Measuring Charged Particle Polarizabilities on the Lattice without Background Fields

by
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I. INTRODUCTION

Electromagnetic polarizabilities are important properties that shed light on the internal structure of hadrons. The quarks respond to probing electromagnetic fields, revealing the charge and current distributions inside the hadron. There is an active community in nuclear physics partaking in this endeavor. Experimentally, polarizabilities are primarily studied by low-energy Compton scattering. On the theoretical side, a variety of methods have been employed to describe the physics involved, from phenomenological models [1, 2], to chiral perturbation theory (ChPT) [3–5] or chiral effective field theory (EFT) [6, 7], to lattice QCD. Reviews of the experimental status can be also found in Refs. [3, 7].

Understanding electromagnetic polarizabilities has been a long-term goal of lattice QCD. The challenge lies in the need to apply both QCD and QED principles. The standard tool to compute polarizabilities is the background field method which has been widely used [8–26]. Methods to study higher-order polarizabilities have also been proposed [27–30] in this approach. Although such calculations are relatively straightforward, requiring only two-point functions, there are a number of unique challenges. First, since weak fields are needed, the energy shift involved is very small relative to the mass of the hadron (on the order of one part in a million depending on field strength). This challenge has been successfully overcome by relying on statistical correlations with or without the field. Second, there is the issue of discontinuities across the boundaries when applying a uniform field on a periodic lattice. This has been largely resolved by using quantized values for the fields. Third and more importantly, a charged hadron accelerates in electric field and and exhibits Landau levels in magnetic field. Such motions are unrelated to polarizability and must be isolated from the deformation due to quark and gluon dynamics inside the hadron. For this reason, most calculations have focused on neutral hadrons. Since standard plateau technique of extracting energy from the large-time behavior of the two-point correlator fails for charged hadrons, special techniques are needed to filter out the collective motion of the system in order to extract polarizabilities [14, 31–33].

In this work, we examine the use of four-point functions to extract polarizabilities. As we shall see, the method is ideally suited to charged hadrons; there is no background field to speak of. Furthermore, the method directly mimics the Compton scattering process on the lattice. Although four-point correlation functions have been applied to various aspects of hadron structure [34–39], not too much attention has been paid to its potential application for polarizabilities. The only work we are aware of are two attempts 25 years ago, one based in position space [40], one in momentum space [41]. Here we want to take a fresh look at the problem.

II. CHARGED PION

A. Electric polarizability

For this part, we follow closely the notations and conventions of Ref. [41]. The central object is the time-ordered Compton scattering tensor defined by the four-point correlation function[1],

$$T_{\mu\nu} = i\int d^4x e^{ik_2\cdot x}(\pi(p_2)|Tj_\mu(x)j_\nu(0)|\pi(p_1)) \tag{1}$$

where the electromagnetic current density

$$j_\mu = q_u\bar{u}\gamma_\mu u + q_d\bar{d}\gamma_\mu d, \tag{2}$$

built from up and down quark fields ($q_u = 2/3$, $q_d = -1/3$). The function is represented in Fig. 1. We work with a special kinematical setup called zero-momentum Breit frame given by,

$$\begin{aligned} &p_1 = (m,\vec{0}),\\ &k_1 = (0,\vec{k}),\ k_2 = (0,\vec{k}),\ \vec{k} = k\hat{z},\ k \ll m,\\ &p_2 = -k_2 + k_1 + p_1 = (m,\vec{0}), \end{aligned} \tag{3}$$

Essentially it can be regarded as forward double virtual Compton scattering. This is different from the real Compton scattering in experiments. They access the same low energy constants including the polarizabilities.

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[1] We use round brackets $(\cdots|\cdots)$ to denote continuum matrix elements, and angle brackets $\langle\cdots|\cdots\rangle$ lattice matrix elements.

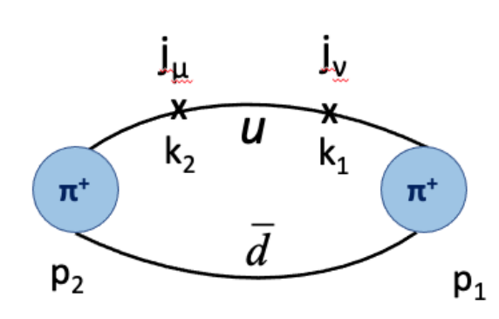

FIG. 1. Pictorial representation of the four-point function in Eq.(1) for π^+ (for proton imagine two u and one d quark lines). Time flows from right to left and the four-momentum conservation is $p_2 + k_2 = k_1 + p_1$.

On the phenomenological level, the process can be described by an effective relativistic theory to expose its physical content. The tensor can be parametrized to second order in photon momentum by the general form,

$$\begin{aligned} \sqrt{2E_12E_2}\,T_{\mu\nu} = &-\frac{T_\mu(p_1+k_1,p_1)T_\nu(p_2,p_2+k_2)}{(p_1+k_1)^2-m^2} - \frac{T_\mu(p_2,p_2-k_1)T_\nu(p_1-k_2,p_1)}{(p_1-k_2)^2-m^2}\\ &+ 2g_{\mu\nu} + A(k_1^2g_{\mu\nu} - k_{1\mu}k_{1\nu} + k_2^2g_{\mu\nu} - k_{2\mu}k_{2\nu}) + B(k_1\cdot k_2g_{\mu\nu} - k_{2\mu}k_{1\nu})\\ &+ C(k_1\cdot k_2Q_\mu Q_\nu + Q\cdot k_1Q\cdot k_2g_{\mu\nu} - Q\cdot k_2Q_\mu k_{1\nu} - Q\cdot k_1Q_\nu k_{2\mu}), \end{aligned} \tag{4}$$

where $Q = p_1 + p_2$ and A, B, C are constants to be characterized. We use a non-covariant normalization

$$\sum_n\int\frac{d^3p}{(2\pi)^3}|n(p))(n(p)| = 1, \tag{5}$$

which is why the square root factor is in front of $T_{\mu\nu}$. The pion electromagnetic vertex with momentum transfer $q = p' - p$ is written as

$$T_\mu(p',p) = (p'_\mu + p_\mu)F_\pi(q^2) + q_\mu\frac{p'^2 - p^2}{q^2}(1 - F_\pi(q^2)). \tag{6}$$

