AMPLITUDES, EUCLIDEAN METRIC, LATTICE QCD: OLD IDEAS & NEW PROSPECTS

M. BRUNO

PRELIMINARIES

$$J(t) = e^{+i\hat{H}t} J(0)\, e^{-i\hat{H}t}$$ $\hat{H}$: Hamiltonian

Wick Rotation $t \to -it$

$$J(t) = e^{+\hat{H}t} J(0)\, e^{-\hat{H}t}$$

$$\langle J(t) J(0) \rangle$$

$$\sum_k \int d\Phi_k \, |\langle 0 | J(0) | k, \mathrm{OUT}, \vec{p}_1 \cdots \vec{p}_{m_k} \rangle|^2 \, e^{-E(\vec{P})t}$$

$$\vec{P} = \vec{p}_1 + \ldots + \vec{p}_{m_k}$$

LATTICE QCD

$$\langle O\rangle = Z^{-1}\int dU\, e^{-S[U]}$$

Wick rotation

$$\approx \frac{1}{N}\sum_i O[U_i]$$

generated from Monte Carlo

i) LATTICE QCD is non-perturbative
ii) statistical errors from MC $\quad \sigma_O \propto O\left(1/\sqrt{N}\right)$
iii) systematic errors
→ finite lattice spacing = UV cutoff $\quad [a^{-1} \sim O(2\,\mathrm{GeV})]$
→ finite volume (HPC large ≠ ∞)
→ quark masses up/down > physical

FOCUS HERE ON ROLE EUCLIDEAN METRIC

same area → importance sampling + Wick rotation

What can we extract from Euclidean correlators?

Euclidean time

zero momentum

$$\langle \bar\psi(t,\mathbf{0})\gamma_5\psi(t,\mathbf{0})\, \bar\psi(0)\gamma_0\gamma_5\psi(0)\rangle$$

→ quantum numbers of pion at rest

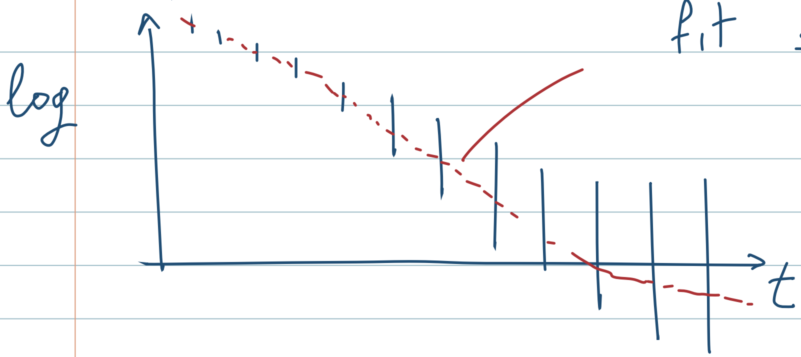
$$= \sum_n \langle 0|\bar\psi(0)\gamma_5\psi(0)|n\rangle\, e^{-E_n t}\, \langle n|\bar\psi(0)\gamma_0\gamma_5\psi(0)|0\rangle$$

$|n\rangle = |\pi\rangle$ we have pion decay constant

only low energetic states survive

Finite volume
↳ quantization of momenta

OBSERVABLES



fit sum of exponentials
→ energies
→ matrix elements

if J_μ = EM current we can get $\langle 0 | J_\mu | \pi\pi \rangle$

but can we?

