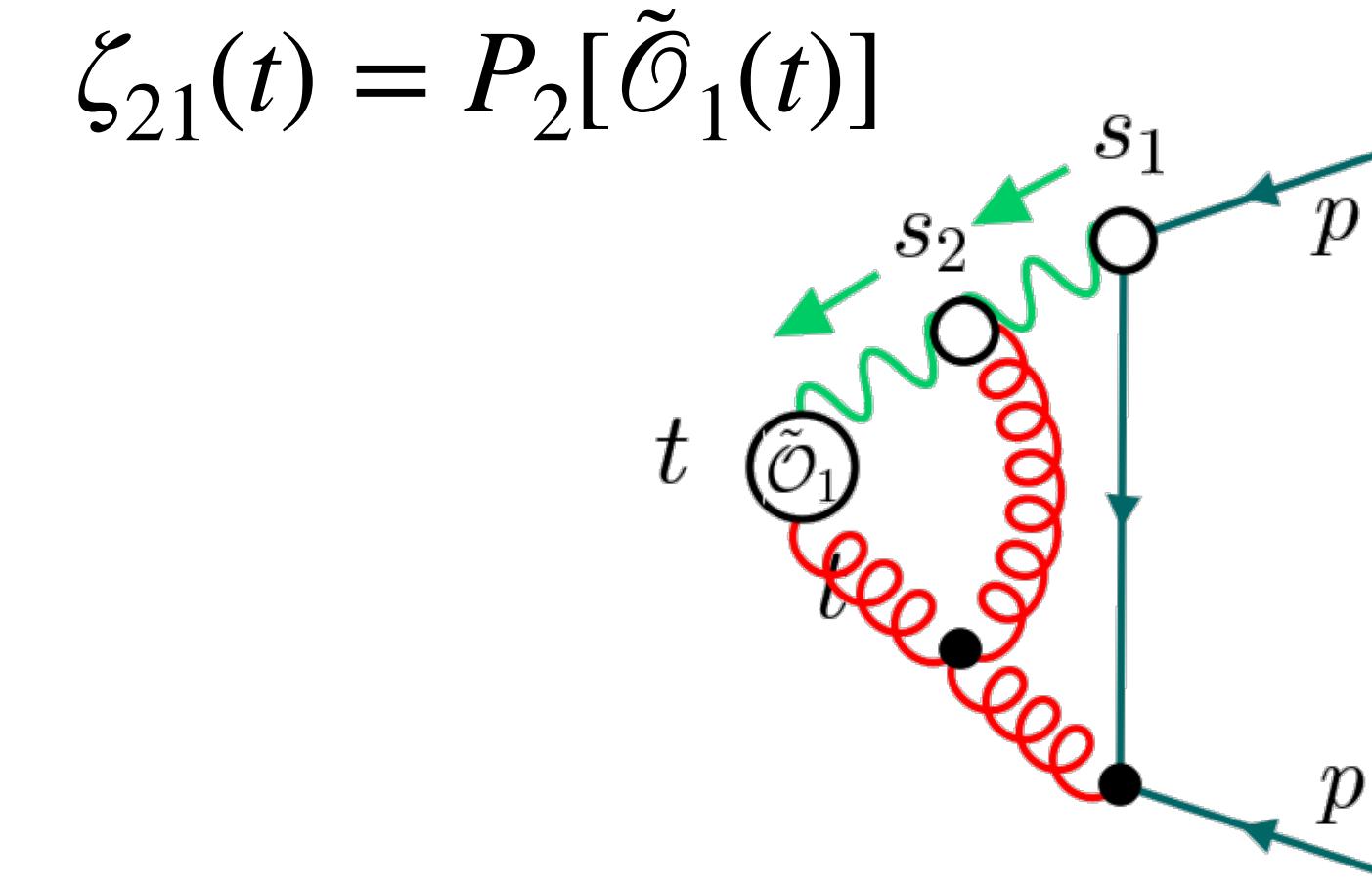
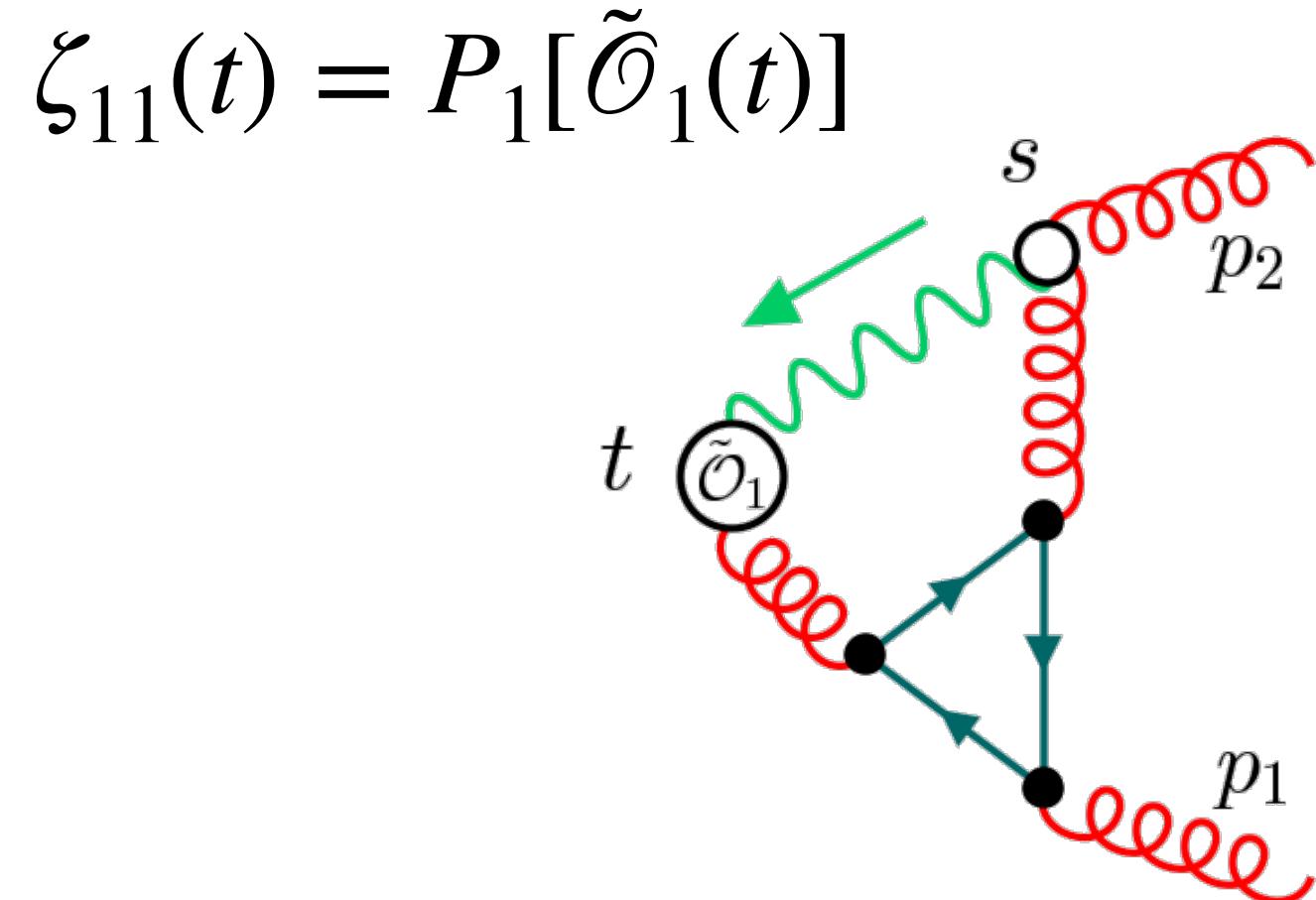
but: $P_k[\tilde{\mathcal{O}}_m(\textcolor{blue}{t})]$ is *not* scale-less!

