P-Wave Two-Body Bound and Scattering States in a Finite Volume including QED

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Abstract

The mass shifts for two-fermion bound and scattering P-wave states subject to the long-range interactions due to QED in the non-relativistic regime are presented. Introducing a short range force coupling the spinless fermions to one unit of angular momentum in the framework of pionless EFT [1, 2], we first report the two-body scattering amplitudes with Coulomb corrections in the infinite-volume context. Motivated by the research on particle-antiparticle bound states, we show the T-matrix elements and the leading scattering parameters for fermions of identical mass and opposite charge. Second, we immerse the system into a cubic box with periodic boundary conditions and we display the finite-volume corrections to the energy of the lowest bound and unbound T_1^- eigenstates. In particular, power law contributions proportional to the fine structure constant and resembling the recent results for S-wave states are found [3]. Higher order terms in α are neglected, since the gapped nature of the momentum operator in the finite-volume environment allows for a perturbative treatment of the QED interactions. Some hints concerning the extension of the analysis to D-wave short-range interactions are eventually given.

Motivation & Objectives

The main goal is the analytical understanding of the distortions induced by a finite cubic configuration space on the energy eigenvalues and eigenfunctions of two fermions subject to both strong and electromagnetic (EM) interactions. The motivations underlying this work are twofold, thus reflecting its internal structure:

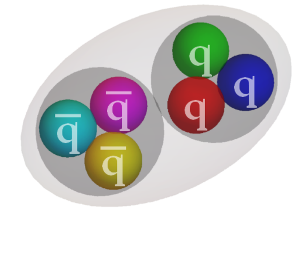
▶ Internal structure of the protonium.

▶ **Infinite Volume**: Extension of the nonperturbative treatment of EM (Coulomb) interactions on top of pionless EFT in ref. [1] to the context of P-wave fermion-fermion and fermion-antifermion scattering

Applications: low energy proton-proton scattering in the **repulsive** case and protonium ($p\bar{p}$ states) in the **attractive** case

▶ **Finite Volume**: Extension of the mass-shift formulae for bound and scattering states in a cubic box with periodic boundary conditions as in ref. [3] to short-range interactions coupling the fermions to one unit of angular momentum

Applications (Lattice QCD + QED) [4]: bound states of like hadrons appearing at unphysical values of the pion masses (currently observed only in the S-wave channel in absence of QED) in the **repulsive** case and hadronic molecules in the **attractive** case. These are bound states found in the vicinity of a two-particle threshold. Such two-hadron systems are observed in S-wave, but the the hidden-charm pentaquark states $P_c(4380)$ and $P_c(4450)$ located below the thresholds at 4385.3 and 4462.2 MeV are candidtates for $\ell = 1$ *molecular* states (cf. ref. [5]).



▶ Representation of a hadronic molecule.

Infinite-volume Formalism

Our analysis is based on pionless Effective Field Theory (EFT) [2, 6, 7]. The theory describes the strong interactions between nucleons at energy scales smaller than the pion mass, M_π [8]. The matter fields are non-relativistic, thus allowing for the introduction of a small expansion parameter $|\mathbf{p}|/M$, where $\mathbf{p}$ is the momentum of the fermion of mass M.

▶ We select the interactions transforming as the $2\ell+1$-dim. irrep of $\mathrm{SO}(3)$,

$$V^{(\ell)}(\mathbf{p},\mathbf{q}) \equiv \langle \mathbf{q}, -\mathbf{q}|\hat{\mathcal{V}}^{(\ell)}|\mathbf{p}, -\mathbf{p}\rangle = \left(c_0^{(\ell)} + c_2^{(\ell)}\mathbf{p}^2 + c_4^{(\ell)}\mathbf{p}^4 + \dots\right)\mathcal{P}_\ell(\mathbf{p}\cdot\mathbf{q})$$

where

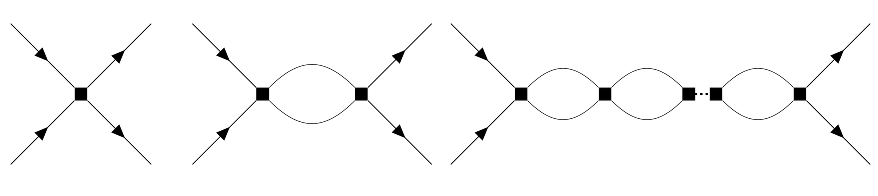
- ▷ $\mathcal{P}_\ell$ is a Legendre polynomial
- ▷ $c_{2j}^{(\ell)}$ are low-energy constants (LECs)
- ▷ $|\mathbf{p},-\mathbf{p}\rangle$ are free 2-body states in the center of mass (CoM) frame.

