A novel nonperturbative renormalization scheme for local operators

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Renormalisation on the lattice

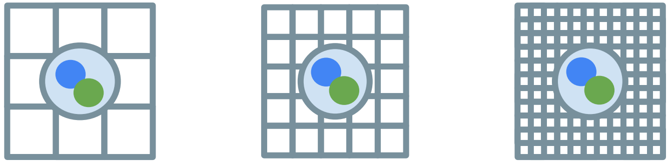
Renormalisation on the lattice poses challenges beyond those encountered in textbook discussions

Renormalisation procedures should be nonperturbative

★ Parameters of QCD Lagrangian typically renormalised by tuning physical parameters

$$a_1 m_{u/d}^{(1)} \rightarrow a_2 m_{u/d}^{(2)} \rightarrow a_3 m_{u/d}^{(3)}$$

$$m_\pi = m_\pi^{\text{phys}}$$



★ Composite operators require different approaches

- connect hadronic and perturbative scales
- matched to the MS-bar scheme via perturbation theory

E.g. Rome-Southampton method *NPB* 445 (1995) 81
or Schrödinger functional *NPB* 384 (1992) 168

Study the feasibility of the gradient flow for defining a renormalisation procedure for local composite operators

Gradient flow and real-space renormalisation group

Block-spin transformation provides a real-space averaging or coarse-graining procedure

$$\Phi_b(x_b) = f(\phi)$$
$$g_i \to g_i^{(b)}$$

- ★ leaves partition function invariant
- ★ modifies parameters of the action and expectation values of operators

In vicinity of a fixed point, scaling operators and two-point functions transform as

$$G_{\mathcal{O}}(g_i, x_0) = \langle \mathcal{O}(0)\mathcal{O}(x_0)\rangle$$
$$x_0 \gg b$$

$$\tilde{\mathcal{O}} = b^{d+\gamma}\mathcal{O} \qquad G_{\mathcal{O}}(g_i, x_0) = b^{-2(d+\gamma)} G_{\mathcal{O}}(g_i^{(b)}, x_0/b)$$

Provides definition of the anomalous dimension of the operator/two-point function

$$\Delta_{\mathcal{O}}(g_i^{(b)}) = b\frac{\mathrm{d}}{\mathrm{d}b}\log G_{\mathcal{O}}(g_i^{(b)}, x_0/b)$$

Gradient flow provides natural tool for coarse-graining - smears fields over a region ~ $\sqrt{8\tau}$

$$\Phi_b(x_b) = b^{-d_\phi - \eta/2}\phi(bx_b;\tau)$$

Carosso, Hasenfratz & Neil, PRL 121 (2018) 201601

Narayanan & Neuberger, JHEP 0603 064
Lüscher, JHEP 1008 071
Lüscher, JHEP 04 (2013) 123

Renormalisation scheme: some preliminaries

Define the bare two-point functions

$$G_{\mathcal{O}}(x_4;\tau) = \int \mathrm{d}^3\mathbf{x}\,\langle \mathcal{O}(\mathbf{x},x_4;\tau)P_{\mathcal{O}}(x)\rangle$$

$$G_V(x_4;\tau) = \frac{1}{3}\sum_{j=1}^{3}\int \mathrm{d}^3\mathbf{x}\,\langle V_j(\mathbf{x},x_4;\tau)P_V(x)\rangle$$

and introduce the bare double ratio

$$\overline{R}_{\mathcal{O}}(x_4;\tau) = \frac{R_{\mathcal{O}}(x_4;\tau=0)}{R_{\mathcal{O}}(x_4;\tau)}$$

$$R_{\mathcal{O}}(x_4;\tau) = \frac{G_{\mathcal{O}}(x_4;\tau)}{G_V(x_4;\tau)}$$

which renormalises as

$$\overline{R}^{\mathrm{R}}_{\mathcal{O}}(x_4;\tau) = \frac{Z_{\mathcal{O}}}{Z_V}\overline{R}_{\mathcal{O}}(x_4;\tau)$$