Method of projectors

$$\tilde{\mathcal{O}}_n(t) \xrightarrow{t \rightarrow 0} \sum_m \zeta_{nm}(t) \mathcal{O}_m$$

$$P_k[\tilde{\mathcal{O}}_n(t)] \xrightarrow{t \rightarrow 0} \sum_m \zeta_{nm}(t) P_k[\mathcal{O}_m]$$

Use 5-dimensional QFT formulation: Lüscher, Weisz '11



produced with FeynGame

Method of projectors

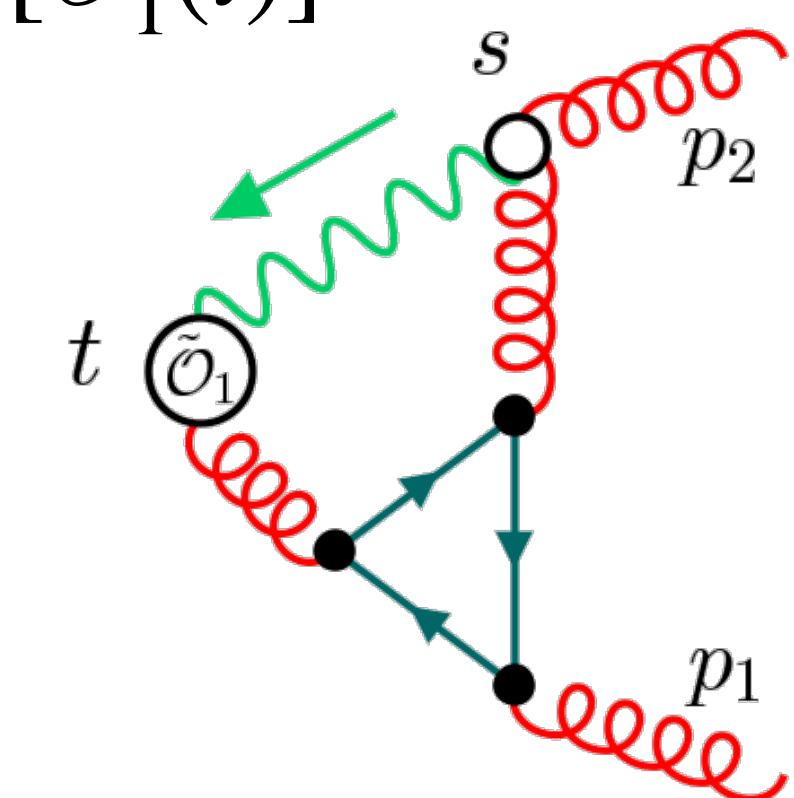
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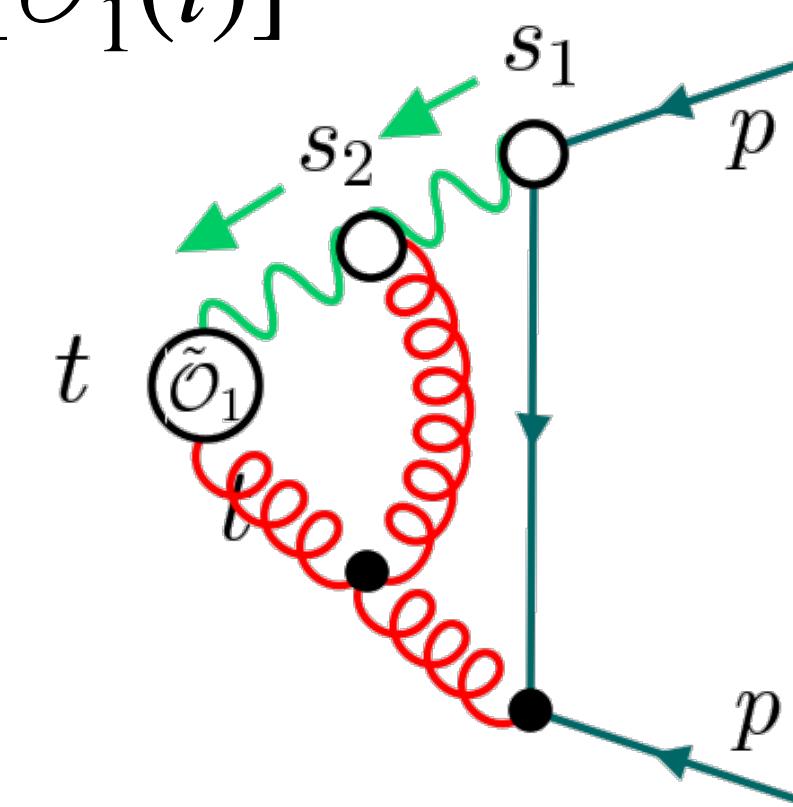
δ_{km}

Use 5-dimensional QFT formulation: Lüscher, Weisz '11

$$\zeta_{11}(t) = P_1[\tilde{\mathcal{O}}_1(t)]$$



$$\zeta_{21}(t) = P_2[\tilde{\mathcal{O}}_1(t)]$$



produced with FeynGame

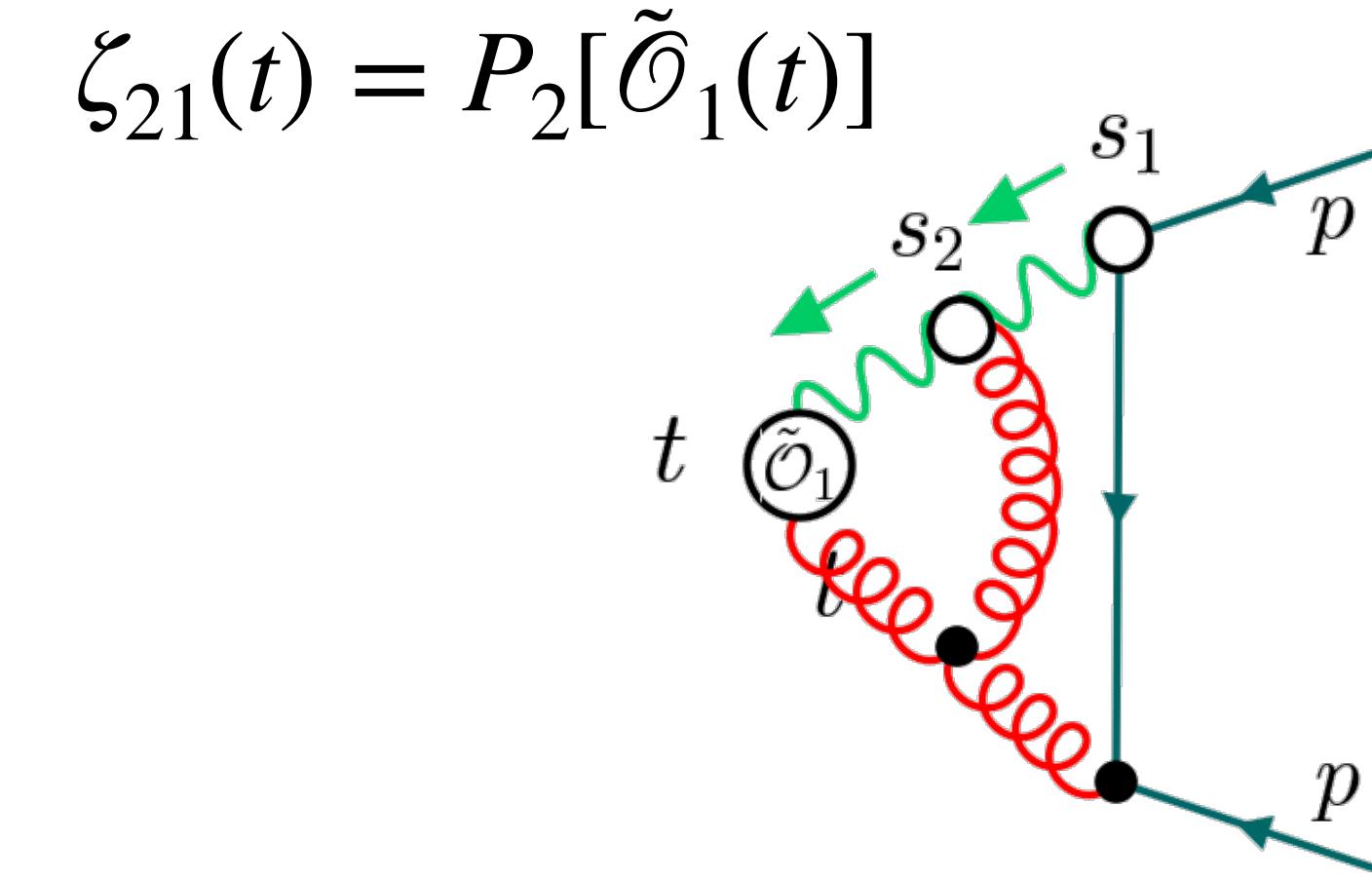
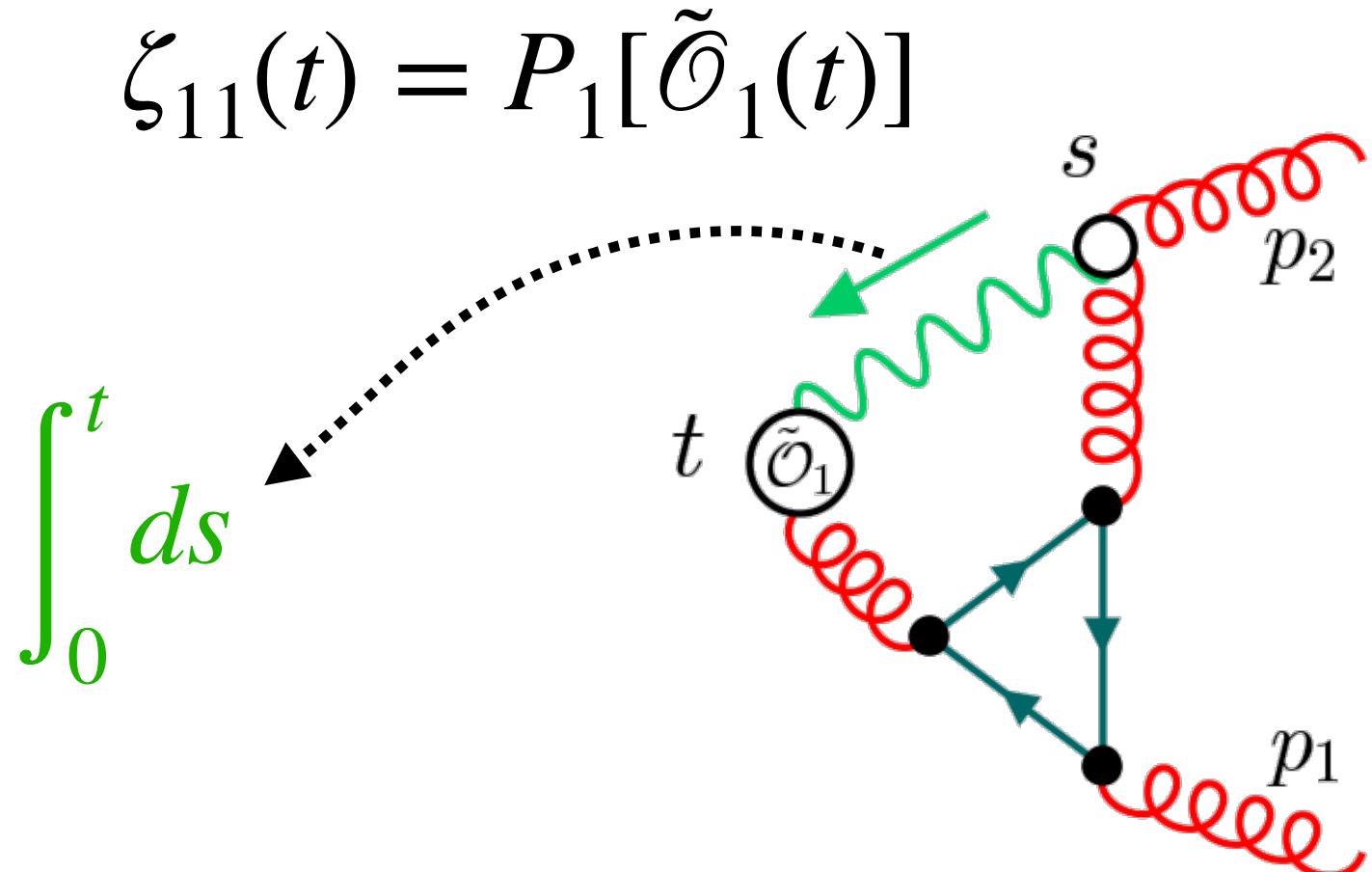
Method of projectors