Finite Box → the 2 pions cannot be asymptotic states

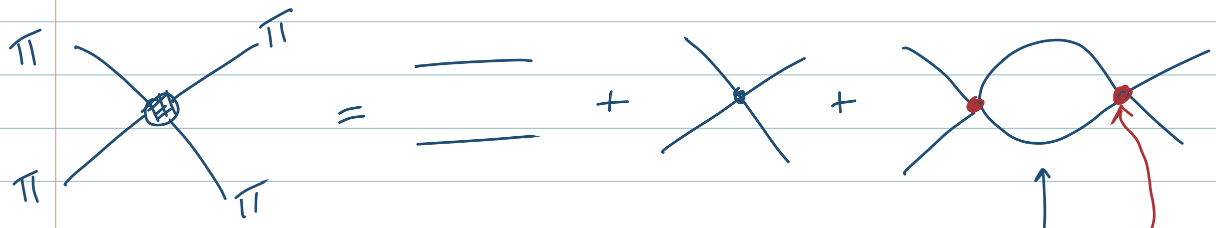
$1/L^k$ corrections to $\langle 0 | J_\mu | \pi\pi \rangle_L \to \langle 0 | J_\mu | \pi\pi \rangle_\infty$

[Lüscher, Lellouch, Meyer, Sharpe, Hansen ....]

Finite Box L → distorsion of spectrum + matrix elements

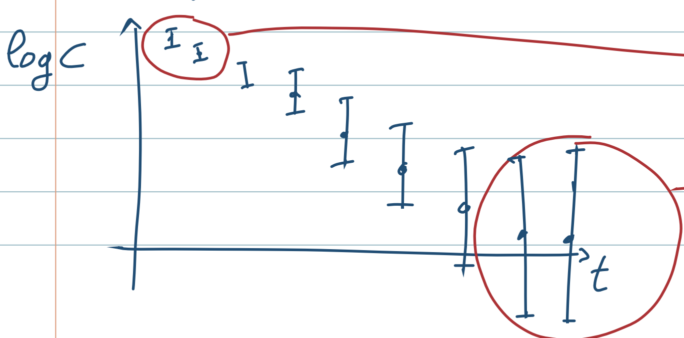
↓

reconstruct the amplitude ← measure distortion LQCD



internal pions can wrap around the torus

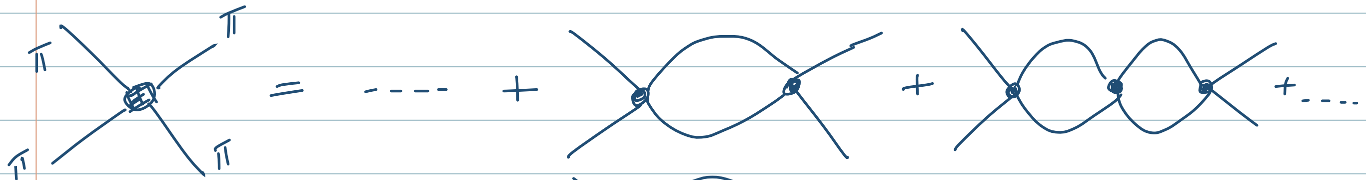
Energy of 2 pions $= 2\sqrt{m_\pi^2 + \vec{p}^{\,2}} + \delta(L)$



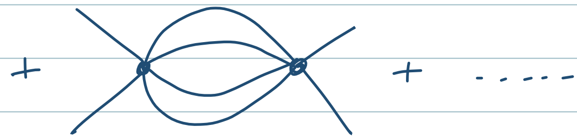
→ high energies vs cutoff effects

→ low energies vs noise from MC sampling

QCD ON A TORUS



IF $\sqrt{s} > 4m_\pi$ THIS IS POSSIBLE



i) $\sqrt{s} < 4m_\pi$ elastic region

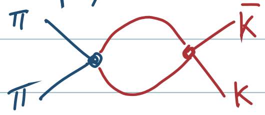
SCATT MATRIX 2→2

ENERGY LEVELS

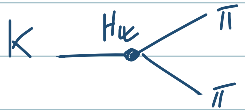
[LÜSCHER] Quantization condition $\det\left[\mathcal{M}_2^{-1}(E_n) + F(E_n, L)\right] = 0$