It satisfies $q_\mu T_\mu(p',p) = p'^2 - p^2$ for off-shell pions, which is needed for the Ward-Takahashi identity. The pion form factor to 4th order in momentum is given by

$$F_\pi(q^2) = 1 + \frac{\langle r^2\rangle}{6}q^2 + \frac{\langle r^4\rangle}{120}q^4, \tag{7}$$

where $\langle r^2\rangle$ is the squared charge radius and $q^2 = -\vec{q}^2 < 0$ is spacelike momentum transfer squared. The form in Eq.(4) can be entirely motivated by general principles of Lorentz invariance, gauge invariance, current conservation, time-reversal symmetry, and crossing symmetry [3]. In fact, current conservation ($k_1^\mu T_{\mu\nu} = k_2^\nu T_{\mu\nu} = 0$) immediately leads to A being related to charge radius by $A = \langle r^2\rangle/3$. The first three terms in Eq.(4) are the Born contributions to scattering from the pion and the remaining three are contact terms. The electric polarizability, α_E, and magnetic polarizability, β_M, terms come from B and C,

$$\alpha_E \equiv -\alpha\left(\frac{B}{2m} + 2mC\right), \beta_M \equiv \alpha\frac{B}{2m}. \tag{8}$$

For electric polarizability, we work with the $\mu = \nu = 0$ component of Eq.(4). Under the special kinematics in Eq.(3), it can be written to order $\vec{k}^2$ in the form,

$$\begin{aligned} T_{00}(\vec{k}) = &\frac{4m_\pi}{\vec{k}^2} + \left(\frac{1}{m_\pi} - \frac{4}{3}m_\pi\langle r^2\rangle\right) + \left[-\frac{\langle r^2\rangle}{3m_\pi} + \frac{1}{9}m_\pi\langle r^2\rangle^2 + \frac{1}{15}m_\pi\langle r^4\rangle + \frac{\alpha_E^\pi}{\alpha}\right]\vec{k}^2\\ \equiv &T_{00}^{Born}(\vec{k}) + \frac{\alpha_E^\pi}{\alpha}\vec{k}^2, \end{aligned} \tag{9}$$

where we separate the Born contribution from the contact term.

The next step is to relate the polarizabilities to lattice matrix elements. To this end, we need to convert from continuum to a lattice of isotropic spacing a with $N_s = N_x \times N_y \times N_z$ number of spatial sites by the following correspondence,

$$|n(p)) \to V^{1/2}|n(p)\rangle,\ j_\mu(x) \to \frac{Z_V}{a^3}j_\mu^L(x),\ \int d^4x \to a^4\int_{-\infty}^{\infty}dt\sum_{\vec{x}}, \tag{10}$$

where $V = N_sa^3$ and the superscript L denotes they are lattice version of the continuum entities. We are still in Minkowski spacetime. We keep the time continuous but dimensionless for convenience in the following discussion. The renormalization factor Z_V for the lattice current $j_\mu^L = (\rho^L,\vec{j}^L)$ can be taken to be unity if conserved currents are used on the lattice. Eq.(1) becomes,

$$T_{\mu\nu} = i\ N_sa\int dt\sum_{\vec{x}}e^{ik_2\cdot x}\langle\pi(p_2)|Tj_\mu^L(x)j_\nu^L(0)|\pi(p_1)\rangle. \tag{11}$$

On the lattice, there is a contribution to this function when $p_1 = p_2$, called a vacuum expectation value (or VEV), that must be subtracted out. The reason is we are interested in differences relative to the vacuum, not the vacuum itself. Formally, this is enforced by requiring normal ordering instead of time ordering in Eq.(11),

$$:j_\mu^L(x)j_\nu^L(0): = Tj_\mu^L(x)j_\nu^L(0) - \langle 0|Tj_\mu^L(x)j_\nu^L(0)|0\rangle. \tag{12}$$

For electric polarizability, the relevant component is T_{00} which amounts to the overlap of charge densities. By inserting a complete set of intermediate states, making use of translation invariance of the lattice current, and integrating over time, we arrive at the subtracted correlator [2]

$$\begin{aligned} T_{00} = &2N_s^2\sum_n\frac{|\langle\pi(\vec{0})|\rho^L(0)|n(\vec{q})\rangle|^2}{E_n - m_\pi} - 2N_s^2\sum_n\frac{|\langle 0|\rho^L(0)|n(\vec{q})\rangle|^2}{E_n}\\ \equiv &T_{00}^{elas} + T_{00}^{inel}, \end{aligned} \tag{13}$$

where the elastic part ($n = \pi$) is separated from the inelastic part as,

$$T_{00}^{elas} \equiv 2N_s^2\frac{|\langle\pi(\vec{0})|\rho^L(0)|\pi(\vec{q})\rangle|^2}{E_\pi - m_\pi}. \tag{14}$$

The matrix element

$$\langle\pi(\vec{0})|\rho^L(0)|\pi(q)\rangle = \frac{1}{N_s}\frac{E_\pi + m_\pi}{\sqrt{2E_\pi2m_\pi}}F_\pi(q^2), \tag{15}$$

is related to the pion form factor F_π given in Eq.(7). It turns out the Born term T_{00}^{Born} in the continuum cancels exactly the elastic term T_{00}^{elas} on the lattice. So the matching produces

$$T_{00}^{inel}(\vec{q}) = \frac{\alpha_E}{\alpha}\vec{q}^2, \tag{16}$$

or a formula for charged pion electric polarizability on the lattice,

$$\alpha_E^\pi = \frac{\alpha}{\vec{q}_1^2}\left[T_{00}(\vec{q}_1) - T_{00}^{elas}(\vec{q}_1)\right], \tag{17}$$

where $\vec{q}_1$ emphasizes that the formula is valid for the smallest non-zero spatial momentum on the lattice.