As in ref. [3], in our case we encode the terms within the backets by a single coefficient, $D(E^*)$, depending on the CoM energy of the colliding particles.

▶ For P-wave strong interactions, the Lagrangian density assumes the form

$$\mathcal{L} = \psi^\dagger\left[\mathrm{i}\hbar\partial_t + \frac{\hbar^2\nabla^2}{2M}\right]\psi + \frac{D(E^*)}{8}(\psi\overleftrightarrow{\nabla}_i\psi)^\dagger(\psi\overleftrightarrow{\nabla}_i\psi)\,,$$

where $\overleftrightarrow{\nabla} = \overleftarrow{\nabla} - \overrightarrow{\nabla}$ denotes the Galilean invariant derivative for spinless fermion fields, ψ. Recalling the Feynman rules, two-body elastic scattering processes without QED are represented by chains of bubbles.



▷ Two-fermion elastic scattering diagrams in absence of QED

▶ We consider QED in the non-relativistic regime, in which the Lagrangian density in ref. [9] for spinor fields Ψ assumes the form,

$$\mathcal{L}^{\mathrm{NRQED}} = -\frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) + \Psi^\dagger\left(\mathrm{i}\partial_t - e\phi + \frac{\mathbf{D}^2}{2M}\right)\Psi + \Psi^\dagger\left[c_1\frac{\mathbf{D}^4}{8M^3} + c_2\frac{e}{2M}\boldsymbol{\sigma}\cdot\mathbf{B} + c_3\frac{e}{8M^2}\nabla\cdot\mathbf{E} + c_4\frac{e}{8M^2}\mathrm{i}\mathbf{D}\times\boldsymbol{\sigma}\right]\Psi + \Psi^\dagger\left[d_1\frac{e}{8M^3}\{\mathbf{D}^2,\ \boldsymbol{\sigma}\cdot\mathbf{B}\}\right]\Psi + \dots$$

where $\mathbf{E} = -\nabla\phi - \partial_t\mathbf{A}$ & $\mathbf{B} = \nabla\times\mathbf{A}$ and

- ▷ $\mathbf{D} = \nabla + \mathrm{i}e\mathbf{A}$ is the covariant derivative

... and the ellipses contain higher order covariant derivatives, $\mathcal{O}(v^8/c^8)$ [9].

> Since the Coulomb force dominates at very low energies and transverse photons couple proportionally to the momentum of the fermions, we retain only the scalar field and its LO coupling to the ψs, as in ref. [1].

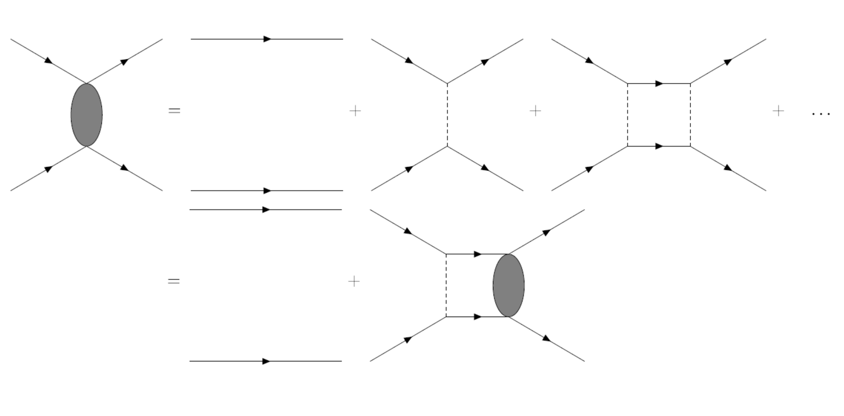
▶ We add to the above Lagrangian density the one involving the electrostatic potential and its leading-order coupling to the spinless fermions,

$$\mathcal{L}^{\mathrm{NRQED\ corr}} = -\frac{1}{2}\nabla\phi\cdot\nabla\phi - e\phi\,\psi^\dagger\psi\,.$$