↳ geometric function

ii) unphysical heavy pions such that $2m_K < 4m_\pi$

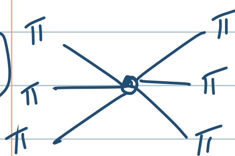


$$\det\left[\begin{pmatrix} \mathcal{M}_{\pi\pi\to\pi\pi} & \mathcal{M}_{\pi\pi\to K\bar{K}} \\ & \mathcal{M}_{K\bar{K}\to K\bar{K}} \end{pmatrix}^{-1} + \begin{pmatrix} F_{\pi\pi} & 0 \\ 0 & F_{KK} \end{pmatrix}\right] = 0$$

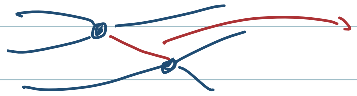
iii) 

$$\langle\pi\pi|H_W|K\rangle_\infty = F(E_{\pi\pi}, L)\langle\pi\pi|H_W|K\rangle_L$$

[LÜSCHER, LELLOUCH]

iv) 

VERY COMPLICATED, BUT SOLVED [HANSEN, SHARPE]

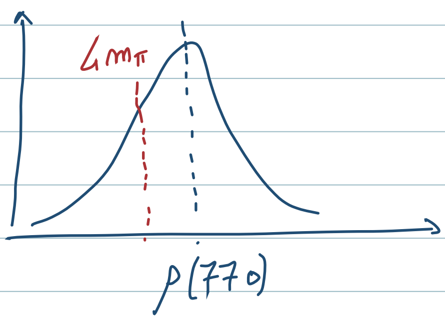


it can go on-shell
divergences to subtract

v) 4π, 6π, ... ? FORMALISM NOT KNOWN!

@ phys pion mass we cannot study ρ RESONANCE!

R-ratio



→ 4π contribution likely small due to phase space

Finite volume "mixes" different asymptotic states

→ partial waves

→ particle content

Analytic Continuation

formally guaranteed by Osterwalder-Schrader theorem
but is it possible? Let's assume ∞-volume

$$G(t) = \langle J_\mu(t)\, J_\mu(0)\rangle = \int d\omega\, e^{-\omega t}\rho(\omega)$$

spectral density: $\rho(\omega) = \delta(\omega - E)\sum_k \int d\Phi_k |\langle 0|J_\mu(0)|k,\mathrm{out},\boldsymbol{p}_1\dots\boldsymbol{p}_{n_k}\rangle|^2$

channel: $\pi\pi$, $\pi\pi\pi$, KK, ... phase space

for e.m. current $\rho(\omega) \to$ R-ratio of $e^+e^- \to \gamma^* \to$ had

$$\int_0^\infty dt\, e^{+\omega_0 t}\, G(t) = \int d\omega\, \rho(\omega) \int_0^\infty dt\, e^{(\omega_0-\omega)t} = \rho(\omega_0)$$

LQCD correlators
→ finite number of t
→ finite non-zero errors

INVERSE LAPLACE TRAFO
ILL-POSED
NUMERICAL PROBLEM

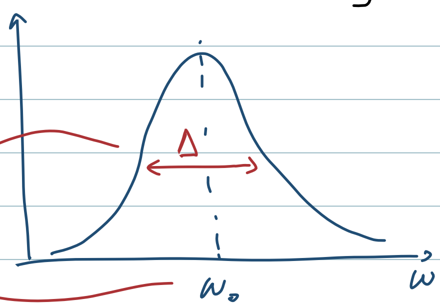
$$\sum_{t=0}^{T} e^{\omega_0 t}\, G(t) \approx \int d\omega\, \rho(\omega)\, \frac{1-e^{(\omega_0-\omega)T}}{\omega_0-\omega}$$