$$\tilde{\mathcal{O}}_n(t) \xrightarrow{t \rightarrow 0} \sum_m \zeta_{nm}(t) \mathcal{O}_m$$

$$P_k[\tilde{\mathcal{O}}_n(t)] \xrightarrow{t \rightarrow 0} \sum_m \zeta_{nm}(t) P_k[\mathcal{O}_m]$$

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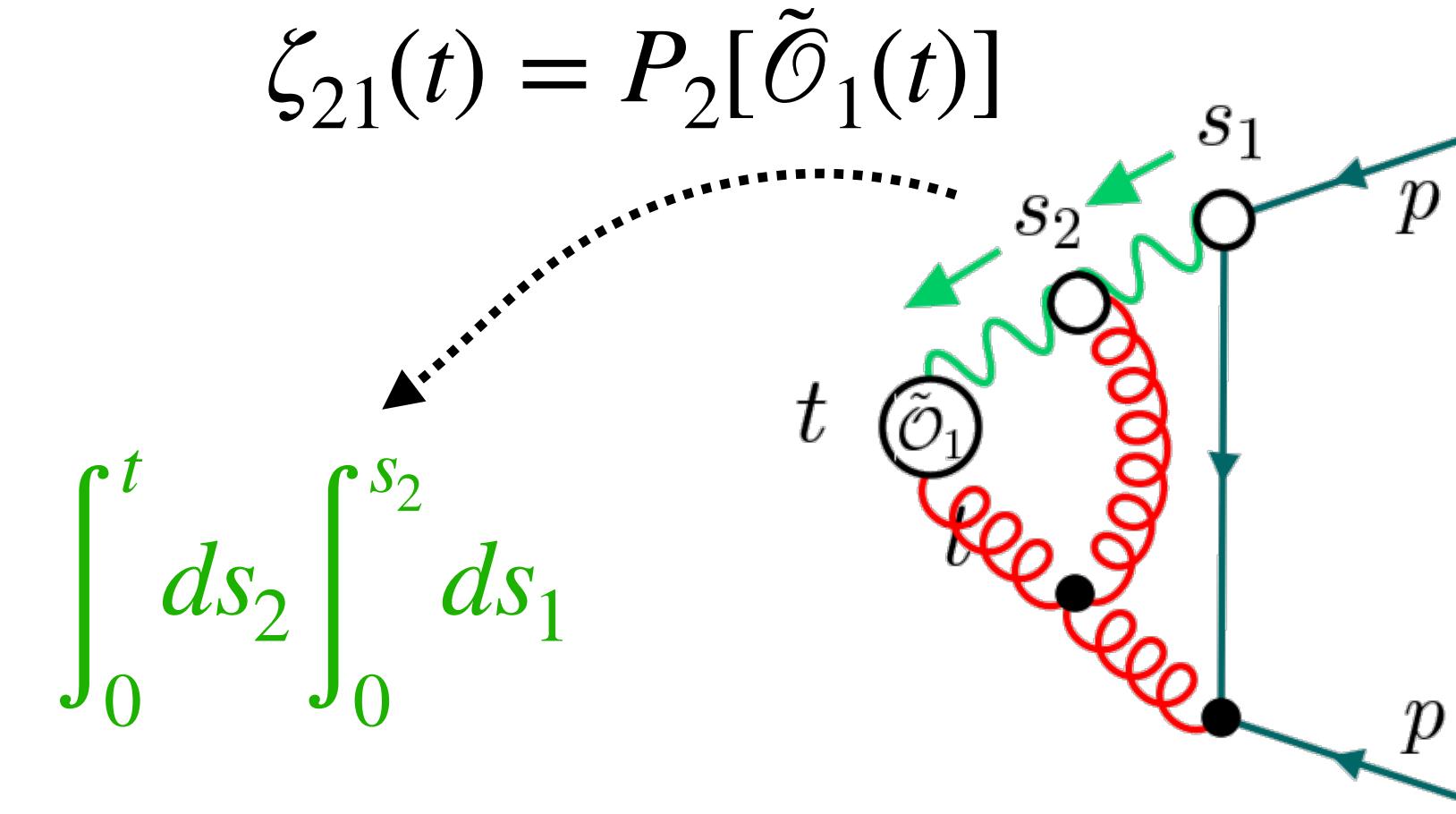
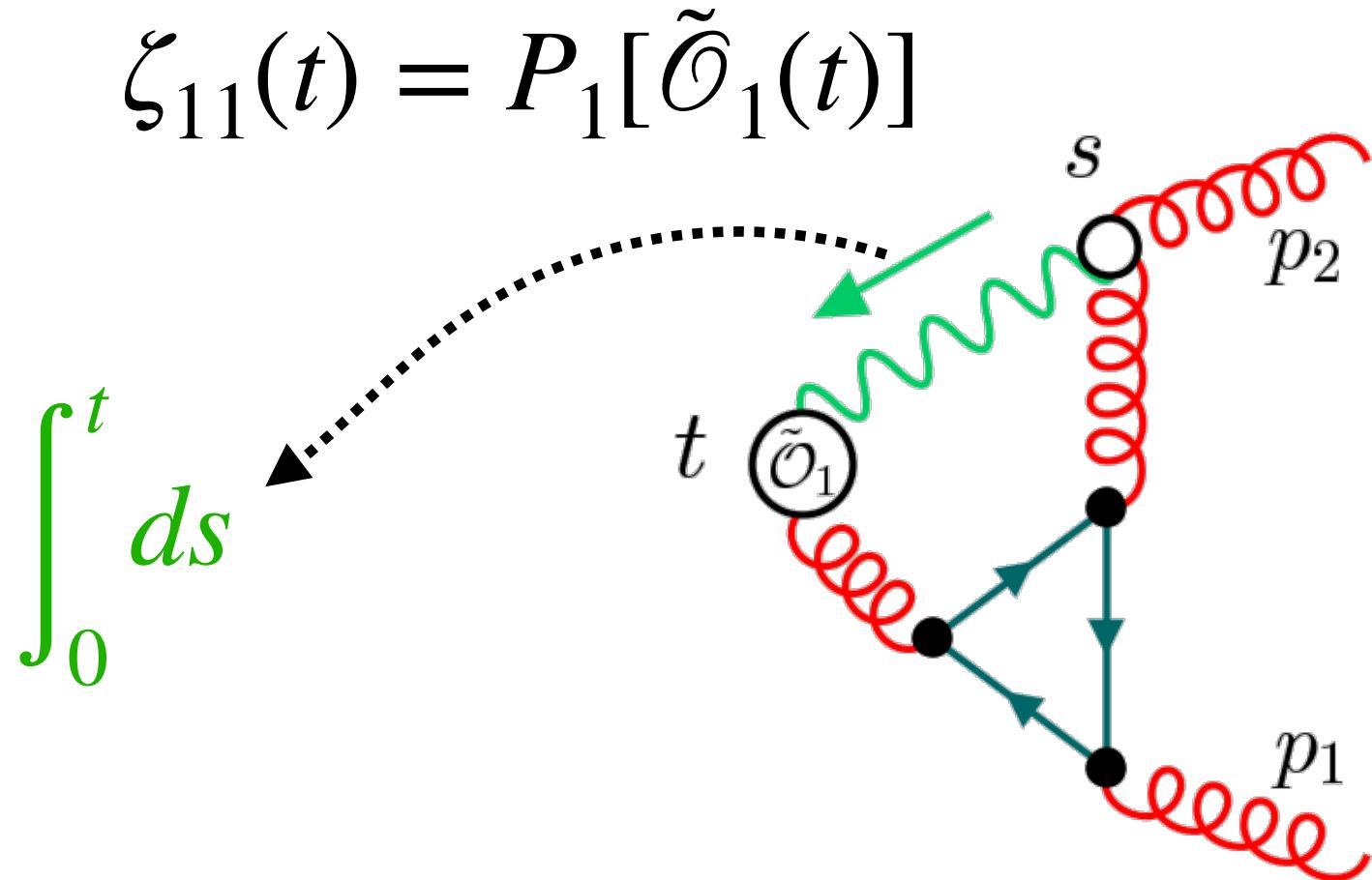
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δ_{km}