▷ In momentum space, the Coulomb force, regulated by an IR cutoff, reads

$$V_{\mathrm{C}}(\mathbf{p},\mathbf{q}) \equiv \langle\mathbf{q},-\mathbf{q}|\hat{V}_{\mathrm{C}}|\mathbf{p},-\mathbf{p}\rangle = \frac{e^2}{(\mathbf{p}-\mathbf{q})^2 + \lambda^2}\,.$$

The T-matrix is enriched by new classes of diagrams, in which the scalar photons either between the external legs and within the loops emerge.
▷ Unlike transverse photons, the Coulomb ones do not propagate between different bubbles and lead only to the appearance of photon *ladders* within the loops or between the external legs in the diagrams for two-body processes.



▷ The Coulomb Green's function as a superposition of *Ladder* diagrams.

▶ The S-matrix element is determined by the overlap between an incoming, $|\chi_{\mathbf{p}}^{(+)}\rangle$, and an outcoming, $|\chi_{\mathbf{p}'}^{(-)}\rangle$, fully-interacting state,

$$S(\mathbf{p}',\mathbf{p}) = \langle\chi_{\mathbf{p}'}^{(-)}|\chi_{\mathbf{p}}^{(+)}\rangle = (2\pi)^3\delta(\mathbf{p}'-\mathbf{p}) - 2\pi\mathrm{i}\,\delta(E'-E)T(\mathbf{p}',\mathbf{p})$$

where $T(\mathbf{p}',\mathbf{p})$ is the superposition of (cf. ref. [**?**])

- ▷ $T_{\mathrm{C}}(\mathbf{p}',\mathbf{p})$, the *electrostatic* scattering amplitude $\equiv \langle\mathbf{p}'|\hat{V}_{\mathrm{C}}|\psi_{\mathbf{p}}^{(+)}\rangle$
- ▷ $T_{\mathrm{SC}}(\mathbf{p}',\mathbf{p})$, the *strong* scattering amplitude modified by the Coulomb *ladders*, equal to $\langle\psi_{\mathbf{p}'}^{(-)}|\hat{V}_{\mathrm{S}}|\chi_{\mathbf{p}}^{(+)}\rangle$

and $\psi_{\mathbf{p}}$ are Coulomb eigenfunctions. Provided the potentials are central, the latter amplitudes admit a partial wave expansion in terms of Legendre polynomials of the cosine of the scattering angle and the phase shifts. In our case

$$T_{\mathrm{SC}}(\mathbf{p}',\mathbf{p}) = -\frac{12\pi}{M}\left[\frac{e^{2\mathrm{i}\delta_1}-1}{2\mathrm{i}|\mathbf{p}|^3}\right]e^{2\mathrm{i}\sigma_1}\mathbf{p}\cdot\mathbf{p}'$$

where

> - ▷ $\sigma_1 = \arg\Gamma(2+\mathrm{i}\eta)$ is the *Coulomb phase shift*
> - ▷ $\eta \equiv Z_1Z_2\alpha M/2|\mathbf{p}|$ governs the viability of the perturbative treatment of the Coulomb interaction.

Repulsive Channel

We consider fermion-fermion scattering. In this case $\eta = \alpha M/2|\mathbf{p}| > 0$ and, thanks to the properties of the scalar photons, the T-matrix can be written in terms of the sum of a geometric series of ratio $D(E^*)\mathbb{J}_{\mathrm{C}}$,

$$T_{\mathrm{SC}}(\mathbf{p}',\mathbf{p}) = \nabla'\psi_{\mathbf{p}'}^{(-)*}(\mathbf{r}')\Big|_{\mathbf{r}'=0}\cdot\frac{D(E^*)}{\mathbb{1} - D(E^*)\mathbb{J}_{\mathrm{C}}}\nabla\psi_{\mathbf{p}}^{(+)}(\mathbf{r})\Big|_{\mathbf{r}=0}$$

where $\mathbb{J}_{\mathrm{C}}$ is the matrix produced by twofold derivation of the retarded (+) two-body Coulomb Green's function,

$$(\mathbb{J}_{\mathrm{C}})_{ij} = \partial_i\partial_j' G_{\mathrm{C}}^{(+)}(\mathbf{r},\mathbf{r}')|_{\mathbf{r},\mathbf{r}'=0}$$

explicitly:

$$\mathbb{J}_{\mathrm{C}} = M\int_{\mathbb{R}^3}\mathrm{d}^3r\,\delta(\mathbf{r})\int_{\mathbb{R}^3}\mathrm{d}^3\mathbf{r}'\,\delta(\mathbf{r}')\int_{\mathbb{R}^3}\frac{\mathrm{d}^3\mathbf{s}}{(2\pi)^3}\frac{\nabla\psi_{\mathbf{s}}^{(+)}(\mathbf{r})\otimes\nabla'\psi_{\mathbf{s}}^{(+)*}(\mathbf{r}')}{\mathbf{p}^2 - \mathbf{s}^2 + \mathrm{i}\varepsilon}$$

where $\psi_{\mathbf{p}}^{(\pm)}$ are the in (-) and out (+) going spherical waves, satisfying $\psi_{\mathbf{p}}^{(-)}(\mathbf{r}) = \psi_{-\mathbf{p}}^{(+)*}(\mathbf{r})$ and susceptible of the angular momentum expansion

$$\psi_{\mathbf{p}}^{(+)}(\mathbf{r}) = \frac{4\pi}{|\mathbf{p}|r}\sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{\ell}\mathrm{i}^\ell e^{i\sigma_\ell}Y_\ell^{m*}(\hat{\mathbf{p}})Y_\ell^m(\hat{\mathbf{r}})F_\ell(\eta,|\mathbf{p}|r)$$

where $\hat{\mathbf{p}}$ ($\hat{\mathbf{r}}$) are unit vectors to $\mathbf{p}$ ($\mathbf{r}$) and

- ▷ $F_\ell(\eta,|\mathbf{p}|r)$ is the Coulomb wavefunction with angular momentum ℓ for unbound states *regular* at the origin (unlike $G_\ell(\eta,|\mathbf{p}|r)$) [10]

▶ Exploiting the P-wave effective range expansion (ERE)

$$\mathbf{p}^2\left(1+\eta^2\right)\left[C_\eta^2|\mathbf{p}|(\cot\delta_1 - \mathrm{i}) + \alpha MH(\eta)\right] = -\frac{1}{a_{\mathrm{C}}^{(1)}} + \frac{1}{2}r_0^{(1)}\mathbf{p}^2 + \dots$$

for the repulsive Coulomb potential, where

$$H(\eta) = \psi(\mathrm{i}\eta) + \frac{1}{2\mathrm{i}\eta} - \log(\mathrm{i}\eta)$$

and $\psi(z)$ is the Digamma function, and evaluating $\mathbb{J}_{\mathrm{C}}$ in dimensional regularization (DR) in the PDS scheme [6, 7], an expression for the scattering length

$$\frac{1}{a_{\mathrm{C}}^{(1)}} = \frac{12\pi}{MD(E^*)} + \frac{\alpha^2M^2\mu}{8}\left(\pi^2 - 3\right) - \frac{\alpha^3M^3}{4}\left[\frac{1}{3-d} + \zeta(3) - \frac{3}{2}\gamma_E + \frac{4}{3} + \log\frac{\mu\sqrt{\pi}}{\alpha M}\right].$$

and the effective range parameter are obtained

$$r_0^{(1)} = \alpha M\left[\frac{2}{3-d} + \frac{8}{3} - 3\gamma_E + 2\log\frac{\mu\sqrt{\pi}}{\alpha M}\right] - 3\mu$$

where

- ▷ μ is a renormalization mass, ▷ $\zeta(3) \approx 1.2021$ is the Apéry constant
- ▷ $\gamma_E \approx 0.5772$ is the Euler-Mascheroni constant.

Attractive Channel

We consider fermion-antifermion scattering. Now $\eta = -\alpha M/2|\mathbf{p}| < 0$ and we define $\overline{D}(E^*)$ as the constant of the strong force. The T-matrix becomes

$$\overline{T}_{\mathrm{SC}}(\mathbf{p}',\mathbf{p}) = \nabla'\psi_{\mathbf{p}'}^{(-)*}(\mathbf{r}')\Big|_{\mathbf{r}'=0}\cdot\frac{\overline{D}(E^*)}{\mathbb{1} - \overline{D}(E^*)\bar{\mathbb{J}}_{\mathrm{C}}}\nabla\psi_{\mathbf{p}}^{(+)}(\mathbf{r})\Big|_{\mathbf{r}=0}$$