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[2] In this work we use $\vec{k}$ to denote continuum momentum and $\vec{q}$ lattice momentum with the same physical unit. When we match the two forms we set $\vec{k} = \vec{q}$ and express the result in terms of $\vec{q}$.

B. Magnetic polarizability

Magnetic polarizability proceeds in a similar fashion, except we consider the spatial component T_{11} (T_{22} gives the same result). Under the same kinematics given in Eq.(3), this component from the general form in Eq.(4) reads

$$T_{11} = -\frac{1}{m_\pi} + \vec{k}^2\left(\frac{\langle r^2\rangle}{3} + \frac{\beta_M}{\alpha}\right). \tag{18}$$

On the other hand, from the lattice four-point function in Eq.(11), we have,

$$T_{11} = i\ N_sa\int_{-\infty}^{\infty}dt\sum_{\vec{x}}e^{ik_2\cdot x}\langle\pi(p_2)|Tj_1^L(x)j_1^L(0)|\pi(p_1)\rangle. \tag{19}$$

Here we examine its context in more detail. Similar steps were used in the electric case [41]. These steps, together with VEV subtraction, lead to

$$T_{11}(\vec{q}) = 2N_s^2\sum_{n,s}\frac{|\langle\pi(\vec{0})|j_1^L(0)|n(s,\vec{q})\rangle|^2}{E_n - m_\pi} - 2N_s^2\sum_{n,s}\frac{|\langle 0|j_1^L(0)|n(s,\vec{q})\rangle|^2}{E_n}. \tag{20}$$

Note that the elastic piece ($n = \pi$) in the sum vanishes under the special kinematics,

$$\langle\pi(\vec{0})|j_1^L(0)|\pi(\vec{q},s)\rangle = 0. \tag{21}$$

The reason is that the matrix element is proportional to $(\vec{0} + \vec{q})_1$ in 1-direction but momentum $\vec{q}$ is in 3-direction.

For the inelastic contributions, the types of intermediate state contributing are vector or axial vector mesons [41]. There is no need to analyze the matrix elements explicitly as done in Ref. [41] for the electric case. We only need to know that the inelastic part can be characterized up to order $\vec{q}^2$ by the form,

$$T_{11}(\vec{q}) \equiv T_{11}(\vec{0}) + \vec{q}^2K_{11}, \tag{22}$$

with $T_{11}(\vec{0})$ and K_{11} to be related to physical parameters and determined on the lattice. Note that we deliberately use the full amplitude label T_{11} instead of T_{11}^{inel} since the elastic part is zero.

Matching the full amplitude on the lattice in Eq.(18) with the continuum version in Eq.(22), we obtain two relations,

$$-\frac{1}{m_\pi} = T_{11}(0), \tag{23}$$

$$\frac{\langle r^2\rangle}{3m_\pi} + \frac{\beta_M}{\alpha} = K_{11}. \tag{24}$$

The first relation is a sum rule at zero momentum. The second leads to a formula for charged pion magnetic polarizability,

$$\beta_M^\pi = \alpha\left[-\frac{\langle r^2\rangle}{3m_\pi} + \frac{T_{11}(\vec{q}_1) - T_{11}(0)}{\vec{q}_1^2}\right], \tag{25}$$

where $\vec{q}_1$ is the lowest momentum on the lattice. Compared to charged pion electric polarizability α_E^π in Eq.(17), we see that instead of subtracting the elastic contribution, we subtract the zero-momentum inelastic contribution in the magnetic polarizability. In other words, there is no zero-momentum contribution in α_E^π, and no elastic contribution in β_M^π.

III. PROTON

A. Electric polarizability

We start with a general proton Compton tensor parameterized to second order in photon momentum,

$$\begin{aligned} \sqrt{2E_12E_2}\,T_{\mu\nu} = &T_{\mu\nu}^{Born} + B(k_1\cdot k_2g_{\mu\nu} - k_{2\mu}k_{1\nu})\\ &+ C(k_1\cdot k_2Q_\mu Q_\nu + Q\cdot k_1Q\cdot k_2g_{\mu\nu}\\ &- Q\cdot k_2Q_\mu k_{1\nu} - Q\cdot k_1Q_\nu k_{2\mu}), \end{aligned} \tag{26}$$

where $Q = p_1 + p_2$. For Born term we take from Ref. [42],

$$T_{\mu\nu}^{Born} = \frac{B_{\mu\nu}(p_2,k_2,s_2|p_1,k_1,s_1)}{m_p^2 - s} + \frac{B_{\nu\mu}(p_2,-k_1,s_2|p_1,-k_2,s_1)}{m_p^2 - u}, \tag{27}$$

where the function is (note a factor of 1/2 difference between our definition and Ref. [42]),

$$B_{\mu\nu}(p_2,k_2,s_2|p_1,k_1,s_1) = \bar{u}(p_2,s_2)\Gamma_\mu(-k_2)(\not{P} + m_p)\Gamma_\nu(k_1)u(p_1,s_1). \tag{28}$$

Here $P = p_2 + k_2 = p_1 + k_1$ is the standard 4-momentum conservation for Compton scattering. There is no A term here because the proton Born terms obey current conservation, unlike the pion case in Eq.(4). The B and C are still related to polarizabilities as in Eq.(8).

