Stout-Smearing/Gradient Flow and c_{SW} at One Loop Order

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Goal:

Calculate the clover coefficient c_{SW} to one loop order with stout-smearing.

Previous work:

- *Sheikholeslami, Wohlert 1985* [`Nucl.Phys.B 259 (1985) 572`]
  $\rightarrow$ introduce clover term, note $c_{SW} = 1$ at tree level.
- *Wohlert 1987* [`DESY-87-069`]
  $\rightarrow$ calculation of c_{SW} to one loop using twisted boundary conditions.
- *Aoki, Kuramashi 2003* [`arXiv:hep-lat/0306015`]
  $\rightarrow$ calculation of c_{SW} to one loop for different improved gauge actions using a small gluon mass as regulator.
- *Horsley, Perlt, Rakow, Schierholz, Schiller 2008* [`arXiv:0807.0345`]
  $\rightarrow$ calculation of c_{SW} to one loop for SLINC-fermions (i.e. stout-smeared links in the Wilson part of the action but not in the clover term).

$\mathcal{O}(a)$-improved action:

$$\mathcal{S}_I = \mathcal{S}_{\text{Wilson}} + c_{\text{SW}} \cdot \sum_x \sum_{\mu<\nu} \bar{\psi}(x) \frac{1}{2} \sigma_{\mu\nu} F_{\mu\nu}(x) \psi(x)$$

$F_{\mu\nu}$: lattice field strength tensor i.e. clover shaped sum of plaquettes.

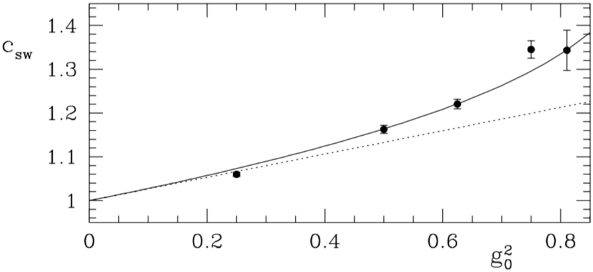
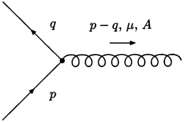


Figure: Non-perturbative determination of c_{SW} (data points), one-loop value (dashed line) and rational fit (solid line). Graphic from *Lüscher et al. 1996* [hep-lat/9609035]

Perturbative determination of $c_{\mathrm{SW}} = c_{\mathrm{SW}}^{(0)} + g_0^2 c_{\mathrm{SW}}^{(1)} + \mathcal{O}(g_0^4)$ via qqg-vertex function



$$\Lambda_\mu^a(p,q) = \sum_{L=0}^{\infty} g_0^{2L+1} \Lambda_\mu^{a(L)}(p,q),$$

At tree level:

$$\Lambda_\mu^{a(0)}(p,q) = (V_1^a)_\mu(p,q)$$
$$= -g_0 T^a \left(i\gamma_\mu + a\left(\frac{1}{2}(p_\mu + q_\mu) - \frac{i}{2} c_{\mathrm{SW}}^{(0)} \sum_\nu \sigma_{\mu\nu}(p_\nu - q_\nu)\right) + \mathcal{O}(a^2)\right)$$

Sandwich with on-shell spinors:

$$\bar{u}(q)\Lambda_\mu^{a(0)}(p,q)u(p) = -g_0 T^a \bar{u}(q)\left(i\gamma_\mu + \frac{a}{2}\left(1 - {c_{\mathrm{SW}}}^{(0)}\right)(p_\mu + q_\mu)\right)u(p) + \mathcal{O}(a^2).$$

$$\Rightarrow {c_{\mathrm{SW}}}^{(0)} = 1 \; (= r)$$

At one-loop level:

$$g_0^3 \Lambda_\mu^{a(1)} = -g_0^3 T^a \left(\gamma_\mu F_1 + a \not{q} \gamma_\mu F_2 + a \gamma_\mu \not{p} F_3 + a(p_\mu + q_\mu) G_1 + a(p_\mu - q_\mu) H_1 \right)$$

F_2 and F_3 do not contribute on-shell, H_1 vanishes due to symmetry arguments.

$$\begin{aligned} & g_0^3 \bar{u}(q) \left(\frac{a}{2} c_{\mathrm{SW}}{}^{(1)} (p_\mu + q_\mu) T^a + \Lambda_\mu^{a(1)}(p, q) \right) u(p) \\ = & g_0^3 \bar{u}(q) \left(i \gamma_\mu F_1 + \frac{a}{2} (p_\mu + q_\mu)(c_{\mathrm{SW}}{}^{(1)} - 2 G_1) T^a \right) u(p) + \mathcal{O}(p^2, q^2) + \mathcal{O}(a^2) \end{aligned}$$

