Study of a lattice 2-group gauge model

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joint work with A. Bochniak, L. Hadasz and P. Korcyl

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Ordinary vs higher gauge theory

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- 2-form gauge theory: surfaces, preferably closed (boundaries of volumes or *Polyakov surfaces*).

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Path ordering $\implies$ nonabelian multiplication along loops.

Surfaces admit no natural ordering

$\implies$ group elements on surfaces necessarily abelian.

Ordinary vs higher gauge theory

Higher category theory $\implies$ another possibility

- degrees of freedom on both links and plaquettes, independent but intertwined.
- described algebraically by 2-*group* or a *crossed module*.

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This talk: general structure.

See the next talk (Piotr Korcyl) for a concrete model.

Some motivations

- Topological Quantum Field Theories.
- Gauge theory EFTs proposed by Gukov and Kapustin.
- Bosonization in arbitrary dimension.
- Applications in quantum gravity.

Algebraic preliminaries

A crossed module is

- $\mathcal{E}, \Phi$ - gauge groups for links and plaquettes
  (both may be non-abelian),
- Binary operation $\triangleright$: $\mathcal{E} \times \Phi \ni (\epsilon, \varphi) \mapsto \epsilon \triangleright \varphi \in \Phi$,
- Unary operation Δ : $\Phi \ni \varphi \mapsto \Delta\varphi \in \mathcal{E}$,

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satisfying the following axioms:

- $\epsilon \triangleright (\varphi_1\varphi_2) = (\epsilon \triangleright \varphi_1)(\epsilon \triangleright \varphi_2), \quad \epsilon \triangleright 1_\Phi = 1_\Phi.$
- $(\epsilon_1\epsilon_2) \triangleright \varphi = \epsilon_1 \triangleright (\epsilon_2 \triangleright \varphi), \quad 1_\mathcal{E} \triangleright \varphi = \varphi.$
- $\Delta\varphi_1 \triangleright \varphi_2 = \varphi_1\varphi_2\varphi_1^{-1}.$
- $\Delta(\epsilon \triangleright \varphi) = \epsilon\Delta\varphi\epsilon^{-1}.$

Degrees of freedom

Degrees of freedom of a lattice 2-group gauge theory:

- $\epsilon_e \in \mathcal{E}$ for every edge (link) e from $s(e)$ to $t(e)$,
- $\varphi_f \in \Phi$ for every face (plaquette) f with a chosen corner $b(f)$,

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subject to the *fake flatness* constraint:

$$\Delta\varphi_f = \epsilon_{\partial f}. \tag{1}$$

$\epsilon_{\partial f} :=$ Wilson loop around the boundary of f,

starting and ending at $b(f)$.

Gauge transformations

Vertex transformations ($\xi_v \in \mathcal{E}$ for a vertex v):

$$\epsilon'_e = \xi_{t(e)} \epsilon_e \xi_{s(e)}^{-1}, \tag{2}$$

$$\varphi'_f = \xi_{b(f)} \triangleright \varphi_f. \tag{3}$$

Gauge transformations

Vertex transformations ($\xi_v \in \mathcal{E}$ for a vertex v):

$$\epsilon'_e = \xi_{t(e)} \epsilon_e \xi^{-1}_{s(e)}, \quad (2)$$

$$\varphi'_f = \xi_{b(f)} \triangleright \varphi_f. \quad (3)$$

- Standard transformations of link variables.
- φ_f transforms as a matter field on the lattice site $b(f)$.

Gauge transformations

Edge transformations ($\psi_e \in \Phi$ for edges e):

$$\epsilon'_e = \Delta\psi_e \epsilon_e, \tag{4}$$

$$\varphi'_f = \psi_{e_n}(\epsilon_{e_n} \rhd \psi_{e_{n-1}}) \cdots (\epsilon_{e_n} \cdots \epsilon_{e_2} \rhd \psi_{e_1})\varphi_f. \tag{5}$$

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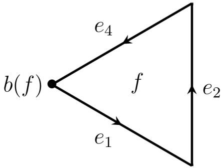
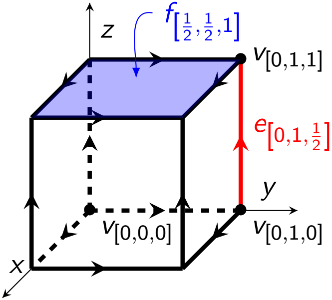


Figure: For this triangular plaquette f we have $\varphi'_f = \psi_{e_3}(\epsilon_{e_3} \triangleright \psi_{e_2})(\epsilon_{e_3}\epsilon_{e_2} \triangleright \psi_{e_1})\varphi_f$.