The Born amplitude has virtual (or off-shell) intermediate hadronic states in the s and u channels, whereas on the lattice we have real (or on-shell) intermediate states. This will produce a difference with the elastic contribution to be discussed later. The vertex function is defined by

$$\Gamma_\mu(k) = \gamma_\mu F_1 + \frac{iF_2}{2m_p}\sigma_{\mu\lambda}k^\lambda, \tag{29}$$

where summation over λ is implied. Specializing to our kinematics in Eq.(3), we have

$$\begin{aligned} &s = (p_1 + k_1)^2 = m_p^2 - \vec{k}^2,\\ &u = (p_1 - k_2)^2 = m_p^2 - \vec{k}^2, \end{aligned} \tag{30}$$

We consider the unpolarized Born expression given by

$$\begin{aligned} 2m_pT_{\mu\nu}^{Born} = &\frac{1}{2}\sum_{s_1,s_2}\frac{1}{\vec{k}^2}\left[\bar{u}(\vec{0},s_2)\left(\gamma_\mu F_1 - \frac{iF_2}{2m_p}\sigma_{\mu\lambda}k_2^\lambda\right)(\not{p}_1 + \not{k}_1 + m_p)\left(\gamma_\nu F_1 + \frac{iF_2}{2m_p}\sigma_{\mu\lambda}k_1^\lambda\right)u(\vec{0},s_1)\right.\\ &\left.+\bar{u}(\vec{0},s_2)\left(\gamma_\nu F_1 + \frac{iF_2}{2m_p}\sigma_{\nu\lambda}k_1^\lambda\right)(\not{p}_1 - \not{k}_2 + m_p)\left(\gamma_\nu F_1 - \frac{iF_2}{2m_p}\sigma_{\nu\lambda}k_2^\lambda\right)u(\vec{0},s_1)\right]. \end{aligned} \tag{31}$$

The Dirac form factors then take the forms,

$$F_1 = \frac{G_E + \tau G_M}{1 + \tau} = 1 + \left(\frac{\kappa}{4m_p^2} - \frac{\langle r_E^2\rangle}{6}\right)\vec{k}^2 + \cdots, \tag{32}$$

$$F_2 = \frac{G_M - G_E}{1 + \tau} = \kappa + \frac{1}{12}\left(-\frac{3\kappa}{m_p^2} + 2\langle r_E^2\rangle - 2(1+\kappa)\langle r_M^2\rangle\right)\vec{k}^2 + \cdots, \tag{33}$$

where $\tau = \vec{k}^2/(4m_p^2) - \vec{k}^4/(16m_p^4)$. The final result is

$$T_{00}^{Born}(\vec{k}) = \frac{4m_p}{\vec{k}^2} - \frac{4}{3}\langle r_E^2\rangle m_p + \left[-\frac{(2+\kappa)\kappa}{4m_p^3} + \frac{m_p}{45}\left(5\langle r_E^2\rangle^2 + 3\langle r_E^4\rangle\right)\right]\vec{k}^2 + \cdots. \tag{34}$$

Including the contact interaction term, the full amplitude in the continuum takes the form,

$$T_{00}(\vec{k}) = T_{00}^{Born}(\vec{k}) + \vec{k}^2\frac{\alpha_E^p}{\alpha}. \tag{35}$$

On the other hand, we consider the unpolarized four-point function of the proton in lattice regularization,

$$T_{\mu\nu} = i\ N_sa\frac{1}{2}\sum_{s_1,s_2}\int_{-\infty}^{\infty}dt\sum_{\vec{x}}e^{ik_2\cdot x}\langle p_2,s_2|\left[Tj_\mu^L(x)j_\nu^L(0) - \langle 0|Tj_\mu^L(x)j_\nu^L(0)|0\rangle\right]|p_1,s_1\rangle, \tag{36}$$

where the VEV subtraction is included. After inserting a complete set of intermediate states,

$$\sum_{N,\vec{p}_N,s_N}|(E_N,\vec{p}_N),s_N\rangle\langle(E_N,\vec{p}_N),s_N| = 1, \tag{37}$$

and specializing to the zero-momentum Breit frame, we have

$$\begin{aligned} T_{\mu\nu} = &N_s^2\sum_{N,s_1,s_2,s_N}\frac{1}{E_N - m_p}\langle(m_p,\vec{0}),s_2|j_\mu^L|(E_N,\vec{q}),s_N)\rangle\langle(E_N,\vec{q}),s_N)|j_\nu^L|(m_p,\vec{0}),s_1\rangle\\ &- N_s^2\sum_{N,s_1,s_2,s_N}\frac{1}{E_N}\langle 0|j_\mu^L(0)|(E_N,\vec{q}),s_N)\rangle\langle(E_N,\vec{q}),s_N)|j_\nu^L(0)|0\rangle. \end{aligned} \tag{38}$$

Due to the vector nature of the electromagnetic current, the only intermediate states that can contribute are spin-1/2 and spin-3/2 states. We separate off the elastic part (N=proton),

$$T_{\mu\nu}^{elas} \equiv N_s^2\sum_{s_1,s_2,s_p}\frac{1}{E_p - m_p}\langle(m_p,\vec{0}),s_2|j_\mu^L|(E_p,\vec{q}),s_p)\rangle\langle(E_p,\vec{q}),s_p)|j_\nu^L|(m_p,\vec{0}),s_1\rangle. \tag{39}$$