Renormalisation scheme

Define the gradient flow scheme by imposing the renormalisation condition

$$\overline{Z}_{\mathcal{O}}^{\mathrm{GF}}(\mu)\overline{R}_{\mathcal{O}}(x_4;\tau)\Big|_{\substack{\mu^2\tau=c\\ x_4^2\gg\tau/c}} = \overline{R}_{\mathcal{O}}^{(\mathrm{tree})}(x_4;\tau)$$

$$\overline{R}_{\mathcal{O}}(x_4;\tau) = \frac{R_{\mathcal{O}}(x_4;\tau=0)}{R_{\mathcal{O}}(x_4;\tau)}$$

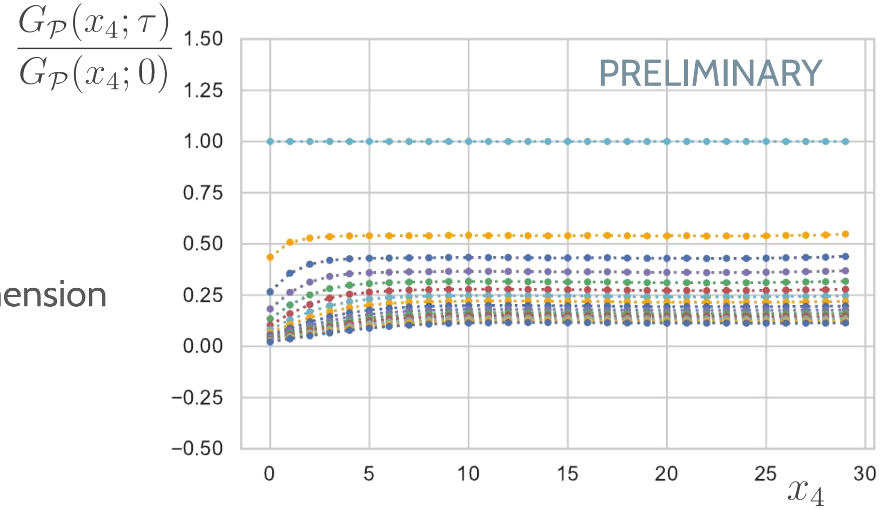
$$R_{\mathcal{O}}(x_4;\tau) = \frac{G_{\mathcal{O}}(x_4;\tau)}{G_V(x_4;\tau)}$$

which allows us to extract

$$\overline{Z}_{\mathcal{O}}^{\mathrm{GF}}(\mu) = \frac{\overline{R}_{\mathcal{O}}^{(\mathrm{tree})}(x_4;\tau)}{\overline{R}_{\mathcal{O}}(x_4;\tau)}\Bigg|_{\substack{\mu^2\tau=c\\ x_4^2\gg\tau/c}}$$

We further define the (nonperturbative) anomalous dimension

$$\gamma_{\mathcal{O}} = -2\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\log Z_{\mathcal{O}}^{\mathrm{GF}}(\mu)$$



Procedure

1. Calculate the renormalisation parameter nonperturbatively

$$\overline{Z}_{\mathcal{O}}^{\mathrm{GF}}(\mu) = \left.\frac{\overline{R}_{\mathcal{O}}^{(\mathrm{tree})}(x_4;\tau)}{\overline{R}_{\mathcal{O}}(x_4;\tau)}\right|_{\substack{\mu^2\tau = c \\ x_4^2 \gg \tau/c}}$$

2. Calculate the anomalous dimension nonperturbatively

$$\gamma_{\mathcal{O}} = -2\tau \frac{\mathrm{d}}{\mathrm{d}\tau} \log Z_{\mathcal{O}}^{\mathrm{GF}}(\mu)$$