Use 5-dimensional QFT formulation: Lüscher, Weisz '11



produced with FeynGame

The perturbative toolbox

[For details, see: Artz, RH, Lange, Neumann, Prausa]

Diagram generation:

qgraf

Nogueira

Diagram analyzation:

q2e/exp

RH, Seidensticker, Steinhauser

Algebraic manipulations:

FORM

Vermaseren

Reduction to masters:

Chetyrkin, Tkachov
Laporta

Kira \otimes **FireFly**

Usovitsch, Uwer, Maierhöfer \otimes Klappert, Klein, Lange

Sector Decomposition:
Binoth, Heinrich

$$\int d^D k \int d^D p \int_0^t \cancel{ds} \frac{e^{-tp^2 - s(k-p)^2}}{k^2 p^2 (k - p^2)} = \frac{A}{\epsilon^2} + \frac{B}{\epsilon} + C + \dots$$

Ex. 1: QCD energy-momentum tensor

Suzuki, Makino '13, '14

$$T_{\mu\nu} = \sum_n C_n \mathcal{O}_{n,\mu\nu}$$

$$\mathcal{O}_{1,\mu\nu} = \frac{1}{g_0^2} F_{\mu\rho}^a F_{\nu\rho}^a$$

$$\mathcal{O}_{2,\mu\nu} = \frac{\delta_{\mu\nu}}{g_0^2} F_{\rho\sigma}^a F_{\rho\sigma}^a$$

$$\mathcal{O}_{3,\mu\nu} = \bar{\psi} \left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu \right) \psi$$

$$\mathcal{O}_{4,\mu\nu} = \delta_{\mu\nu} \bar{\psi} \overleftrightarrow{D} \psi$$

$$C_1 \equiv 1$$

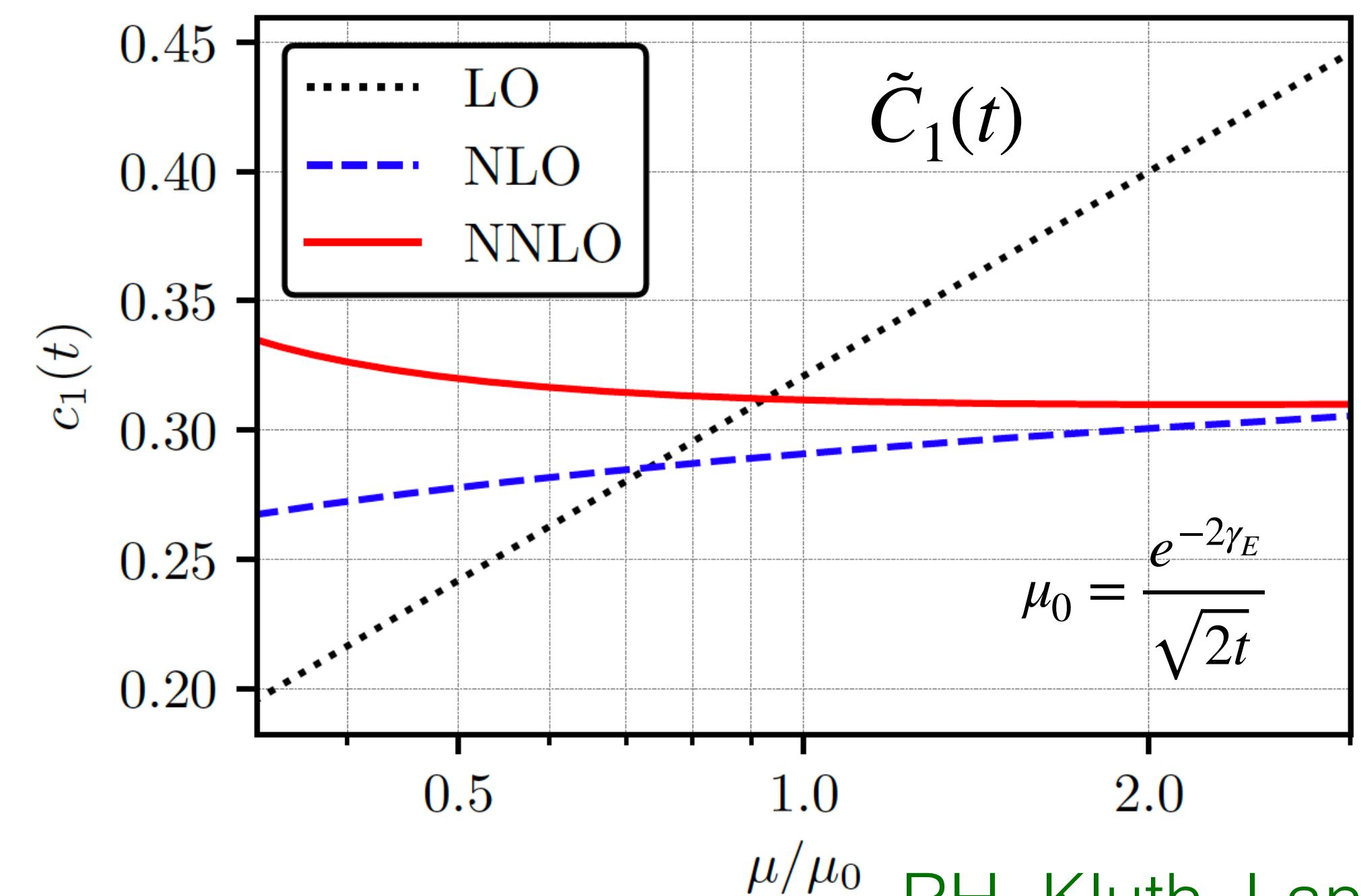
$$C_2 \equiv -\frac{1}{4}$$

$$C_3 \equiv \frac{1}{4}$$

$$C_4 \equiv 0$$

$$T_{\mu\nu} = \sum_n \tilde{C}_n(\textcolor{blue}{t}) \tilde{\mathcal{O}}_{n,\mu\nu}(\textcolor{blue}{t})$$