The remaining inelastic part will be related to polarizabilities. The connection between the lattice and continuum matrix elements is

$$\langle p',s'|j_\mu^L(0)|p,s\rangle = \frac{1}{N_s}\frac{(p',s'|j_\mu(0)|p,s)}{\sqrt{2E_p2E_{p'}}}. \tag{40}$$

Using the continuum definition of form factors ($q = p' - p$),

$$(p',s'|j_\mu|p,s) = \bar{u}(p',s')\left(\gamma_\mu F_1 + \frac{iF_2}{2m_p}\sigma_{\mu\lambda}q^\lambda\right)u(p,s), \tag{41}$$

the elastic part can be written as

$$T_{\mu\nu}^{elas} = \sum_{s_1,s_2}\frac{1}{4m_pE_p(E_p - m_p)}\bar{u}(\vec{0},s_2)\left(\gamma_\mu F_1 - \frac{iF_2}{2m_p}\sigma_{\mu\lambda}q^\lambda\right)(\not{q} + m_p)\left(\gamma_\nu F_1 + \frac{iF_2}{2m_p}\sigma_{\nu\lambda}q^\lambda\right)u(\vec{0},s_2). \tag{42}$$

For electric polarizability, we are interested in the $\mu = \nu = 0$ component of Eq.(42),

$$T_{00}^{elas} = \sum_{s_1,s_2}\frac{1}{4m_pE_p(E_p - m_p)}\bar{u}(\vec{0},s_2)\left(\gamma_0F_1 - \frac{iF_2}{2m_p}\sigma_{03}q\right)(\gamma_0E_p + \gamma_3q + m_p)\left(\gamma_0F_1 + \frac{iF_2}{2m_p}\sigma_{03}q\right)u(\vec{0},s_2), \tag{43}$$

where q refers to the spatial momentum in the z-direction $\vec{q} = q\hat{z}$. It evaluates to order $\vec{q}^2$ as,

$$T_{00}^{elas}(\vec{q}) = \frac{4m_p}{\vec{q}^2} - \frac{4}{3}\langle r_E^2\rangle m_p + \left[\frac{1}{4m_p^3} + \frac{1}{45}\left(5\langle r_E^2\rangle^2 + 3\langle r_E^4\rangle\right)\right]\vec{q}^2 + \cdots.$$

Matching the lattice and continuum forms and subtracting off the elastic contribution, we have

$$T_{00}(\vec{q}) - T_{00}^{elas}(\vec{q}) = T_{00}^{Born}(\vec{q}) - T_{00}^{elas}(\vec{q}) + \vec{q}^2\frac{\alpha_E^p}{\alpha}. \tag{44}$$

Many terms cancel between T_{00}^{Born} and T_{00}^{elas}, leaving the difference,

$$T_{00}(\vec{q}) - T_{00}^{elas}(\vec{q}) = -\frac{(1+\kappa)^2}{4m_p^3}\vec{q}^2 + \frac{\alpha_E^p}{\alpha}\vec{q}^2, \tag{45}$$

from which we arrive at a final formula for proton electric polarizability,

$$\alpha_E^p = \alpha\left[\frac{(1+\kappa)^2}{4m_p^3} + \frac{T_{00}(\vec{q}_1) - T_{00}^{elas}(\vec{q}_1)}{\vec{q}_1^2}\right]. \tag{46}$$

Here we emphasize that the expression must be evaluated using the smallest non-zero momentum $\vec{q}_1$ on the lattice. Compared to charged pion electric polarizability α_E^π in Eq.(17), proton α_E^p has an extra term that has its magnetic moment and mass. In this sense, proton's electric and magnetic properties are coupled. Both m_p and κ have to be measured at the same time as T_{00} in order to extract α_E^p.

B. Magnetic polarizability

For the Compton amplitude in the continuum, we start with the $\mu = \nu = 1$ component of Eq.(31) (22 component gives the same result),

$$\begin{aligned} 2m_pT_{11}^{Born} = &\frac{1}{2k^2}\sum_{s_1,s_2}\left[\bar{u}(\vec{0},s_2)\left(\gamma_1F_1 - \frac{iF_2}{2m_p}\sigma_{13}k\right)\right.\\ &(\gamma_0m_p + \gamma_3k + m_p)\left(\gamma_1F_1 + \frac{iF_2}{2m_p}\sigma_{13}k\right)u(\vec{0},s_1)\\ &+\bar{u}(\vec{0},s_2)\left(\gamma_1F_1 + \frac{iF_2}{2m_p}\sigma_{13}k\right)\\ &\left.(\gamma_0m_p - \gamma_3k + m_p)\left(\gamma_1F_1 - \frac{iF_2}{2m_p}\sigma_{13}k\right)u(\vec{0},s_1)\right]. \end{aligned} \tag{47}$$

It evaluates to

$$T_{11}^{Born}(\vec{k}) = \frac{\kappa(2+\kappa)}{m_p} + \vec{k}^2\left(-\frac{\kappa}{2m_p^3} + \frac{\langle r_E^2\rangle}{3m_p} - \frac{\langle r_M^2\rangle}{3m_p}(1+\kappa)^2\right). \tag{48}$$

Including the contact interaction term, the full amplitude in the continuum becomes

$$T_{11}(\vec{k}) = T_{11}^{Born}(\vec{k}) + \vec{k}^2\frac{\beta_E^p}{\alpha}. \tag{49}$$

On the lattice, we start with the $\mu = \nu = 1$ component of Eq.(42),

$$T_{11}^{elas} = \sum_{s_1,s_2}\frac{1}{4m_pE_p(E_p - m_p)}\bar{u}(\vec{0},s_2)\left(\gamma_1F_1 - \frac{iF_2}{2m_p}\sigma_{13}q\right)(\gamma_0E_p + \gamma_3q + m_p)\left(\gamma_1F_1 + \frac{iF_2}{2m_p}\sigma_{13}q\right)u(\vec{0},s_2). \tag{50}$$

It evaluates to

$$T_{11}^{elas}(\vec{q}) = \frac{(1+\kappa)^2}{m_p} + \vec{q}^2\left(-\frac{(1+\kappa)}{2m_p^3} - \frac{\langle r_M^2\rangle}{3m_p}(1+\kappa)^2\right). \tag{51}$$

We see that unlike charged pion, there is an elastic contribution for the proton magnetic case.

