From Scattering to the Dynamics of Black Holes and Fluids

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double copy: (1612.00868, 1612.03927, 1709.04932)
gravitational waves: (1808.02489, 1901.04424, 1908.01493, 2003.08351, 2006.06665)
fluid dynamics: (2010.15970)

action

↓

amplitudes

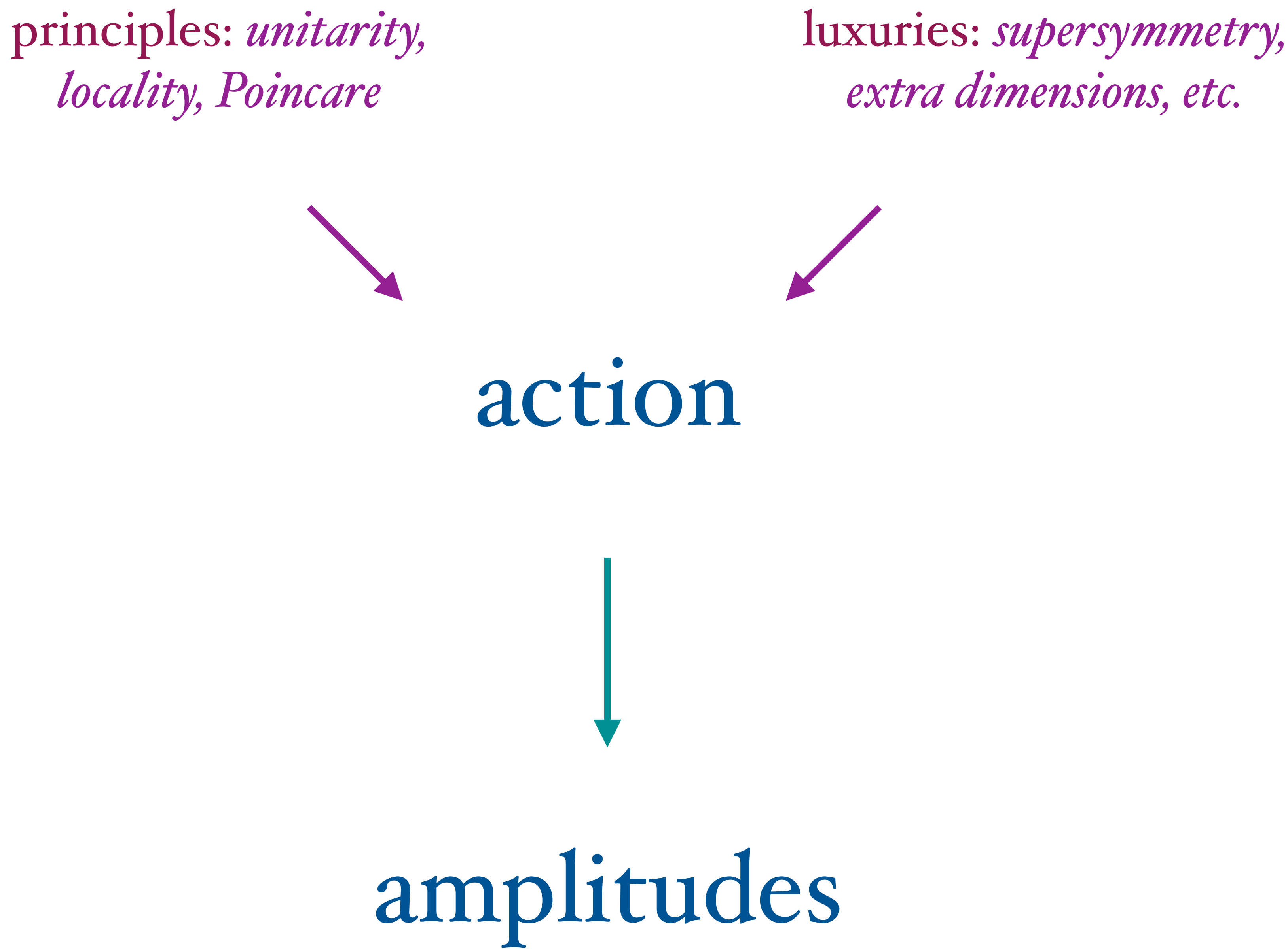
“ *the theory* ”

action

↓

amplitudes

“ *the observables* ”


principles: *unitarity, locality, Poincare*
luxuries: *supersymmetry, extra dimensions, etc.*
action
amplitudes