$$\mu_0 = 3 \text{ GeV}$$



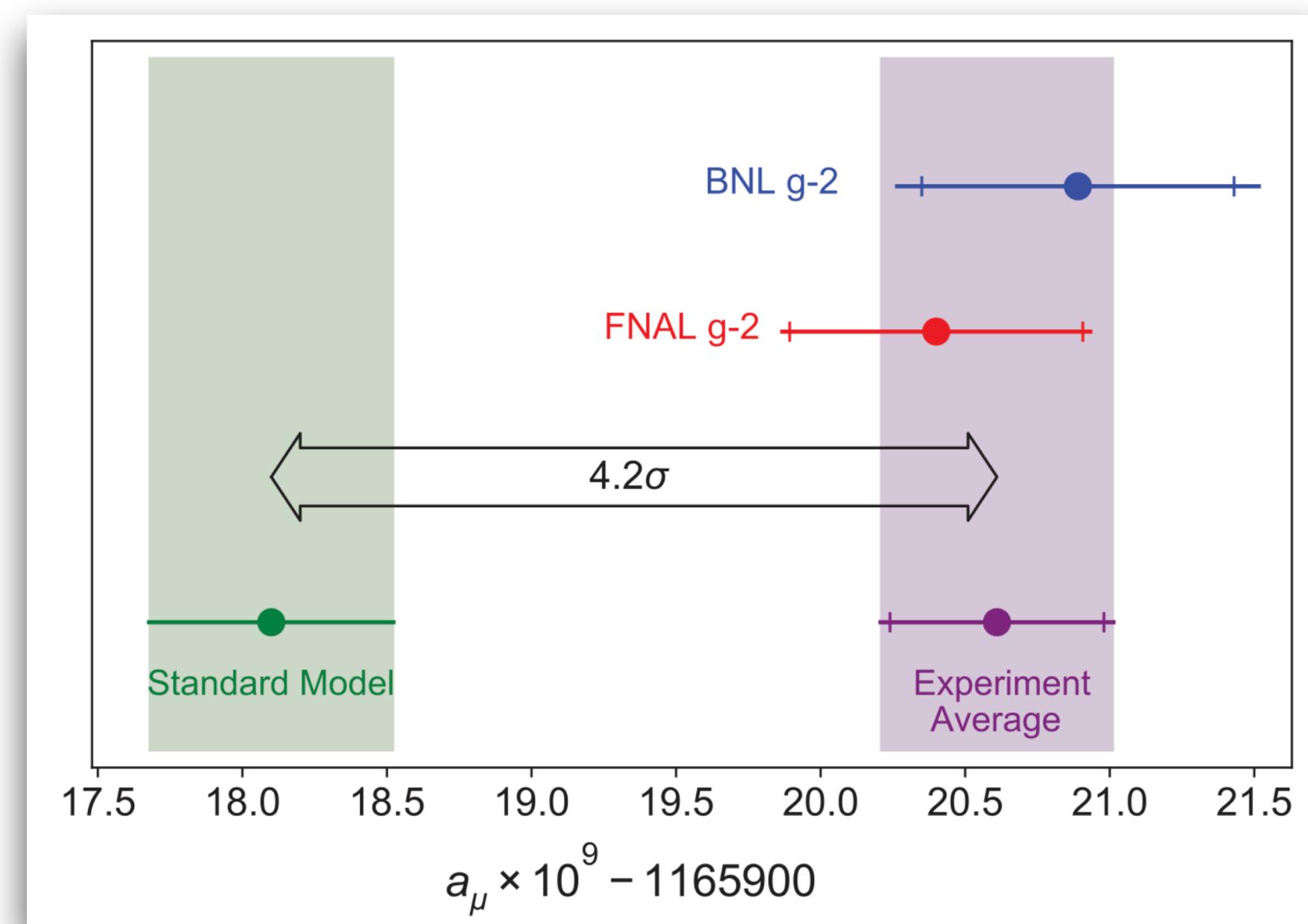
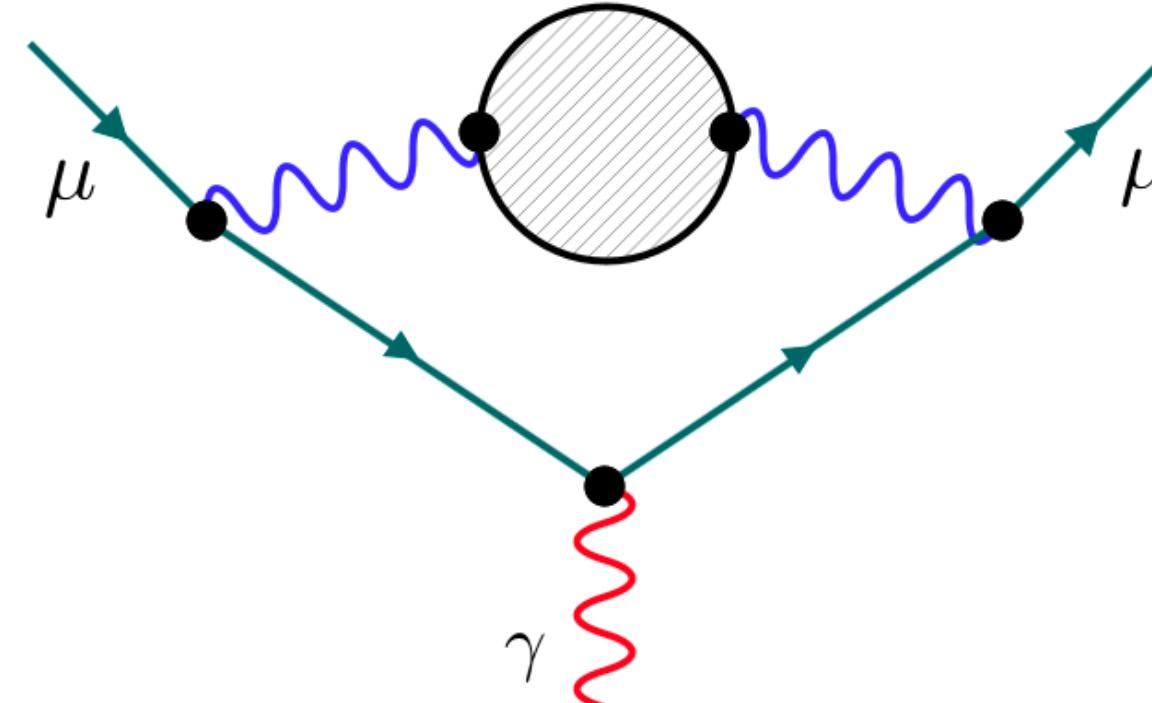
RH, Kluth, Lange '18