The inelastic 11 component in Eq.(38) can be formally characterized as a constant plus a linear term in $\vec{q}^2$,

$$T_{11}^{inel}(\vec{q}) \equiv T_{11}^{inel}(\vec{0}) + \vec{q}^2K_{11}, \tag{52}$$

with $T_{11}^{inel}(\vec{0})$ and K_{11} to be matched with physical parameters. The difference between the Born term in the continuum and the elastic term on the lattice is ($\vec{k} \to \vec{q}$ in Born)

$$T_{11}^{Born} - T_{11}^{elas} = -\frac{1}{m_p} + \vec{q}^2\left(\frac{1}{2m_p^3} + \frac{\langle r_E^2\rangle}{3m_p}\right), \tag{53}$$

where the κ terms in the zero-momentum part cancel, as well as the magnetic charge radius terms in the $\vec{q}^2$ part.

By matching the full T_{11} in the continuum and on the lattice, we have,

$$T_{11}^{elas}(\vec{q}) + T_{11}^{inel}(\vec{0}) + \vec{q}^2K_{11} = T_{11}^{Born}(\vec{q}) + \vec{q}^2\frac{\beta_M^p}{\alpha}. \tag{54}$$

Using Eq.(53), we obtain two relations,

$$T_{11}^{inel}(0) = -\frac{1}{m_p}, \tag{55}$$

$$K_{11} = \frac{1}{2m_p^3} + \frac{\langle r_E^2\rangle}{3m_p} + \frac{\beta_M^p}{\alpha}. \tag{56}$$

We see the same sum rule in the first relation as Eq.(23) for charged pion. The second relation produces an expression for proton magnetic polarizability on the lattice,

$$\beta_M^p = \alpha\left[-\frac{1}{2m_p^3} - \frac{\langle r_E^2\rangle}{3m_p} + \frac{T_{11}^{inel}(\vec{q}_1) - T_{11}^{inel}(\vec{0})}{\vec{q}_1^2}\right], \tag{57}$$

where we have used Eq.(52) for K_{11}.

It turns out there is no elastic part to the zero momentum amplitude $T_{11}(\vec{0})$. There is a subtlety here. If we do the analytic time integral first, then set $\vec{q} = 0$, we get $T_{11}^{elas}(\vec{0}) = (1+\kappa)^2/m_p$ from Eq.(51). However, if we first set $\vec{q} = 0$, the integrand itself vanishes, so $T_{11}^{elas}(\vec{0}) = 0$. This is the way it is done on the lattice in a numerical sense as we will see in Eq.(69). So we can drop the reference to the inelastic part $T_{11}^{inel}(\vec{0}) \to T_{11}(\vec{0})$. Using the full amplitude T_{11} defined in Eq.(38), we write the final lattice formula for proton magnetic polarizability as,

$$\beta_M^p = \alpha\left[-\frac{1}{2m_p^3} - \frac{\langle r_E^2\rangle}{3m_p} + \frac{T_{11}(\vec{q}_1) - T_{11}^{elas}(\vec{q}_1) - T_{11}(\vec{0})}{\vec{q}_1^2}\right]. \tag{58}$$

Compared to charged pion magnetic polarizability β_M^π in Eq.(25), proton β_M^p has two extra terms: a mass contribution and an elastic contribution. Both terms, along with the r_E term, must be measured at the same time as T_{11} in order to extract β_M^p.

IV. LATTICE MEASUREMENT

Having obtained polarizability formulas in Eq.(17) and Eq.(25) for charged pion, and Eq.(46) and Eq.(58) for proton, we now discuss how to measure them in lattice QCD. First, we need to match the kinematics used in deriving the expressions, *i.e.*, with hadrons at rest and photons having spacelike momentum in the z-direction [3],

$$\begin{aligned} &p_1 = (m_h,\vec{0}),\\ &q_1 = (0,q\hat{z}),\ q_2 = (0,-q\hat{z}),\ q \ll m_h,\\ &p_2 = q_2 + q_1 + p_1 = (m_h,\vec{0}), \end{aligned} \tag{59}$$

as illustrated in Fig 2. It is the same kinematics as in Eq. (3) but expressed differently to match what is being done on the lattice. One may think of Fig 2 as having 'internal' photons whereas Fig 1 as having 'external' photons.

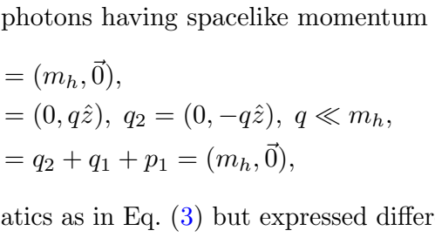

FIG. 2. Zero-momentum Breit frame in Eq.(59) used in extracting charged pion polarizabilities from four-point functions on the lattice (for proton imagine three quark lines). Time flows from right to left and the four momentum conservation is expressed as $p_2 = q_2 + q_1 + p_1$.