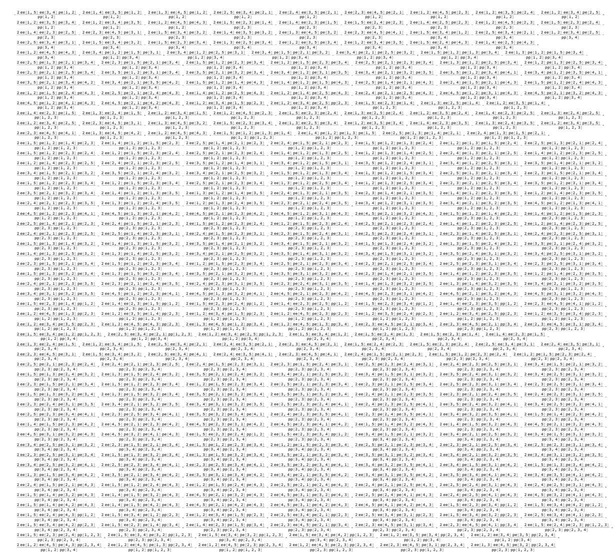
action

“ *S-matrix program* ”

amplitudes

Gauge symmetry manifests Poincare invariance and locality at the cost of redundancy.

$$A(1^{h_1}2^{h_2}3^{h_3}4^{h_4}5^{h_5}) \quad = $$



Feynman diagrams
(factorization manifest)

$$A(1^+2^+3^+4^+5^+) \quad = \quad A(1^-2^+3^+4^+5^+) \quad = 0$$

$$A(1^-2^+3^-4^+5^+) \quad = \quad \frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

$$A(1^-2^-3^+4^+5^+) \quad = \quad \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

modern tools
(factorization obscure)

Gravity suffers also, due to diffeomorphisms.

$$\frac{\delta^3 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}}\longrightarrow$$

$$\begin{aligned}&\mathrm{Sym}\big[-\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda})-\tfrac{1}{4}P_6(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\lambda})+\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda})+\tfrac{1}{2}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda})+P_3(p^\sigma p^\lambda\eta^{\mu\nu}\eta^{\tau\rho})\\&-\tfrac{1}{2}P_3(p^\tau p'^\mu\eta^{\nu\sigma}\eta^{\rho\lambda})+\tfrac{1}{2}P_3(p^\rho p'^\lambda\eta^{\mu\sigma}\eta^{\nu\tau})+\tfrac{1}{2}P_6(p^\rho p^\lambda\eta^{\mu\sigma}\eta^{\nu\tau})+P_6(p^\sigma p'^\lambda\eta^{\tau\mu}\eta^{\nu\rho})+P_3(p^\sigma p'^\mu\eta^{\tau\rho}\eta^{\lambda\nu})\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad-P_3(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu})\big],\end{aligned}\tag{2.6}$$

3pt graviton *vertex*

$$\frac{\delta^4 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}\delta\varphi_{\iota'''\kappa'''}}\longrightarrow$$

$$\begin{aligned}&\mathrm{Sym}\big[-\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})-\tfrac{1}{8}P_{12}(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\lambda}\eta^{\iota\kappa})-\tfrac{1}{4}P_6(p^\sigma p'^\mu\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})+\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})\\&+\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})+\tfrac{1}{4}P_{12}(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\iota}\eta^{\lambda\kappa})+\tfrac{1}{2}P_6(p^\sigma p'^\mu\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})-\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})\\&+\tfrac{1}{4}P_{24}(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda}\eta^{\iota\kappa})+\tfrac{1}{4}P_{24}(p^\sigma p^\tau\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\iota\kappa})+\tfrac{1}{4}P_{12}(p^\rho p'^\lambda\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\iota\kappa})+\tfrac{1}{2}P_{24}(p^\sigma p'^\rho\eta^{\tau\mu}\eta^{\nu\lambda}\eta^{\iota\kappa})\\&-\tfrac{1}{2}P_{12}(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu}\eta^{\iota\kappa})-\tfrac{1}{2}P_{12}(p^\sigma p'^\mu\eta^{\tau\rho}\eta^{\lambda\nu}\eta^{\iota\kappa})+\tfrac{1}{2}P_{12}(p^\sigma p^\rho\eta^{\tau\lambda}\eta^{\mu\nu}\eta^{\iota\kappa})-\tfrac{1}{2}P_{24}(p\cdot p'\eta^{\mu\nu}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\sigma})\\&-P_{12}(p^\sigma p^\tau\eta^{\nu\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})-P_{12}(p^\rho p'^\lambda\eta^{\nu\iota}\eta^{\kappa\sigma}\eta^{\tau\mu})-P_{24}(p_\sigma p'^\rho\eta^{\tau\iota}\eta^{\kappa\mu}\eta^{\nu\lambda})-P_{12}(p^\rho p'^\iota\eta^{\lambda\sigma}\eta^{\tau\mu}\eta^{\nu\kappa})\\&+P_6(p\cdot p'\eta^{\nu\rho}\eta^{\lambda\sigma}\eta^{\tau\iota}\eta^{\kappa\mu})-P_{12}(p^\sigma p^\rho\eta^{\mu\nu}\eta^{\tau\iota}\eta^{\kappa\lambda})-\tfrac{1}{2}P_{12}(p\cdot p'\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\sigma\iota}\eta^{\tau\kappa})-P_{12}(p^\sigma p^\rho\eta^{\tau\lambda}\eta^{\mu\iota}\eta^{\nu\kappa})\\&\qquad-P_6(p^\rho p'^\iota\eta^{\lambda\kappa}\eta^{\mu\sigma}\eta^{\nu\tau})-P_{24}(p^\sigma p'^\rho\eta^{\tau\mu}\eta^{\nu\iota}\eta^{\kappa\lambda})-P_{12}(p^\sigma p'^\mu\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\nu})+2P_6(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})\big].\end{aligned}\tag{2.7}$$