Now $\bar{\mathbb{J}}_{\mathrm{C}} = \bar{\mathbb{J}}_{\mathrm{C}}^{\mathrm{d}} + \bar{\mathbb{J}}_{\mathrm{C}}^{\mathrm{c}}$, where $\bar{\mathbb{J}}_{\mathrm{C}}^{\mathrm{d}}$ comes from the eigenstates $\phi_{n,\ell,m}(\mathbf{r})$,

$$\phi_{n,\ell,m} = \sqrt{\left(\frac{\alpha M}{n}\right)^3\frac{n-\ell-1!}{n+\ell!\,2n}}\,e^{-\frac{\alpha M}{2n}r}\left(\frac{\alpha Mr}{n}\right)^\ell L_{n-\ell-1}^{2\ell+1}\left(\frac{\alpha M}{n}r\right)Y_\ell^m$$

of the discrete spectrum with energy $E_n = -\frac{\alpha^2M}{4n^2}$, namely

$$\bar{\mathbb{J}}_{\mathrm{C}}^{\mathrm{d}} = \sum_{\substack{n=1\\ \ell=0}}^{+\infty}\sum_{m=-\ell}^{+\ell}\int_{\mathbb{R}^3}\mathrm{d}^3r'\,\delta(\mathbf{r}')\int_{\mathbb{R}^3}\mathrm{d}^3r\,\delta(\mathbf{r})\frac{\nabla'\phi_{n,\ell,m}(\mathbf{r}')\otimes\nabla\phi_{n,\ell,m}^*(\mathbf{r})}{E - E_n + \mathrm{i}\varepsilon}$$

where $L_k^n(x)$ are the associated Laguerre polynomials. By means of an anlogous procedure based on the attractive P-wave ERE, from DR in the PDS scheme we obtain expressions for $1/\bar{a}_{\mathrm{C}}^{(1)}$ and $\bar{r}_0^{(1)}$. The latter are found to coincide with the repulsive counterparts upon

- ▷ sign reversal of α in the polynomial terms
- ▷ exchange of $\overline{D}(E^*)$ with $D(E^*)$.

Finite-volume Formalism

▶ Aware of the role of numerical simulations for QFT in finite regions of the configuration space, we transpose our system onto a cubic finite volume with side L. We continue analytically fields and wavefunctions outside the cubic box by means of periodic boundary conditions (PBCs).

▶ The new topology yelds manifold consequences:

- ▷ a free particle carries a momentum $\mathbf{p} = 2\pi\mathbf{n}/L$, where $\mathbf{n}$ is a dimensionless 3D vector of integers;
- ▷ the validity of Ampère's and Gauss's law is compromised ⇝ the zero modes of the photon are removed [3];
- ▷ the masses of composite spinless particles are modified as [11]

$$\Delta M \equiv M^L - M = \frac{\alpha}{2\pi L}\left[\sum_{\mathbf{n}\neq 0}^{\Lambda_n}\frac{1}{|\mathbf{n}|^2} - 4\pi\Lambda_n\right] + \mathcal{O}\left(\alpha^2;\tfrac{\alpha}{L^2}\right).$$

With reference to the latter case, the finite-volume ERE takes the form

$$\mathbf{p}^2(1+\eta^2)[C_\eta^2|\mathbf{p}|(\cot\delta_1^L - \mathrm{i}) + \alpha MH(\eta)] = -\frac{1}{a_{\mathrm{C}}'^{(1)}} + \frac{1}{2}r_0'^{(1)}\mathbf{p}^2 + \dots$$

where the primed quantities are the *shifted* lattice parameters:

$$\frac{1}{a_{\mathrm{C}}'^{(1)}} = \frac{1}{a_{\mathrm{C}}^{(1)}} - \frac{\alpha r_0^{(1)}M}{2\pi L}\mathcal{I}^{(0)} \qquad r_0'^{(1)} = r_0^{(1)} + \frac{4\alpha r_1^{(1)}M}{\pi L}\mathcal{I}^{(0)}$$

and terms of $\mathcal{O}\left(\alpha^2;\tfrac{\alpha}{L^2}\right)$ are omitted.

Remark

▶ Without zero modes, in FV the momenta are $|\mathbf{p}| \geq 2\pi/L$, i.e. $\eta\sim\alpha ML$

- ▷ if $ML \ll 1/\alpha$ then $\eta \ll 1$: large volume required
- ▷ in LQCD $ML \gg 1$: QED can be treated *perturbatively*.