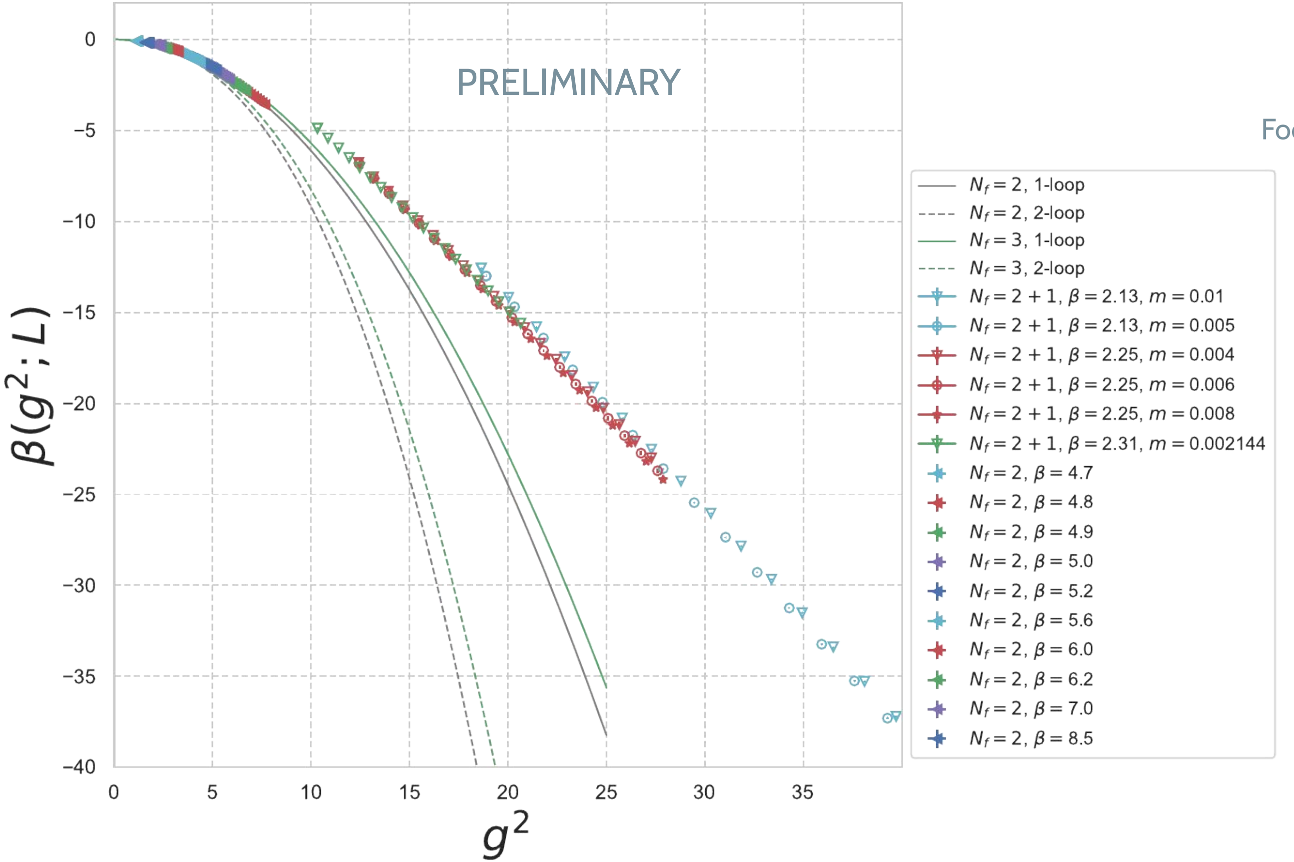
to move from low to high scales

3. Match to the MS-bar scheme using perturbation theory

Preliminary nonperturbative results

nf=2 data from Hasenfratz & Witzel, PRD 101 (2020) 034514
nf=2+1 RBC/UKQCD DWF ensembles JHEP 1712 (2017) 008
PRD 83 (2011) 074508
PRD 78 (2008) 114509

Similar approach proposed in
Fodor *et al.*, EPJ Web Conf. 175 (2018) 08027



See also:

R. Harlander Tuesday 07:15

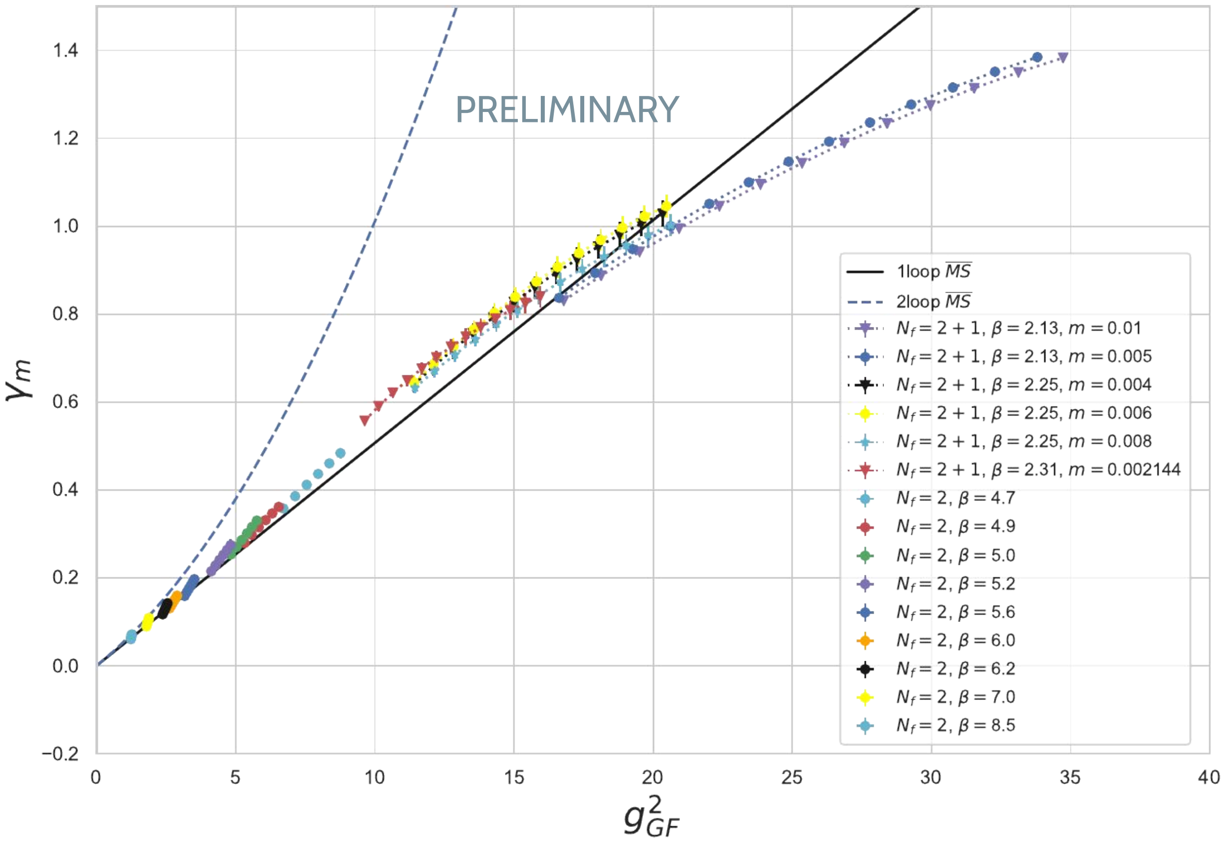
C. Peterson Wednesday 22:30

K. Holland Wednesday 22:45

J. Kuti Thursday 21:00

A. Hasenfratz Thursday 21:15

Preliminary nonperturbative results



See: M. Rizik Wednesday 07:30

Perturbative analysis

Parallel perturbative calculations

- ★ Perturbative matching to MS-bar scheme
- ★ Perturbative analysis of tree-level discretisation effects