We construct the Euclidean four-point current-current correlation function,

$$P_{\mu\nu}(\vec{x}_2,\vec{x}_1,t_2,t_1) \equiv \frac{\sum_{\vec{x}_3,\vec{x}_0}\langle 0|\psi^\dagger(x_3):j_\mu^L(x_2)j_\nu^L(x_1):\psi(x_0)|0\rangle}{\sum_{\vec{x}_3,\vec{x}_0}\langle 0|\psi^\dagger(x_3)\psi(x_0)|0\rangle}, \tag{60}$$

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[3] For general discussion, we use h to represent either charged pion or proton.

where the two-point function is for normalization, ψ the interpolating field of the hadron, and normal ordering is used to include the VEV contribution. In the case of proton, sum over final spin and average over initial spin are assumed for unpolarized measurement. The spatial sums over $\vec{x}_3$ and $\vec{x}_0$ project to zero momentum at the sources which are located at fixed times t_3 and t_0. Time flows from right to left $t_3 > t_{1,2} > t_0$ and $t_{1,2}$ indicates the two possibilities of time ordering. When the times are well separated (defined by the time limits $t_3 \gg t_{1,2} \gg t_0$) the correlator is dominated by the ground state,

$$\begin{aligned} &P_{\mu\nu}(\vec{x}_2,\vec{x}_1,t_2,t_1) \to \langle h(\vec{0})|:j_\mu^L(x_2)j_\nu^L(x_1):|h(\vec{0})\rangle\\ &= \langle h(\vec{0})|Tj_\mu^L(r)j_\nu^L(0)|h(\vec{0})\rangle - \langle 0|Tj_\mu^L(r)j_\nu^L(0)|0\rangle, \end{aligned} \tag{61}$$

where translation invariance has been used to shift the bilinear to $r = x_2 - x_1$ and 0. To implement the special kinematics in Fig. 2 we consider the Fourier transform

$$Q_{\mu\nu}(\vec{q},t_2,t_1) \equiv N_s\sum_{\vec{r}}e^{-i\vec{q}\cdot\vec{r}}P_{\mu\nu}(\vec{x}_2,\vec{x}_1,t_2,t_1), \tag{62}$$

where $\vec{q}$ is lattice momentum and $\vec{r} = \vec{x}_2 - \vec{x}_1$ is the relative distance between the current insertions. The need for Fourier transform is natural in the sense that the polarizability formulas are derived in momentum space. In this work we only consider the diagonal components ($\mu = \nu$) of $Q_{\mu\nu}(\vec{q},t_2,t_1)$. Assuming the time separation $t = t_2 - t_1 > 0$ and inserting a complete set of intermediate states, the expression in the same time limits develops the time dependence,

$$\begin{aligned} Q_{\mu\mu}(\vec{q},t) = &N_s^2\sum_n|\langle h(\vec{0})|j_\mu^L(0)|n(\vec{q})\rangle|^2e^{-a(E_n - m_h)t}\\ &- N_s^2\sum_n|\langle 0|j_\mu^L(0)|n(\vec{q})\rangle|^2e^{-aE_nt}. \end{aligned} \tag{63}$$

The elastic contribution ($n = h$) in the expression can be separately defined,

$$Q_{\mu\mu}^{elas}(\vec{q},t) \equiv N_s^2|\langle h(\vec{0})|j_\mu^L(0)|h(\vec{q})\rangle|^2e^{-a(E_h - m_h)t}. \tag{64}$$

We see that the elastic piece in the four-point function has information on the form factors of the hadron through the amplitude and can be isolated at large time separations of the currents.

Charged pion electric polarizability in Eq.(17) is measured on the lattice by

$$\alpha_E^\pi = \frac{2\alpha a}{\vec{q}_1^2}\int_0^\infty dt\left[Q_{00}(\vec{q}_1,t) - Q_{00}^{elas}(\vec{q}_1,t)\right]. \tag{65}$$

Charged pion magnetic polarizability in Eq.(25) is measured on the lattice by

$$\beta_M^\pi = \alpha\left\{-\frac{\langle r_E^2\rangle}{3m_\pi} + \frac{2a}{\vec{q}_1^2}\int_0^\infty dt\left[Q_{11}(\vec{q}_1,t) - Q_{11}(\vec{0},t)\right]\right\}, \tag{66}$$

where $Q_{11}(\vec{q}_1,t)$ is the 11 component of Eq.(63). Unlike the electric case where the elastic contribution is subtracted in the time integral, the magnetic case has the zero-momentum inelastic contribution subtracted. The expression contains the electric charge radius r_E contribution which has to be added to the time integral. This makes the extraction of β_M^π more complicated than α_E^π. Fortunately, four-point function $Q_{00}(\vec{q}_1,t)$ already contains information on the form factor in its elastic limit [44, 45].

We now turn to the proton. The electric polarizability in Eq.(46) can be measured on the lattice by

$$\alpha_E^p = \alpha\left\{\frac{(1+\kappa)^2}{4m_p^2} + \frac{2a}{\vec{q}_1^2}\int_0^\infty dt\left[Q_{00}(\vec{q}_1,t) - Q_{00}^{elas}(\vec{q}_1,t)\right]\right\}, \tag{67}$$

and the magnetic polarizability in Eq.(58) by

$$\beta_M^p = \alpha\left\{-\frac{1}{2m_p^3} - \frac{\langle r_E^2\rangle}{3m_p} + \frac{2a}{\vec{q}_1^2}\int_0^\infty dt\left[Q_{11}(\vec{q}_1,t) - Q_{11}^{elas}(\vec{q}_1,t) - Q_{11}(\vec{0},t)\right]\right\}. \tag{68}$$