Quantization Conditions

▶ The quantization conditions (QC) determine the counterpart of the $\ell = 1$ energy eigenvalues in finite volume (irrep T_1 of the cubic group).

> The eigenvalues of the full Hamiltonian of the system can be identified with the singularities of the two-point correlation function.

In infinite volume, the full two-particle Green's function reads

$$G_{\mathrm{SC}}^{(+)}(\mathbf{r}',\mathbf{r}) = G_{\mathrm{C}}^{(+)}(\mathbf{r}',\mathbf{r}) + \nabla_{\mathbf{r}_1}G_{\mathrm{C}}^{(+)}(\mathbf{r}',\mathbf{r}_1)\Big|_{\mathbf{r}_1=\mathbf{0}}\cdot\frac{D(E^*)}{1 - D(E^*)\mathbb{J}_{\mathrm{C}}}\nabla_{\mathbf{r}_2}G_{\mathrm{C}}^{(+)}(\mathbf{r}_2,\mathbf{r})\Big|_{\mathbf{r}_2=\mathbf{0}}$$

In the finite volume environment:

- ▷ momentum integrals are replaced by 3D sums
- ▷ $D(E^*)\mapsto D^L(E^*)$ but the above expression remains valid.

From the denom. of the 2^{nd} term, the QC are drawn and *regulated* as in ref. [3]

$$\frac{\mathbb{1}}{D^L(E^*)} - \mathfrak{Re}\mathbb{J}_{\mathrm{C}}^{\{\mathrm{DR}\}}(\mathbf{p}) = \mathbb{J}_{\mathrm{C}}^L(\mathbf{p}) - \mathfrak{Re}\mathbb{J}_{\mathrm{C}}^{\{\Lambda\}}(\mathbf{p})$$

where

- ▷ $\mathbb{J}_{\mathrm{C}}^L$ is the finite-volume counterpart of $\mathbb{J}_{\mathrm{C}}$, here truncated at $\mathcal{O}(\alpha)$

$$\mathbb{J}_{\mathrm{C}}^L(\mathbf{p}) = -\frac{M}{L^3}\sum_{\mathbf{n}}^{\Lambda_n}\frac{\mathbf{n}\otimes\mathbf{n}}{|\mathbf{n}|^2 - \tilde{p}^2} + \frac{\alpha(M)^2}{4\pi^3L^2}\sum_{\mathbf{n}}^{\Lambda_n}\sum_{\mathbf{m}\neq\mathbf{n}}^{\infty}\frac{1}{|\mathbf{n}|^2-\tilde{p}^2}\frac{1}{|\mathbf{m}|^2-\tilde{p}^2}\frac{\mathbf{n}\otimes\mathbf{m}}{|\mathbf{n}-\mathbf{m}|^2}$$

where $\Lambda = 2\pi\Lambda_n/L$, $|\mathbf{p}| = 2\pi\tilde{p}/L$ and $\mathbf{n},\mathbf{m}\in\mathbb{Z}^3$

- ▷ $\mathfrak{Re}\mathbb{J}_{\mathrm{C}}^{\{\mathrm{DR}\}}$ is the dimensionally-regularized Hessian matrix of the retarded Coulomb Green's function,

$$\mathbb{J}_{\mathrm{C}}^{\{\mathrm{DR}\}}(\mathbf{p}) = \mathbb{1}\frac{\alpha M^2}{4\pi}\frac{\mathbf{p}^2}{3}\left[\frac{1}{\epsilon} - \frac{\gamma_E}{2} + \frac{4}{3} + \mathrm{i}\pi + \log\left(\frac{\mu\sqrt{\pi}}{2|\mathbf{p}|}\right)\right]$$

- ▷ $\mathfrak{Re}\mathbb{J}_{\mathrm{C}}^{\{\Lambda\}}$ is the cutoff-regularized version of the same matrix,

$$\mathbb{J}_{\mathrm{C}}^{\{\Lambda\}}(\mathbf{p}) = -\frac{M\Lambda}{2\pi^2}\frac{\mathbb{1}}{3}\left(\frac{\Lambda^2}{3} + \mathbf{p}^2\right) + \frac{\alpha M^2}{16\pi}\frac{\mathbb{1}}{3}\left[\Lambda^2 - \frac{2\mathrm{i}}{\pi}|\mathbf{p}|\Lambda + 4\mathbf{p}^2\log\frac{\Lambda}{|\mathbf{p}|}\right] + \mathcal{O}(\alpha^2).$$