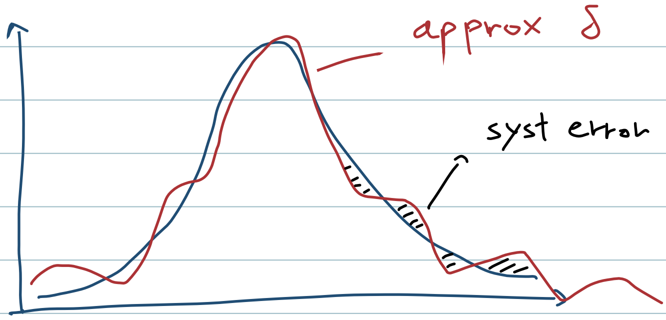
$\omega < \omega_0$ EXPLODES

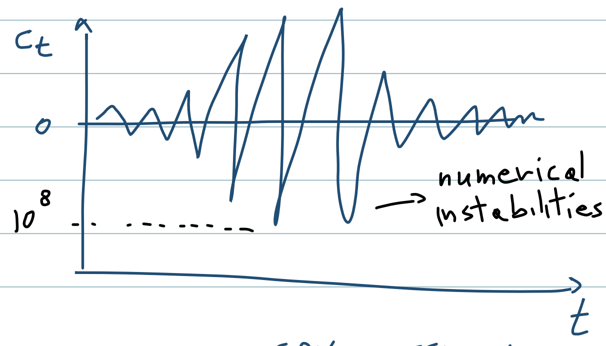
APPROX. SOLUTION [Backus-Gilbert '68]

$$\sum_t c_t e^{-\omega t} \approx \delta(\omega-\omega_0, \Delta)$$

from numerical minimization





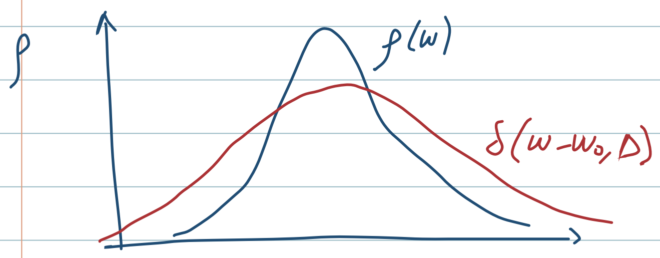


@ the end of the day: physics @ $\Delta = 0$ VERY DIFFICULT TO ACHIEVE

SMEARED SPECTR. DENSITY

$$\sum_t c_t \langle J_i(t)\, J_i(0)\rangle \propto \int d\omega\, \rho(\omega)\, \delta(\omega-\omega_0, \Delta)$$

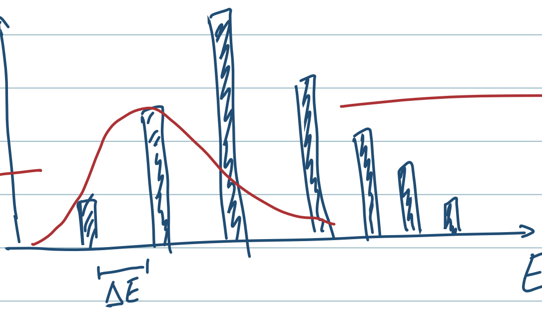
↳ coefficients determine ω_0, Δ



if $\Delta > \Gamma$ width of resonance we cannot extract it

but $\Delta \gg 0$ to regulate numerical instabilities

FINITE BOX



spectrum quantized by L

$L\langle m|J_i(0)|0\rangle$ has finite size correction $1/L^k$

$$\langle J_i(t)\, J_i(0)\rangle = \sum_m |\langle 0|J_i(0)|m\rangle|^2 e^{-E_m t} \qquad \mathcal{O}(e^{-m_\pi L})$$

if $\Delta > \Delta E$ then $\int d\omega\, \rho(\omega)\, \delta(\omega-\omega_0, \Delta)$

@ fixed volume $\Delta \approx m_\pi$ 140 MeV
for current lattices $\rho(770) \to \Gamma \sim 145$ MeV ✗

IDEALLY @ fixed Δ, $L\to\infty$, then $\Delta \to 0$

↳ IN PRACTICE VERY EXPENSIVE

NOTE. $\delta(\omega-\omega_0, \Delta) \longrightarrow \mathrm{Im}\, \dfrac{1}{\omega-\omega_0+i\Delta}$