Most of the above-mentioned arguments for charged pion apply also to the proton. The difference is they additionally involve the proton mass (m_p) and its anomalous magnetic moment (κ). Both need to be measured along with the time integral on the same lattice. The mass can be readily obtained from the two-point function which is already used in Eq.(60) for normalization. Excellent signal is expected for the mass measurement since well-separated zero momentum sources are used. Although only the Q_{00} component is needed for the time integral for α_E^p, the elastic part of Q_{11} component is required for the anomalous magnetic moment term in α_E^p,

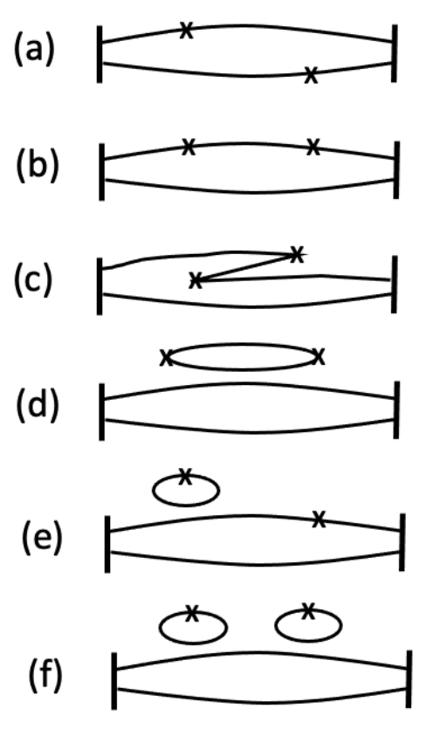

FIG. 3. Quark-line diagrams of a four-point function contributing to polarizabilities of a meson: (a) different flavor, (b) same flavor, (b) same flavor Z-graph, (d) single-flavor double-current loop, (e) single disconnected loop, (f) double disconnected loops. In each diagram, flavor permutations are assumed as well as gluon lines that connect the quark lines.

$$Q_{11}^{elas}(\vec{q}_1,t) \xrightarrow[t\gg 1]{} \frac{(1+\kappa)^2}{4m_p^2}\vec{q}_1^2e^{-a(E_p - m_p)t}. \tag{69}$$

Since Q_{11} component is needed anyway in the calculation of β_M^p, the two measurements complement each other. The same is true of the charge radius term in β_M^p which can be accessed through the unpolarized elastic part of Q_{11},

$$Q_{00}^{elas}(\vec{q}_1,t) \xrightarrow[t\gg 1]{} \left[1 - \frac{\vec{q}_1^2}{12m_p^2}(3 + 4m_p^2\langle r_E^2\rangle)\right]e^{-a(E_p - m_p)t}. \tag{70}$$

The close coupling between the electric and magnetic suggests that it is most efficient to measure the two polarizabilities together, with associated mass, charge radius, and magnetic moment in the same simulation. In practice, this should be done on a configuration by configuration basis to maintain correlations.

Wick contractions of quark-antiquark pairs in the unsubtracted part lead to topologically distinct quark-line diagrams shown in Fig. 3. Diagrams (a), (b), and (c) are connected. Diagram (d) has a loop that is disconnected from the hadron, but connected between the two currents. Diagrams (e) has one disconnected loop (also known as all-to-all propagator) and diagram (f) has two such loops. Furthermore, diagrams (d), (e) and (f) must have associated VEV subtracted. However, if conserved lattice current density is used, there is no need for subtraction in diagram (e) since the VEV vanishes in the configuration average [46].

According to Eq.(2), the full hadron polarizabilities can be broken down to contributions from various quark flavor current-current correlations. Assuming isospin symmetry in u and d quarks, we have

$$\alpha_E^\pi = \frac{5}{9}\alpha_E^{uu} + \frac{4}{9}\alpha_E^{u\bar{d}},\ \beta_M^\pi = \frac{5}{9}\beta_M^{uu} + \frac{4}{9}\beta_M^{u\bar{d}}, \tag{71}$$

for charged pion, and

$$\alpha_E^p = \frac{4}{9}\alpha_E^{uu} + \frac{1}{9}\alpha_E^{dd} - \frac{4}{9}\alpha_E^{ud}, \beta_M^p = \frac{4}{9}\beta_M^{uu} + \frac{1}{9}\beta_M^{dd} - \frac{4}{9}\beta_M^{ud}, \tag{72}$$

for charged pion, and

$$\alpha_E^p = \frac{4}{9}\alpha_E^{uu} + \frac{1}{9}\alpha_E^{dd} - \frac{4}{9}\alpha_E^{ud}, \beta_M^p = \frac{4}{9}\beta_M^{uu} + \frac{1}{9}\beta_M^{dd} - \frac{4}{9}\beta_M^{ud}, \tag{72}$$

for proton. Specifically, the quark flavor labels uu, dd, ud, and $u\bar{d}$ refer to contributions in Eq.(63) and Eq.(64) without the charge factors which have been pulled out in Eq.(71) and Eq.(72).

V. CONCLUSIONS AND ACKNOWLEDGEMENTS

Computing polarizability of charged hadrons has been a challenge for lattice QCD due to the acceleration and Landau levels in the background field method. In this work we lay out a program for the use of four-point correlation functions as an alternative, by revitalizing an earlier study on electric polarizability of charged pions and expanding the formalism to include magnetic polarizability and the proton. The approach bears a close resemblance to Compton scattering process with a transparent physical picture and conceptual clarity.

Note that four-point function techniques are also useful for hadron structure function calculations leading to parton distribution functions. The same Compton meson and (unpolarized) baryon quark-line diagrams are evaluated, except now at high momentum transfer. The key to this evaluation is the implementation of the inverse Laplace transform [47], such as the Bayesian reconstruction method employed in Ref. [34]. Using this technique, useful comparisons on proposed continuum forms can be examined.

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