Lüscher functions

▶ Exploiting the expression for $\mathbb{J}_{\mathrm{C}}$, together with the one of T_{SC} and the $\ell = 1$ component of the partial wave expansion for the same T-matrix, we obtain

$$\mathbf{p}^2(1+\eta^2)[C_\eta^2|\mathbf{p}|(\cot\delta_1^L - \mathrm{i}) + \alpha MH(\eta)] = -\frac{12\pi}{MD^L(E^*)} + \alpha M\mathbf{p}^2\left[\frac{4}{3} - \frac{3}{2}\gamma_E + \log\left(\frac{\mu\sqrt{\pi}}{\alpha M}\right)\right]$$

where the $\mathcal{O}(\alpha^2)$ contributions from the FV mass are neglected.

▶ Replacing $D^L(E^*)$ with its espression arising from the trace of the QC in the last equation, the finite-volume ERE can be rewritten as

$$-\frac{1}{a_C'^{(1)}} + \frac{1}{2}r_0'^{(1)}\mathbf{p}^2 + r_1'^{(1)}\mathbf{p}^4 + r_2'^{(1)}\mathbf{p}^6 + r_3'^{(1)}\mathbf{p}^8 + \dots = \frac{4\pi}{L^3}\mathcal{S}_0(\tilde{\mathbf{p}}) + \frac{\mathbf{p}^2}{\pi L}\mathcal{S}_1(\tilde{\mathbf{p}}) - \frac{\alpha M\mathbf{p}^2}{4\pi^4}\mathcal{S}_2(\tilde{\mathbf{p}}) - \frac{\alpha M}{\pi^2L^2}\mathcal{S}_3(\tilde{\mathbf{p}}) + \dots + \alpha M\mathbf{p}^2\left[\log\left(\frac{4\pi}{\alpha ML}\right) - \gamma_E\right]$$

where

- ▷ $\mathcal{S}_0(\tilde{\mathbf{p}}) = \sum_{\mathbf{n}}^{\Lambda_n}1 - \frac{4\pi}{3}\Lambda_n^3\,,\qquad \mathcal{S}_1(\tilde{\mathbf{p}}) = \sum_{\mathbf{n}}^{\Lambda_n}\frac{1}{\mathbf{n}^2-\tilde{\mathbf{p}}^2} - 4\pi\Lambda_n$
- ▷ $\mathcal{S}_2(\tilde{\mathbf{p}}) = \sum_{\mathbf{n}}^{\Lambda_n}\sum_{\mathbf{m}\neq\mathbf{n}}^{\infty}\frac{1}{\mathbf{n}^2-\tilde{\mathbf{p}}^2}\frac{1}{\mathbf{m}^2-\tilde{\mathbf{p}}^2}\frac{1}{|\mathbf{n}-\mathbf{m}|^2} - 4\pi^4\log\Lambda_n$
- ▷ $\mathcal{S}_3(\tilde{\mathbf{p}}) = \sum_{\mathbf{n}}^{\Lambda_n}\sum_{\mathbf{m}\neq\mathbf{n}}^{\infty}\frac{1}{\mathbf{n}^2-\tilde{\mathbf{p}}^2}\frac{1}{\mathbf{m}^2-\tilde{\mathbf{p}}^2}\frac{\mathbf{m}\cdot\mathbf{n}-\tilde{\mathbf{p}}^2}{|\mathbf{n}-\mathbf{m}|^2} - \pi^4\Lambda_n^2\,.$

are *Lüscher functions* [12] and the $\tilde{p}$-independent sum $\mathcal{S}_0$ vanishes.

The lowest unbound state

The lowest unbound 2-body state with $\ell^{\mathrm{P}} = 1^-$ in the infinite volume $\mapsto$ the T_1^- state in finite volume. Its total energy is $2M + \xi/M$ where $\xi \equiv 4\pi^2/L^2$.

▶ Recipe for the energy eigenvalue $E_{\mathrm{S}}^{(1,T_1)}$:

- ▷ expand the Lüscher functions about $\delta\tilde{p}^2 \equiv \tilde{p}^2 - 1$
- ▷ rewrite the ERE as a polynomial of $\delta\tilde{p}^2$
- ▷ truncate the ERE to order 0 in $\delta\tilde{p}^2$ and obtain the first expr. for $E_{\mathrm{S}}^{(1,T_1)}$
- ▷ improve the solution iteratively, by including higher powers of $\delta\tilde{p}^2$.