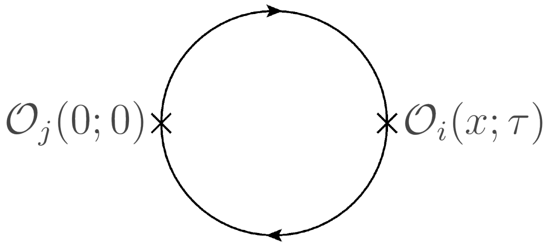
Proof-of-principle matching calculation: calculate matching coefficient

$$\Delta Z_{\mathcal{O}} = Z_{\mathcal{O}}^{\mathrm{GF}}(\mu) - Z_{\mathcal{O}}^{\overline{\mathrm{MS}}}(\mu)$$

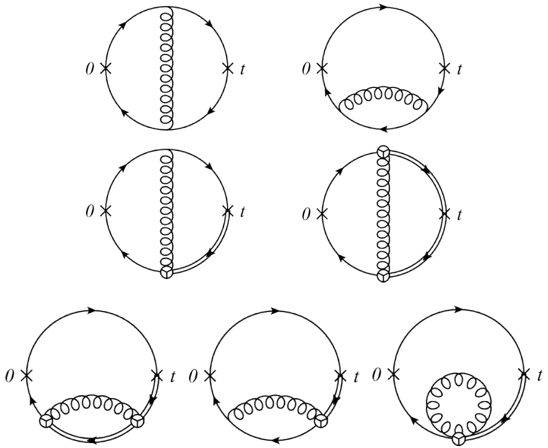
Standard momentum-space perturbative representation of correlators

- leads to challenging integrals at NLO
- oscillatory factor complicate numerical integration

Tree level



One loop



Preliminary perturbative calculations

Consider first

$$\Gamma_{ij}(\tau) = \int \mathrm{d}^4x \, \langle \mathcal{O}_i(x;\tau)\mathcal{O}_j(0;0)\rangle$$

Recall: nonperturbative scheme defined via

$$\Gamma_{ij}(x_4;\tau) = \int \mathrm{d}^3\mathbf{x} \, \langle \mathcal{O}_i(\mathbf{x},x_4;\tau)\mathcal{O}_j(0;0)\rangle$$

Results depend on choice of bilinear operator, but can be written

$$\Gamma_{i,i}(\tau) = \Gamma_{i,i}^{(0)} \left\{ 1 + g^2 \frac{C_2(F)}{(4\pi)^2} \left[(B_i^2 - 4)\left(\frac{1}{\epsilon} + \log(2\bar{\mu}^2\tau) + \gamma_E \right) + C_i + \mathcal{O}(\epsilon) \right] + \mathcal{O}(g^4) \right\}$$

LO correlator

operator-dependent constant

scheme-dependent finite part

Generates correct leading-order anomalous dimension, e.g. for the pseudoscalar case

$$\gamma_P = -2\tau\partial_\tau \log \left[\frac{R_P(\tau)}{R_P^{(0)}(\tau)} \right] = -6g^2 \frac{C_2(F)}{(4\pi)^2} + \mathcal{O}(g^4)$$

Conclusions

Gradient flow provides controlled, continuous smearing for fields on the lattice

Introduced the gradient flow scheme to nonperturbatively renormalise local composite operators

★ Nonperturbative
★ Gauge-invariant
★ Provides nonperturbative step-scaling procedure
★ Defined for both small- and large-volume regimes

Preliminary nonperturbative results available and perturbative analysis underway

Thank you!