application: see WHOT collaboration

Ex. 2: Hadronic vacuum polarization

$$\int d^4x e^{iQx} \langle T j(x) j(0) \rangle$$

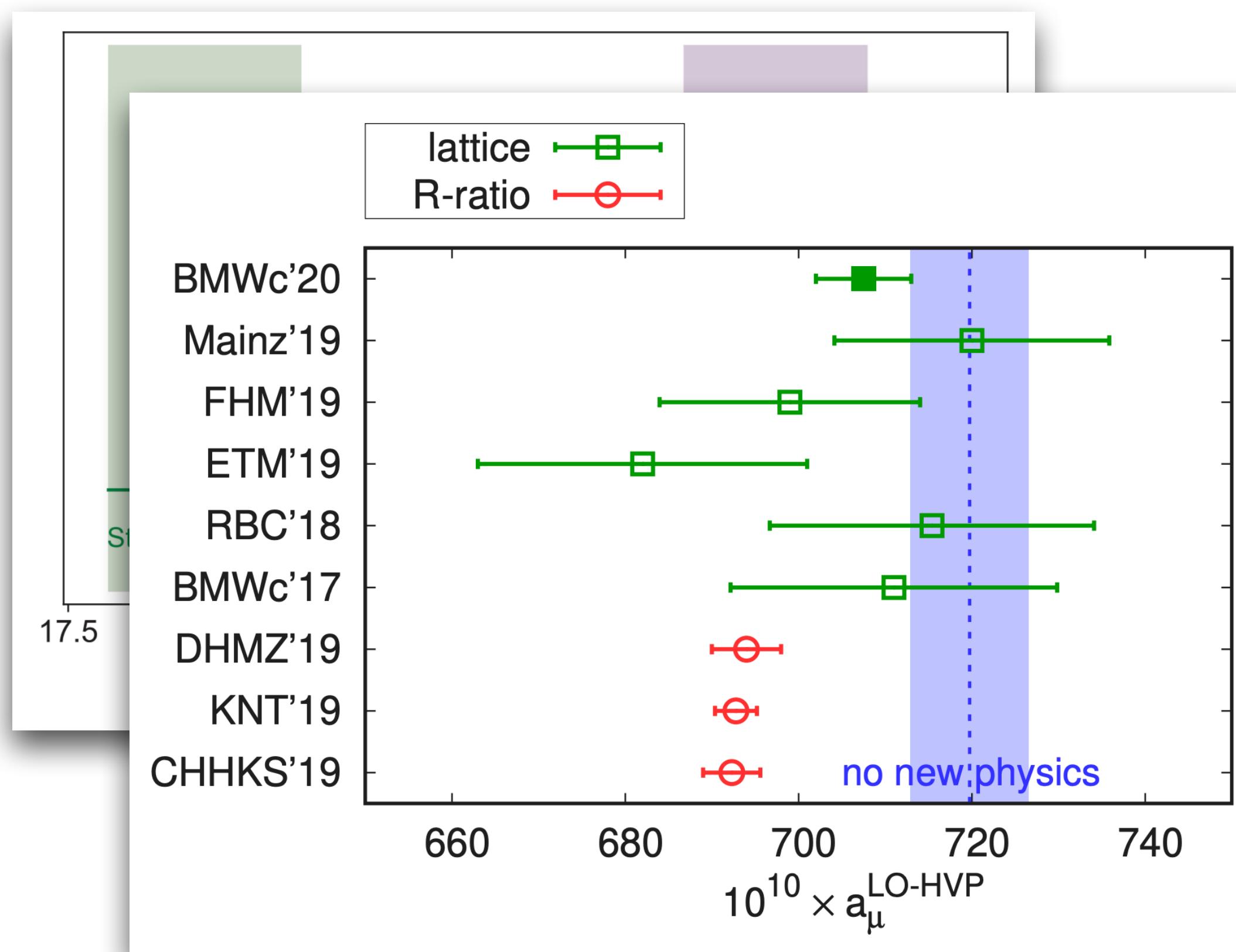
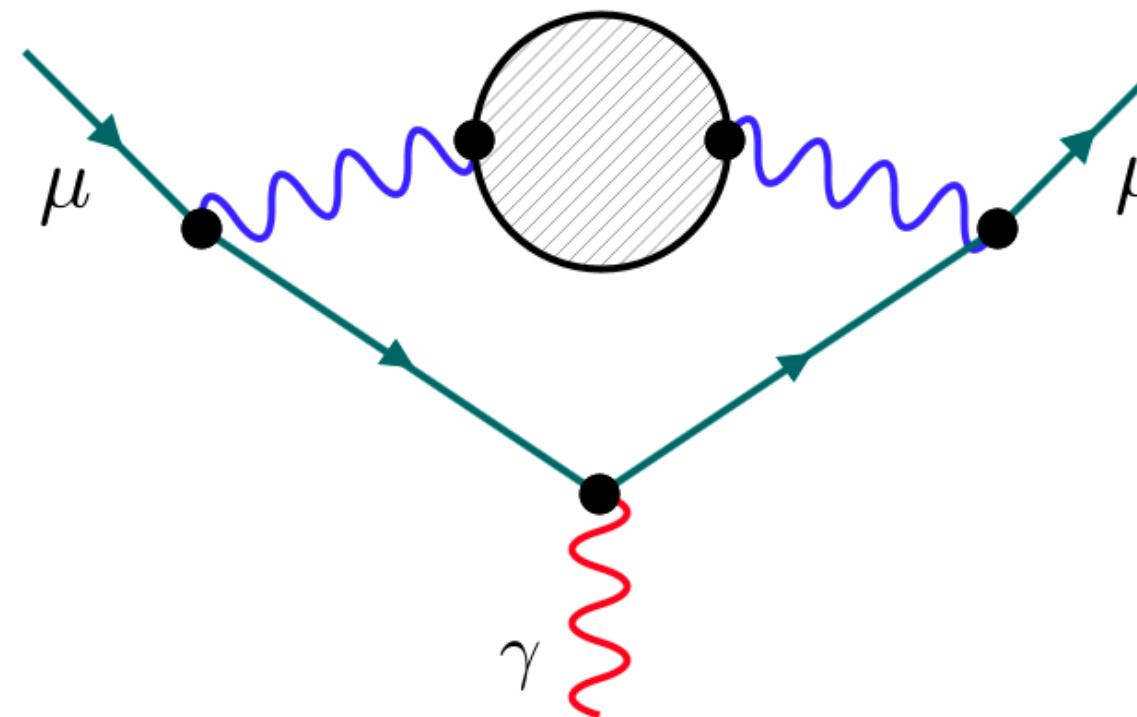
contribution to $(g - 2)_\mu$



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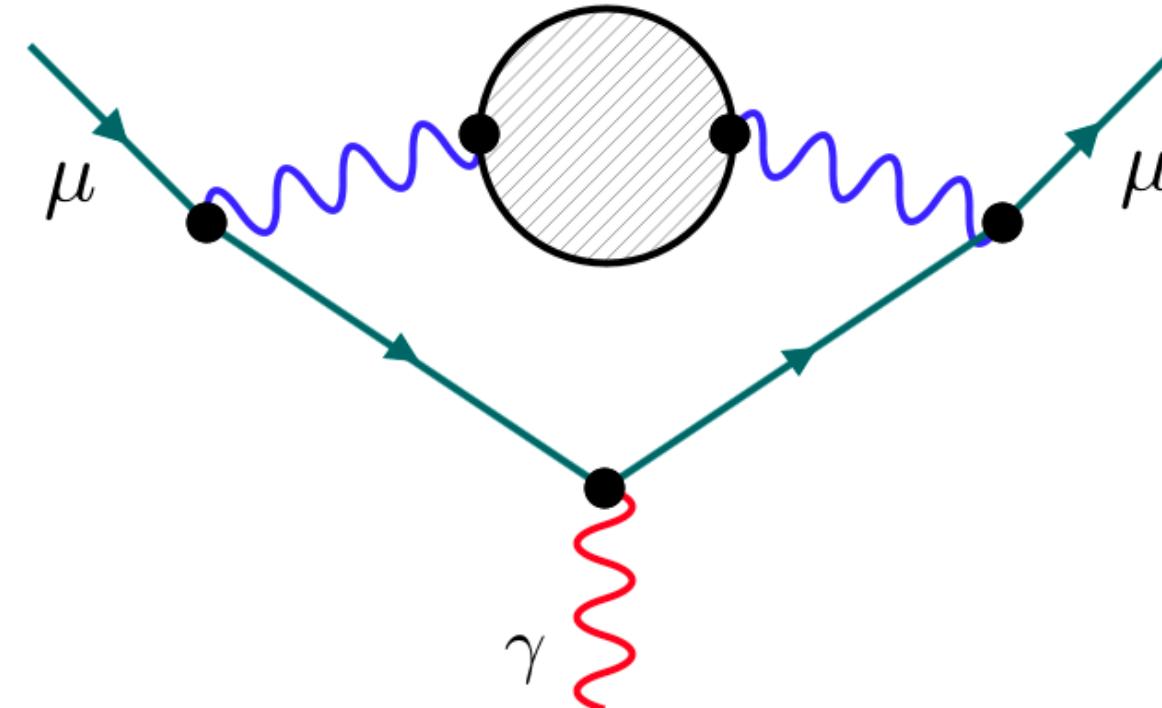
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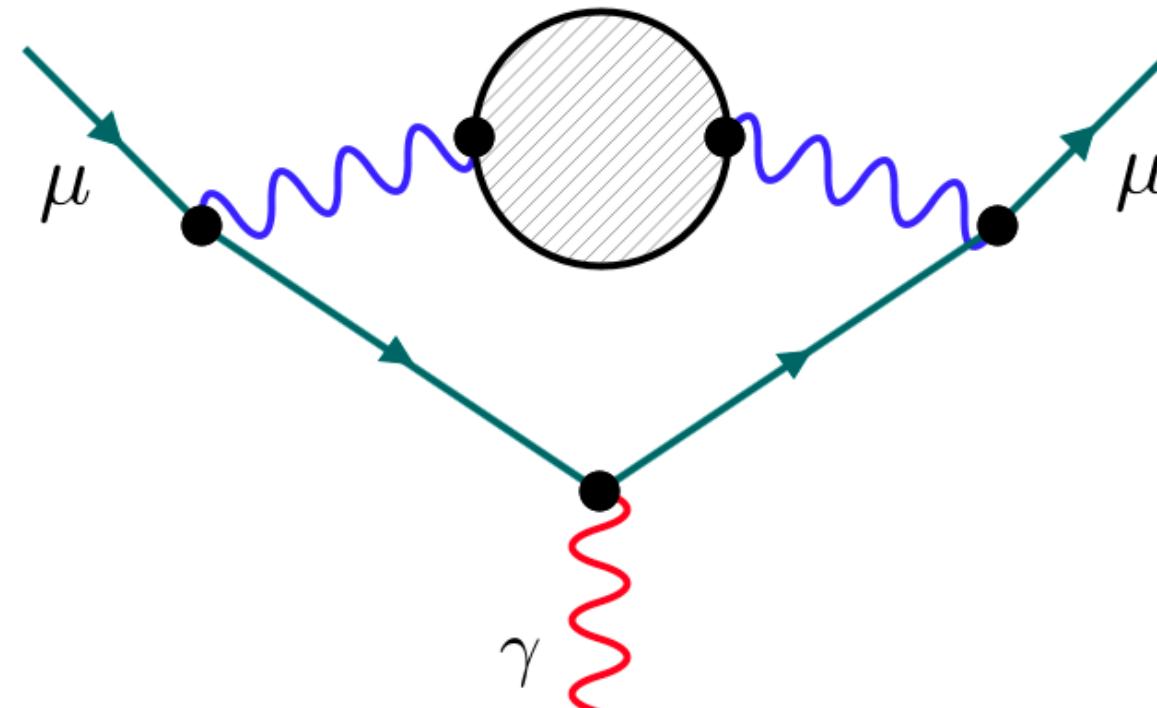
contribution to $(g - 2)_\mu$