Elementary surface observables



Invariant cube observable:

$$\varphi_{\text{cube}} = \varphi_{[\frac{1}{2},\frac{1}{2},0]}\varphi_{[\frac{1}{2},0,\frac{1}{2}]}(\epsilon^{-1}_{[0,0,\frac{1}{2}]} \triangleright \varphi_{[\frac{1}{2},\frac{1}{2},0]})\varphi^{-1}_{[\frac{1}{2},0,\frac{1}{2}]}(\epsilon^{-1}_{[\frac{1}{2},0,0]} \triangleright \varphi_{[1,\frac{1}{2},\frac{1}{2}]})$$

$$\varphi^{-1}_{[\frac{1}{2},\frac{1}{2},0]}(\epsilon^{-1}_{[0,\frac{1}{2},0]} \triangleright \varphi_{[\frac{1}{2},1,\frac{1}{2}]})\varphi^{-1}_{[0,\frac{1}{2},\frac{1}{2}]}\varphi^{-1}_{[\frac{1}{2},0,\frac{1}{2}]}\varphi^{-1}_{[\frac{1}{2},\frac{1}{2},0]}.$$

Unnamed slide

- Formulas for any surface observables of spherical topology may be derived systematically using relative homotopy groups.

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- Formulas for any surface observables of spherical topology may be derived systematically using relative homotopy groups.
- $\varphi_{\text{closed surface}}$ is valued in the abelian subgroup $\ker(\Delta) \subset \Phi$, but it is constructed from not necessarily abelian objects.
- Regarding all edge transformations as gauge redundancies $\implies$ Wilson loops are not invariant.

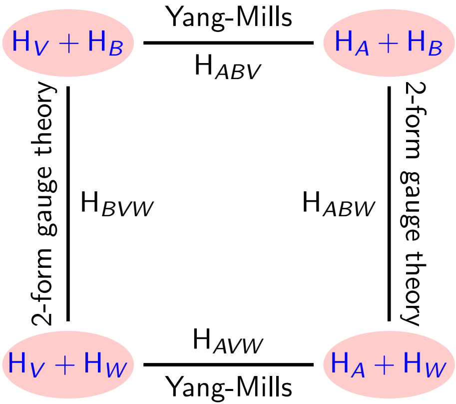
  Restricting gauge parameters to $\psi_e \in \ker(\Delta)$ yields generalized Yang-Mills with loop and surface observables.

Structure of the Hamiltonian

Hamiltonian $H = H_A + H_B + H_V + H_W$:

- H_A magnetic,
- H_B higher magnetic,
- H_V electric,
- H_W higher electric.

Special cases



Corners $\implies$ integrable, gapped limits (TQFT).

Edges $\implies$ known dynamical models (up to global twists).

Topological limits

- $H_V + H_W$: "toric code" with gauge group $\mathrm{coker}(\Delta)$.
- $H_A + H_W$: "toric code" with gauge group $\mathcal{E}$.
- $H_B + H_V$: UV completion of Yetter's TQFT.
- $H_A + H_B$: as above, with group Φ reduced to $\ker(\Delta)$.

Outlook

- Path integral version: proved and checked in simulations that interaction between $\mathcal{E}$ and Φ sectors is seen only in nonlocal observables (see next talk).
  - Not clear in Hamiltonian version.
  - How to escape this?
- What matter can be coupled to 2-group gauge fields?
- Are there more general 2-group gauge theories out there?

Thank you for your attention

For details, see:

- A. Bochniak, L. Hadasz and B. Ruba, *Dynamical generalization of Yetter's model based on a crossed module of discrete groups*, Journal of High Energy Physics **2021** (2021).
- A. Bochniak, L. Hadasz, P. Korcyl and B. Ruba, *Dynamics of a lattice 2-group gauge theory model*, arXiv:2105.05671 (2021).