Omitting some higher order terms present in the original formula, we write:

$$E_{\mathrm{S}}^{(1,T_1)} = \frac{4\pi^2}{ML^2} + \frac{4\pi^2\delta\tilde{p}^2}{ML^2} = \frac{4\pi^2}{ML^2} + 6\xi\frac{4\pi a_{\mathrm{C}}'^{(1)}}{ML^3}\left[1 + \xi^2a_{\mathrm{C}}'^{(1)}r_0'^{(1)} - \xi\left(\frac{a_{\mathrm{C}}'^{(1)}}{\pi L}\right)(\mathcal{I}^{(1)} - 6) + \dots\right] + \xi\frac{\alpha\,a_{\mathrm{C}}'^{(1)}}{L^2\pi^2}\Big\{-(2\chi_1 + 2\mathcal{Q}_0)\cdot\left[1 + \xi^2a_{\mathrm{C}}'^{(1)}r_0'^{(1)} + \xi^3a_{\mathrm{C}}'^{(1)}r_1'^{(1)}\right] + \xi\left(\frac{a_{\mathrm{C}}'^{(1)}}{\pi L}\right)\left[(\mathcal{I}^{(1)} - 6)(2\mathcal{Q}_0 + 2\chi_1) + 6(\mathcal{Q}_1 - \chi_1 - 2\chi_2 + \tilde{\mathcal{R}}^{(1)} + 6\mathcal{J}^{(1)})\right] + \dots\Big\} + \dots$$

the ellipsis denotes higher order terms in $1/L$ times the scattering parameters.

The lowest bound state

▶ We search now for the most tightly bound state. The momentum turns imaginary and large ⇝ $\mathbf{p} = i\boldsymbol{\kappa}$ and $L|\boldsymbol{\kappa}|/2\pi = \tilde{\kappa} \gg 1$. In this limit:

$$\mathcal{S}_1(\mathrm{i}\tilde{\kappa}) \to -2\pi^2\tilde{\kappa} \qquad \mathcal{S}_2(\mathrm{i}\tilde{\kappa}) \to -4\pi^4\log(2\tilde{\kappa}) + \frac{\pi^2}{\tilde{\kappa}}\mathcal{I}^{(0)}$$

$$\mathcal{S}_3(\mathrm{i}\tilde{\kappa}) \to \frac{\pi^2}{2\tilde{\kappa}} - 2\pi^2\tilde{\kappa}\mathcal{I}^{(0)} - 2\pi^4\tilde{\kappa}^2$$

▶ We highlight the dependence of the ERE on α, by expanding κ in power series

$$\kappa = \kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 + \dots$$

and obtain an expression for κ_1 from the ERE in the large vol. limit ($|\kappa| \lesssim 1$). The binding energy of the lowest eigenvalue follows ▷ ▷ ▷

$$E_B^{(1,T_1)}(L) = \frac{\kappa^2}{M} = \frac{\kappa_0^2}{M} + 2\frac{\kappa_0\kappa_1}{M} + \dots = \frac{\kappa_0^2}{M} + \frac{2\alpha\kappa_0^3}{3\kappa_0^2 - r_0^{(1)}\kappa_0}\cdot\left[\log\left(\frac{4\kappa_0}{\alpha M}\right) - \gamma_E + \frac{1}{2}\right] + \frac{\alpha\mathcal{I}^{(0)}}{\pi L} - \frac{\alpha}{\pi^3L^3}\frac{2\pi^4}{\kappa_0^2}\frac{1}{3k_0 - r_0^{(1)}}\,.$$

Approximating the result further, we obtain an expression for the mass-shift

$$\Delta E_B^{(1,T_1)} \equiv E_B^{(1,T_1)}(\infty) - E_B^{(1,T_1)}(L) \approx -\frac{\alpha\mathcal{I}^{(0)}}{\pi L} + \frac{\alpha}{\pi^3L^3}\frac{2\pi^4}{3\kappa_0^3}\,.$$

Remark

▶ $\Delta E_B^{(1,T_1)}$ has the same modulus and opposite sign w.r.t. the one for S-wave bound states in ref. [3]. The same relation (cf. ref. [13]) holds in absence of QED!

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