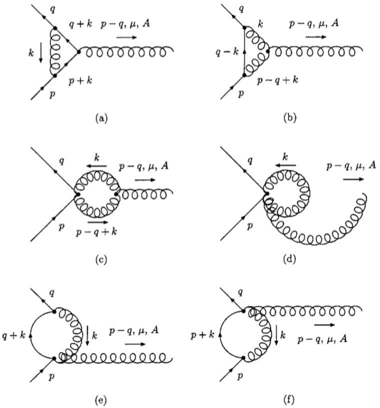
$$\Rightarrow c_{\mathrm{SW}}{}^{(1)} = 2 G_1$$

Extract G_1:

$$G_1 = -\frac{1}{8} \mathrm{Tr} \left(\left(\frac{\partial}{\partial p_\mu} + \frac{\partial}{\partial q_\mu} \right) \Lambda_\mu^{a(1)} - \left(\frac{\partial}{\partial p_\nu} - \frac{\partial}{\partial q_\nu} \right) \Lambda_\mu^{a(1)} \gamma_\nu \gamma_\mu \right)_{p,q \to 0}^{\mu \neq \nu},$$

Diagrams:

$$\Lambda_\mu^{a\,(1)} = \sum_{i=a,\dots,f} \Lambda_\mu^{a\,(1)(i)} = \sum_{i=a,\dots,f} \int_{-\pi}^{\pi} \frac{\mathrm{d}^4 k}{(2\pi)^4} I_\mu^{a\,(i)}(p,q,k).$$



from *Aoki, Kuramashi* [arXiv:hep-lat/0306015]

Example: Diagram (c)

$$I_\mu^{a\,(c)} = \sum_{b,c}\sum_{\nu,\rho} V_{2\,\nu\rho}^{bc}(p,q,q-p-k)\ G(k)\ G(p-q+k) V_{g3\,\nu\rho\mu}^{bca}(p-q+k,-k,q-p)$$

Extract divergent part:

$$G_1^{(i)} = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4}\left[I_\mu^{(i)}(k) - \theta(\pi^2-k^2)J_\mu^{(i)}(k,0)\right] + \frac{\Omega_3}{(2\pi)^4}\int_0^{\pi} J_\mu^{(i)}(k,\mu^2)k^3 dk$$

$$\mathrm{J}^{(c)} = \frac{3}{4}N_C {c_{\mathrm{SW}}}^{(0)}\frac{2\pi^2}{(2\pi)^4}\int_0^{\pi}\frac{k^3}{(k^2+\mu^2)^2}dk = \frac{3}{4}N_C {c_{\mathrm{SW}}}^{(0)}\frac{1}{16\pi^2}\left(\ln\left(\frac{\pi^2}{\mu^2}\right)-1\right) + \mathcal{O}(\mu^2)$$

$$\Rightarrow 2G_1^{(c)} = \frac{9}{2}L - 0.0813095$$

Diagram	Divergent part	Constant part	Aoki, Kuramashi
(a)	$-L/3$	0.00457196	0.004572(2)
(b)	$-9L/2$	0.0830768	0.08311(3)
(c)	$9L/2$	−0.0813095	−0.08133(3)
(d)	0	0.297394537	0.29739454(1)
(e)	$L/6$	−0.0175746	−0.017574(1)
(f)	$L/6$	−0.0175746	−0.017574(1)
Sum	0	0.268588292	0.26858825(1)

- Stout-smearing

$$U_\mu^{(1)}(x) = e^{i\rho Q_\mu(x)} U_\mu(x)$$

with

$$Q_\mu(x) = \frac{1}{2i}\left(W_\mu(x) - \frac{1}{3}\mathrm{Tr}[W_\mu(x)]\right)$$
$$W_\mu(x) = \sum_{\nu\neq\mu}(P^\dagger_{\mu\nu} - P_{\mu\nu})$$

- Iterated smearing $\rightarrow U_\mu^{(n)}(x)$
- Wilson flow is generated by infinitesimal stout-smearings

$$\partial_t U_\mu(x,t) = -g_0^2\left\{\partial_{x,\mu}\mathcal{S}_W[U(x,t)]\right\} U_\mu(x,t) = iQ_\mu(x,t)U_\mu(x,t)$$
$$U_\mu(x,0) = U_\mu(x)$$

- Limit $n \rightarrow \infty$, $\rho \rightarrow 0$ with $n\rho = t =$ const. (t in lattice units, i.e. t/a^2).