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Gradient flow

Continuous one parameter mapping - evolves fields to classical minimum

$$\partial_\tau B_\mu = D_\nu G_{\nu\mu} + \alpha_0 D_\mu \partial_\nu B_\nu \qquad D_\mu G_{\nu\sigma} = \partial_\mu G_{\nu\sigma} + [B_\mu, G_{\nu\sigma}]$$

$$\partial_\tau \chi = D_\nu D^\nu \chi - \alpha_0 \partial_\nu B_\nu \chi \qquad D_\mu \chi = \partial_\mu \chi + B_\mu \chi \qquad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

Provides controlled, continuous smearing

- ★ Gauge invariant
- ★ Nonperturbative
- ★ Renormalised correlation functions remain finite, up to a multiplicative wavefunction renormalisation

Solving the flow equations at leading order

$$\widetilde{B}_\mu(p) = e^{-p^2\tau} \widetilde{A}_\mu(p) + \mathcal{O}(g) \qquad B_\mu|_{\tau=0} = A_\mu$$

$$\widetilde{\chi}(p) = e^{-p^2\tau} \widetilde{\psi}(p) + \mathcal{O}(g) \qquad \chi|_{\tau=0} = \psi$$

Preliminary perturbative calculations

For any bilinear i, we will need the NLO calculation for i and the vector:

$$\tilde{\Gamma}^{(1)}_{i,i} = \frac{C_2(F)}{(4\pi)^2}\tilde{\Gamma}^{(0)}_{i,i}\left\{(B_i^2-4)\left[\frac{1}{\epsilon}+\log(8\pi t)\right]-6\log 3+20\log 2+2B_i^2-2B_i-2+\mathcal{O}(\epsilon)\right\},$$

where $B_i=(2,-2,1,-1,0)$ and

$$\tilde{\Gamma}^{(0)}_{ij}(t,0)=\begin{cases} k_S^2\frac{\dim(F)}{(4\pi)^2 t}(8\pi t)^{2-d/2}\frac{4d}{d(d-2)}+\mathcal{O}(m), & i,j=S,S \\ k_P^2\frac{\dim(F)}{(4\pi)^2 t}(8\pi t)^{2-d/2}\frac{4(d-8)}{d(d-2)}+\mathcal{O}(m), & i,j=P,P \\ -k_V^2\frac{\dim(F)}{(4\pi)^2 t}(8\pi t)^{2-d/2}\frac{4(d-2)}{d(d-2)}\delta_{\mu\nu}+\mathcal{O}(m), & i,j=V,V \quad (\gamma_\mu,\gamma_\nu) \\ k_A^2\frac{\dim(F)}{(4\pi)^2 t}(8\pi t)^{2-d/2}\frac{4(d-6)}{d(d-2)}\delta_{\mu\nu}+\mathcal{O}(m,\hat{\delta}), & i,j=A,A \quad (\gamma_\mu\gamma_5,\gamma_\nu\gamma_5) \\ k_T^2\frac{\dim(F)}{(4\pi)^2 t}(8\pi t)^{2-d/2}\frac{4(d-4)}{d(d-2)}\delta_\mu^{[\rho}\delta_\nu^{\sigma]}+\mathcal{O}(m), & i,j=T,T \quad (\sigma_{\mu\nu},\sigma_{\rho\sigma}) \end{cases}.$$

Renormalizing only the coupling, so that $g_0^2=\mu^{2\epsilon}g^2+\mathcal{O}(g^4)$, we have

$$\tilde{\Gamma}_{i,i}=\tilde{\Gamma}^{(0)}_{i,i}\left\{1+g^2\frac{C_2(F)}{(4\pi)^2}\left[(B_i^2-4)\left(\frac{1}{\epsilon}+\log(2\bar{\mu}^2 t)+\gamma_E\right)-6\log 3+20\log 2+2B_i^2-2B_i-2+\mathcal{O}(\epsilon)\right]+\mathcal{O}(g^4)\right\}.$$

Using

$$\partial_t\frac{\tilde{\Gamma}_{i,i}}{\tilde{\Gamma}^{(0)}_{i,i}}=g^2\frac{C_2(F)}{(4\pi)^2}\frac{B_i^2-4}{t},$$

We have

$$\gamma_P=-2t\partial_t\log[R_P(t)/R_P^{(0)}(t)]=-6g^2\frac{C2F}{(4\pi)^2}+\mathcal{O}(g^4).$$