MAIANI-TESTA RESULT

→ can we bypass analytic continuation / inverse Laplace?

$$\langle \tilde{\pi}_{\mathbf{q}_1}(t_1)\tilde{\pi}_{\mathbf{q}_2}(t_2)J(0)\rangle$$

is there a fit function in $t_1 t_2$ which returns the form factor?

insert complete set of states

↳ $\exp[-E(t_2-t_1)]$ filters $\langle \pi, \vec{q}_1|$

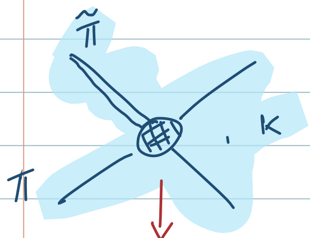
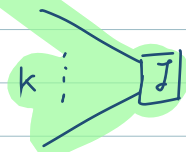
→ we consider $\vec{q}_1 = -\vec{q}_2 = -\vec{q}$

$$\langle \pi, -\mathbf{q}|\tilde{\pi}_{\mathbf{q}}(0)\, e^{-(\hat{H}-2\omega_q)t_2}|J(0)|0\rangle$$

$\hat{H}$ = Hamiltonian

$\omega_q = \sqrt{m_\pi^2 + \vec{q}^{\,2}}$

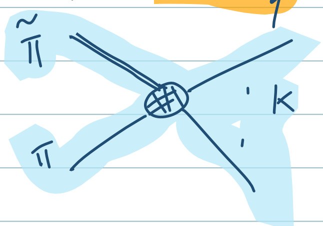
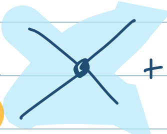
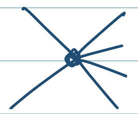
↳ insert $\sum_k \int d\Phi_k\, |k, \text{OUT}, p_1 \ldots p_n\rangle\langle k, \text{OUT}, p_1 \ldots p_n|$

$\int dE$ ⊗  ⊗ $e^{-(E-2\omega_q)t_2}$ ⊗  → physical form factor

limit $\tilde{\pi}$ → on-shell then physical $2\to k$ amplitude

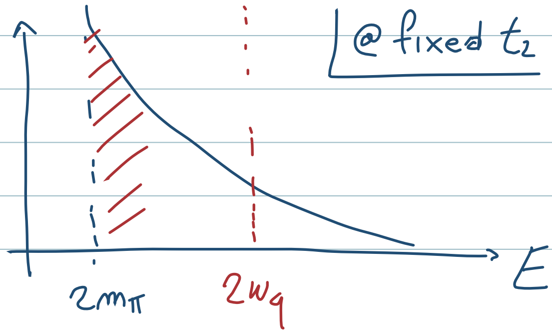
→ projection of 3-momentum $\vec{q}$, time coordinate t_2 free

↳ if $E = 2\omega_q$ then $\tilde{\pi}$ on-shell (η is virtuality)

 $=$  $+ \dfrac{\sqrt{Z_\pi}}{\eta + i\varepsilon}$  $+ \dfrac{\sqrt{Z_\pi}}{\eta + i\varepsilon}$  $+ \ldots$

FROM LSZ THEOREM

$$\int_{2m_\pi}^{\infty} dE\, \bar{\mathcal{M}}_{2\to k}(E)\, e^{-(E-2\omega_q)t_2}\, \mathcal{F}_k(E)$$



integral dominated by $E \approx 2m_\pi$ where $\bar{\mathcal{M}}$ not physical

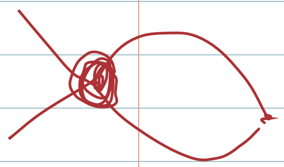
MAIANI-TESTA SOLUTION

$2\omega_q \longrightarrow 2m_\pi$, $\vec{q} = \vec{0}$ 2π @ threshold

$$\int d\Phi_{2\pi} \, \bar{\mathcal{M}}_{2\to 2} \, e^{-(E(\vec{p})-2m_\pi)t_2} \, \mathcal{F}_2$$