Ex. 2: Hadronic vacuum polarization

$$\int d^4x e^{iQx} \langle T j(x) j(0) \rangle \rightarrow \sum_n C_n(Q) \langle \mathcal{O}_n \rangle$$

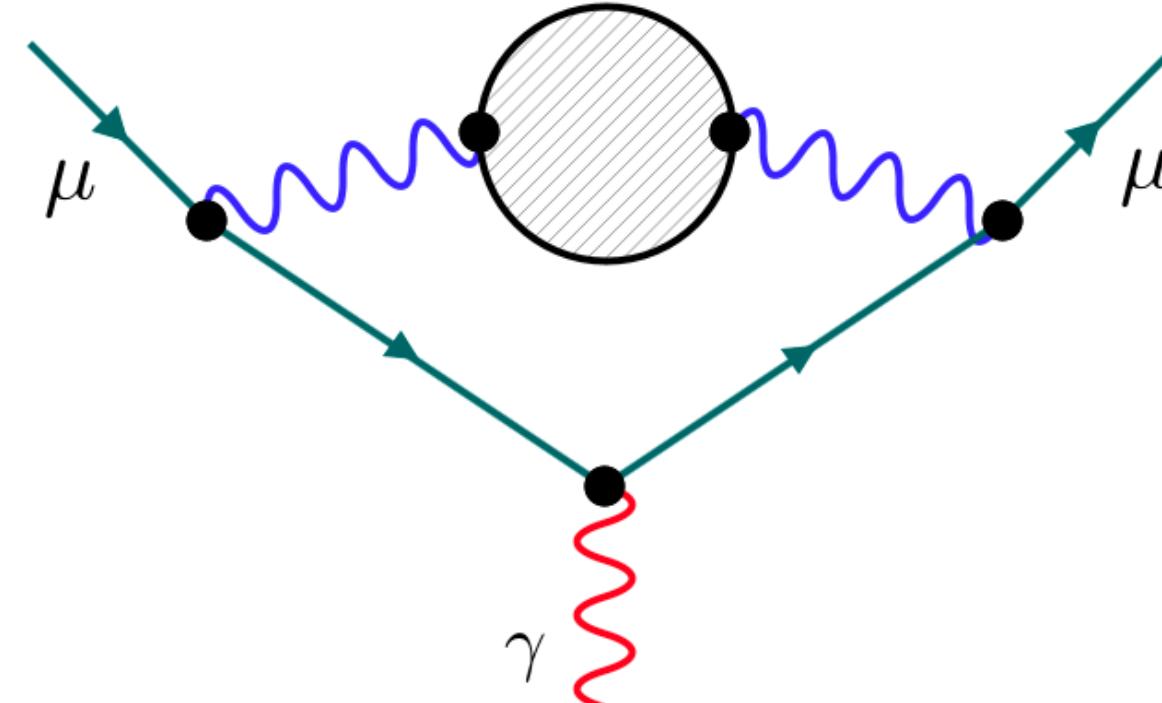
contribution to $(g - 2)_\mu$



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contribution to $(g - 2)_\mu$

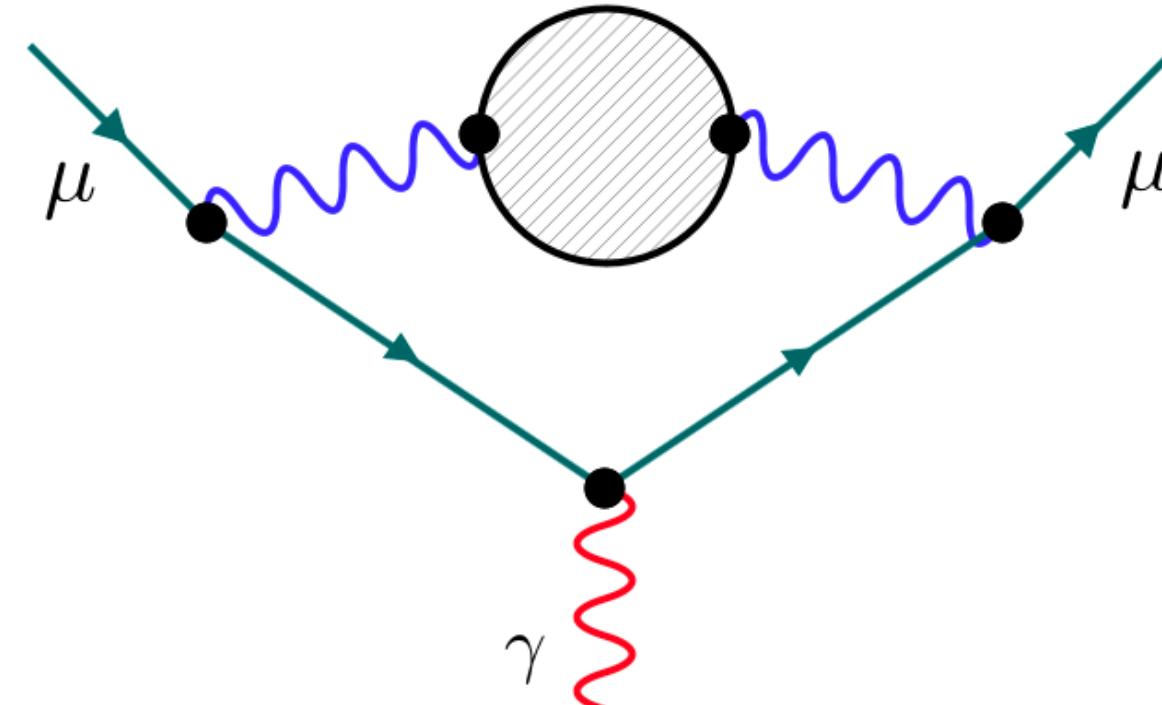


Known to
high orders in
perturbation theory

Ex. 2: Hadronic vacuum polarization

$$\int d^4x e^{iQx} \langle T j(x) j(0) \rangle \rightarrow \sum_n C_n(Q) \langle \mathcal{O}_n \rangle = \sum_n \tilde{C}_n(Q, \textcolor{red}{t}) \langle \tilde{\mathcal{O}}_n(\textcolor{red}{t}) \rangle$$