Perturbative expansion of the smeared link variable:
At leading order:

$$U_\mu^{(1)}(x) = 1 + ig_0 T^a A_\mu^{a(1)}(x) + \mathcal{O}(g_0^2)$$
$$A_\mu^{a(1)}(x) = A_\mu^a(x) + \rho(Q_\mu^a(x))^{\mathrm{LO}}$$

Fourier transform: $(\hat{k}_\mu = 2\sin(\frac{1}{2}k_\mu))$

$$A_\mu^{a(1)}(k) = \sum_\nu \left(f(k)\delta_{\mu\nu} - (f(k)-1)\frac{\hat{k}_\mu \hat{k}_\nu}{\hat{k}^2} \right) A_\nu^a(k)$$
$$A_\mu^{a(n)}(k) = \sum_\nu \left(f(k)^n\delta_{\mu\nu} - (f(k)^n-1)\frac{\hat{k}_\mu \hat{k}_\nu}{\hat{k}^2} \right) A_\nu^a(k)$$
$$A_\mu^{a}(k,t) = \sum_\nu \left(e^{-t\hat{k}^2}\delta_{\mu\nu} - (e^{-t\hat{k}^2}-1)\frac{\hat{k}_\mu \hat{k}_\nu}{\hat{k}^2} \right) A_\nu^a(k,0)$$

$$f(k) = 1 - \rho\hat{k}^2$$

Perturbative expansion of the smeared link variable:
At next-to-leading order:

$$U_\mu^{(1)}(x) = 1 + ig_0 T^a A_\mu^{a(1)}(x) - \frac{g_0^2}{2}\left(\frac{1}{2}\{T^a, T^b\} A_\mu^{a(1)}(x) A_\mu^{b(1)}(x) + \frac{1}{2}[T^a, T^b] A_\mu^{ab(1)}(x)\right) + \mathcal{O}(g_0^3)$$

$$A_\mu^{ab(1)}(x) = 4\rho \cdot \left[\frac{1}{2}(Q_\mu^a(x))^{\mathrm{LO}} A_\mu^b(x) + (Q_\mu^{ab}(x))^{\mathrm{NLO}}\right]$$

Fourier transform:

$$A_\mu^{ab(1)}(k_1, k_2) = 4\rho \sum_{\nu,\rho} g_{\mu\nu\rho}(k_1, k_2) A_\nu^a(k_1) A_\rho^b(k_2),$$

with

$$\begin{aligned} g_{\mu\nu\rho}(k_1, k_2) = & \delta_{\mu\nu} \sin(\tfrac{1}{2}(2k_{1\rho} + k_{2\rho})) \cos(\tfrac{1}{2} k_{2\mu}) \\ & - \delta_{\mu\rho} \sin(\tfrac{1}{2}(2k_{2\nu} + k_{1\nu})) \cos(\tfrac{1}{2} k_{1\mu}) \\ & - \delta_{\nu\rho} \sin(\tfrac{1}{2}(k_{1\mu} - k_{2\mu})) \cos(\tfrac{1}{2}(k_{1\nu} + k_{2\nu})). \end{aligned}$$

Perturbative expansion of the smeared link variable:
At next-to-leading order:

$$A_\mu^{ab(1)}(k_1, k_2) = 4\rho \sum_{\nu,\rho} g_{\mu\nu\rho}(k_1, k_2) A_\nu^a(k_1) A_\rho^b(k_2)$$

$$A_\mu^{ab(n)}(k_1, k_2) = 4\rho \sum_{m=0}^{n-1} \left[\sum_{\nu,\rho} g_{\mu\nu\rho}(k_1, k_2) A_\nu^{a(m)}(k_1) A_\rho^{b(m)}(k_2) \right].$$

$$\begin{aligned} A_\mu^{ab}(k_1, k_2, t) = 4 \sum_{\nu,\rho} & g_{\mu\nu\rho}(k_1, k_2) \\ & \times \sum_{\alpha\beta} \left[(e^{-t\hat{k}_1^2} - 1)\delta_{\nu\alpha} - (e^{-t\hat{k}_1^2} - 1) \frac{\hat{k}_{1\nu}\hat{k}_{1\alpha}}{\hat{k}_1^2} \right] \\ & \quad \times \left[(e^{-t\hat{k}_2^2} - 1)\delta_{\rho\beta} - (e^{-t\hat{k}_2^2} - 1) \frac{\hat{k}_{2\rho}\hat{k}_{2\beta}}{\hat{k}_2^2} \right] A_\alpha^a(k_1, 0) A_\beta^b(k_2, 0) \\ & = 4 \sum_{\nu,\rho} g_{\mu\nu\rho}(k_1, k_2) [A_\nu^a(k_1, t) - A_\nu^a(k_1, 0)][A_\rho^b(k_2, t) - A_\rho^b(k_2, 0)] \end{aligned}$$

Next steps:

- Smearing/GF at order g_0^3.
- Derive Feynman-rules with smearing (especially from the clover term).
- Calculate ${c_{\mathrm{SW}}}^{(1)}$ depending on ρ, n or t.