expansion large t_2 $\longleftrightarrow$ focus integrand @ $E \simeq 2m_\pi$

$$\langle \tilde{\pi}_0(t_1)\tilde{\pi}_0(t_2)J(0)\rangle = Z_\pi \frac{e^{-m_\pi(t_1+t_2)}}{4m_\pi^2} \mathcal{F}_2(4m_\pi^2)\left[1 - a_{\pi\pi}\sqrt{\frac{m_\pi}{\pi t_2}} + O(t_2^{-3/2})\right]$$



Form factor ↲ scattering length

Became NO-GO THEOREREM for $\vec{q} \neq \vec{0}$

→ what about $K \to \pi\pi$?

FINITE VOLUME FORMALISM , $m_K < 4m_\pi$ ✓

→ what about $D \to \pi\pi$?

attempt inverse Laplace [Hansen, Bulava]

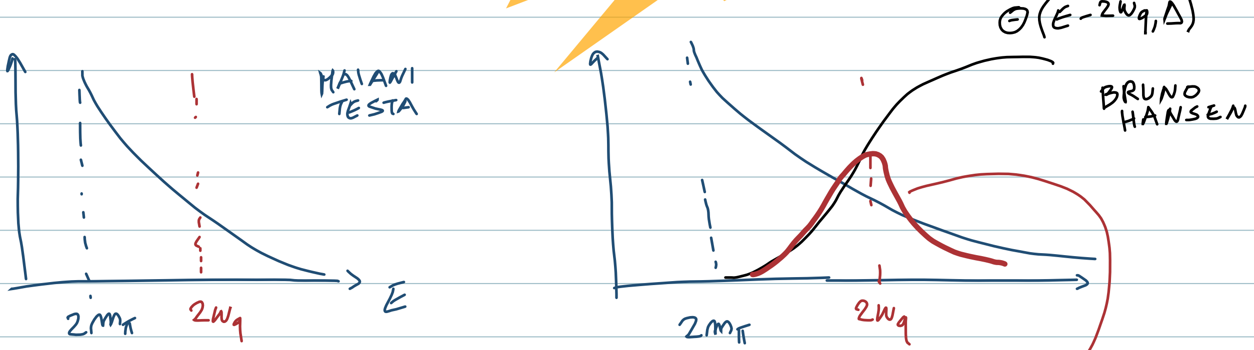
OR

NEW IDEAS!

Bruno-Hansen

$$\langle \pi, -\boldsymbol{q} | \tilde{\pi}_{\boldsymbol{q}}(t_2) \Theta(\hat{H} - 2\omega_q, \Delta) J(0) \rangle$$

smooth step function



COMBO = $e^{-(E-2\omega_q)t_2} \otimes \Theta(E-2\omega_q,\Delta) \approx \underline{\delta(E-2\omega_q,\Delta)}$

→ if we expand large t_2 focalization @ $E \sim 2\omega_q$

$\int d\mathcal{E}\, \delta(\mathcal{E}-E)$ $\quad \sum_k \int d\Phi_k\, \mathcal{M}_{2k}\, e^{-(E-2\omega_q)t_2}\, \Theta(E-2\omega_q,\Delta)\, \mathcal{F}_k$

$\int d\mathcal{E}\, \mathcal{G}(\mathcal{E})\, e^{-(\mathcal{E}-2\omega_q)t_2}\, \Theta(\mathcal{E}-2\omega_q,\Delta)$ $\quad \mathcal{G}(\mathcal{E}) = \sum_k \int d\Phi_k (2\pi)\delta(\mathcal{E}-E) M_{2k}\mathcal{F}_k$