contribution to $(g - 2)_\mu$

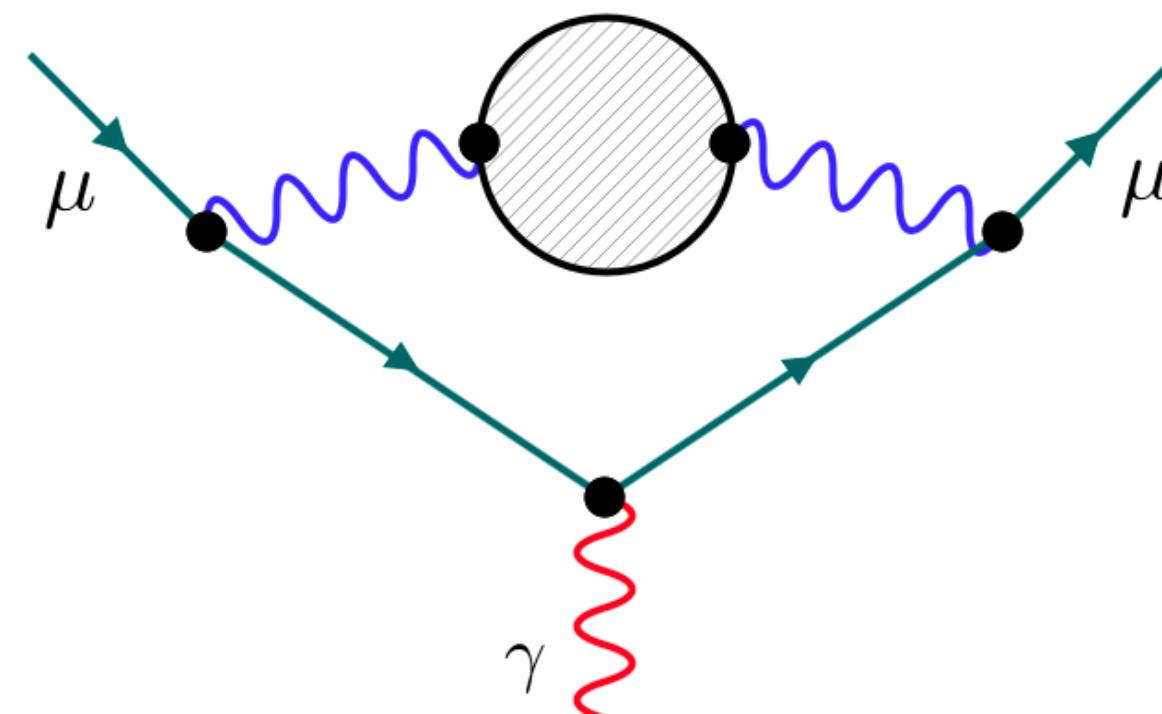


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contribution to $(g - 2)_\mu$



Known to
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$$\mathcal{O}_1 = 1$$

$$\mathcal{O}_2 = m^2$$

$$\mathcal{O}_3 = m^4$$

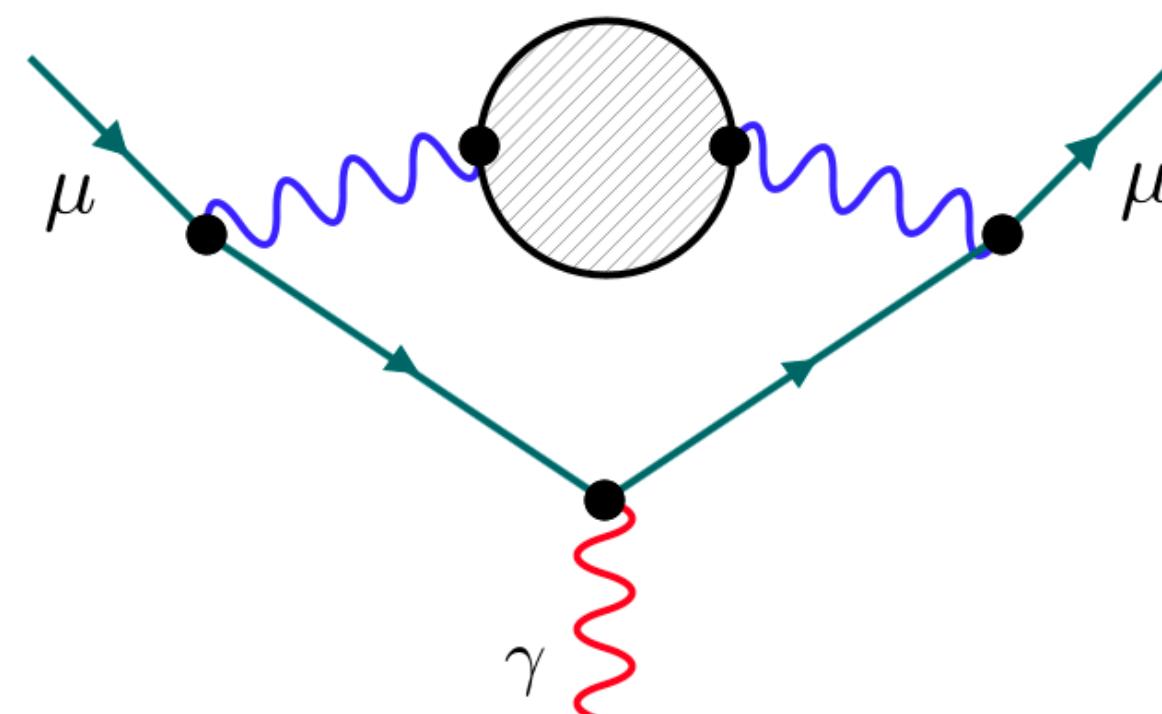
$$\mathcal{O}_4 = F_{\mu\nu}^a F_{\mu\nu}^a$$

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Known to
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$$\begin{aligned}\mathcal{O}_1 &= 1 & \mathcal{O}_2 &= m^2 & \mathcal{O}_3 &= m^4 \\ \mathcal{O}_4 &= F_{\mu\nu}^a F_{\mu\nu}^a & \mathcal{O}_5 &= m \bar{\psi} \psi\end{aligned}$$

$\tilde{C}_n(Q, \textcolor{blue}{t})$ to NNLO

RH, Lange, Neumann '20

Flow-time evolution

$t \rightarrow 0 :$

$$\tilde{\mathcal{O}}_n(\textcolor{blue}{t}) = \sum_m \zeta_{nm}(\textcolor{blue}{t}) \mathcal{O}_m$$

matrix notation:

$$\tilde{\mathcal{O}}(\textcolor{blue}{t}) = \zeta(\textcolor{blue}{t}) \mathcal{O}$$
$$t \frac{\partial}{\partial t} \tilde{\mathcal{O}}(\textcolor{blue}{t}) = \left(t \frac{\partial}{\partial t} \zeta(\textcolor{blue}{t}) \right) \mathcal{O} = \left(t \frac{\partial}{\partial t} \zeta(\textcolor{blue}{t}) \right) \zeta^{-1}(\textcolor{blue}{t}) \tilde{\mathcal{O}}(\textcolor{blue}{t})$$

$$t \frac{\partial}{\partial t} \tilde{\mathcal{O}}(t) = \tilde{\gamma}(t) \tilde{\mathcal{O}}(t)$$

with

$$\tilde{\gamma}(t) = \left(t \frac{\partial}{\partial t} \zeta(t) \right) \zeta^{-1}(t)$$

RH, Lange, Neumann '20

Conclusions and Outlook

$$R = \sum_n C_n \langle \mathcal{O}_n \rangle = \sum_n \tilde{C}_n(\textcolor{blue}{t}) \langle \tilde{\mathcal{O}}_n(\textcolor{blue}{t}) \rangle$$

unique possibilities for lattice/p.t. synergies

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unique possibilities for lattice/p.t. synergies

Applications so far:

- energy-momentum tensor (NNLO)
- hadronic vacuum polarization (NNLO)

RH, Kluth, Lange '18

RH, Lange, Neumann '20

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See also:

- ~~CP~~ operators (NLO) SymLat '20
- Bag parameter (NLO) A. Suzuki, Taniguchi, H. Suzuki, Kanada '20

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3-loop gradient flow coupling / beta-function

RH, Neumann, Lange '16, '20

see talks by K. Holland, J. Kuti, C. Peterson, ...