→ expand about large t_2 → $\sum_m g_m \cdot \mathcal{J}^{(m)}(t_2, \omega_q, \Delta)$

geometric functions

$g_0 = \mathcal{G}(2\omega_q) = 2\,\mathrm{Im}\,\mathcal{F}_2(4\omega_q^2)$

$$\mathcal{N}\langle \pi, \boldsymbol{q} | \tilde{\pi}_{-\boldsymbol{q}}(t) \Theta(\hat{H} - \sqrt{s}, \Delta) J(0) \rangle =$$

$$e^{-\omega_q t}\left[\Theta(0,\Delta)\mathrm{Re}\mathcal{F}_2(\sqrt{s}) - 2\mathcal{J}^{(0)}(t,s,\Delta)\mathrm{Im}F_2(\sqrt{s}) + \cdots\right]$$

we measure this with LQCD

we fit time dependence using new basis $\mathcal{J}^{(m)}(t)$ instead of $\sum_m c_m e^{-E_m t}$

fit parameters are Re, Im parts of form factor

→ ∞-volume up to $O(e^{-\Delta L})$

→ valid @ all energies $\sqrt{s} > 4m_\pi$

What's the catch?

How can we plug the Θ inside the correlator?

A) measure several $\langle \pi\pi, m | J(0) | 0 \rangle$, $E_{\pi\pi, m}$

$$\langle \pi, \vec{q} \,|\, \tilde{\pi}_{-\vec{q}}(t)\, \Theta(\hat{H} - 2\omega_q, \Delta)\, J(0) | 0 \rangle =$$

$$\langle \pi, \vec{q} | \tilde{\pi}_{-\vec{q}}(t) J(0) | 0 \rangle - \sum_{m=0}^{N} \Theta(2\omega_q - E_{\pi\pi,m}, \Delta) \langle \pi\pi, m | J(0) \rangle\, e^{-E_{\pi\pi,m} t}$$

B) approx solution $\sum_{t'} c_{t'} e^{-Et'} \approx \Theta(E - \sqrt{s}) e^{-Et}$

less severe Inverse Problem compared to $\delta(E - \sqrt{s}, \Delta)$

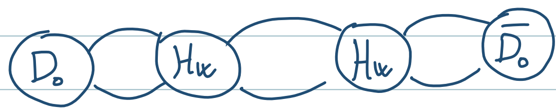
Conclusions:

Extension of Maiani-Testa $\longrightarrow$ New Formalism for amplitudes in LQCD

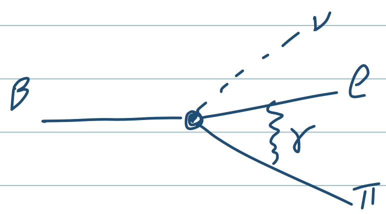
we have also studied $\pi N \to \pi N$ new determination of πN scattering lenght

we have formalism for $D \to \pi\pi$, numerical calculation still challenging though

possible future extensions to observables not doable $\longrightarrow$



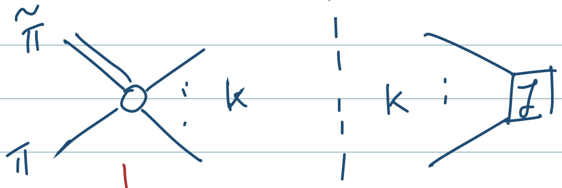
$\longrightarrow$



THANKS FOR YOUR ATTENTION!!

BACKUP

Re $F_2(\sqrt{s})$?



$$= \otimes = F_2(\sqrt{s}) \quad \text{complex}$$

$$\frac{\sqrt{Z_\pi}}{m+i\varepsilon} \otimes \xrightarrow{\text{Im}} \delta(m) = \text{Im}\, F_2$$

$$F_2(\sqrt{s}) - \text{Im}[F_2] + PV \int \frac{1}{m}$$

$$\text{Re}[F_2] + J^{(0)} \cdot \text{Im}[F_2] + J^{(1)} \ldots\ldots$$