4pt graviton *vertex*

$$M(1^-2^-3^+) \quad = \quad \frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 32\rangle^2}$$

3pt graviton *amplitude*

$$M(1^-2^-3^+4^+) \quad = \quad \frac{\langle 12\rangle^4[34]^4}{stu}$$

4pt graviton *amplitude*

Redundancy is not an affliction of spin. Not even scalars are safe. Consider on-shell amplitudes in

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 g(\phi)$$

At 3pt, 4pt, 5pt, ... you will find they are all zero!

Redundancy is not an affliction of spin. Not even scalars are safe. Consider on-shell amplitudes in

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 g(\phi) \quad \longleftrightarrow \quad \mathscr{L} = \frac{1}{2}(\partial\phi)^2$$

At 3pt, 4pt, 5pt, ... you will find they are all zero!

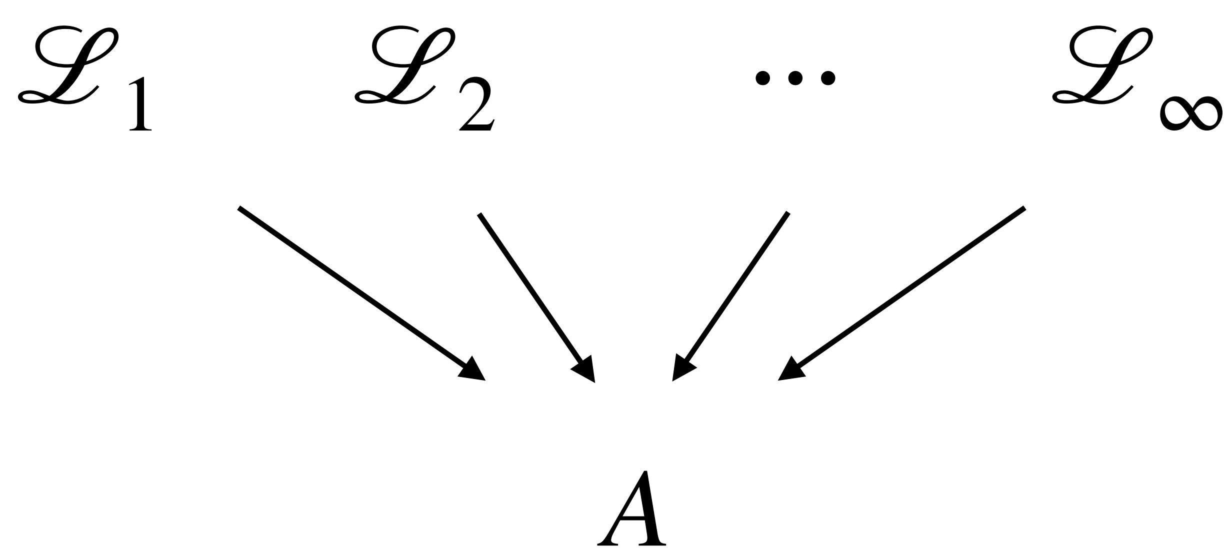
$$f(\phi) \longleftrightarrow \phi \quad \text{where} \quad f'(\phi)^2 = g(\phi)$$

Field redefinitions: a non-symmetry of the action that leaves the S-matrix invariant.

Quantum fields are integration variables of the path integral. You can always change variables.

$$Z[J] \sim \int [d\phi]\, e^{iS[\phi]+i\int J\phi}$$

Infinitely many actions describe the same physics.



Outline

I) **Hidden Structures** are lurking in old theories and the amplitudes program has found them.

II) **Black Hole Binary Dynamics** are amenable to these tools, yielding now state-of-the-art results.

III) **Fluid Dynamics** can be analyzed via modern amplitudes tools, revealing new structures.

1) Hidden Structures

New fact: color and kinematics are dual!

3pt gluon

3pt graviton

$$A(1_a^- 2_b^- 3_c^+) = \frac{\langle 12\rangle^3}{\langle 13\rangle\langle 32\rangle} f_{abc} \qquad M(1^- 2^- 3^+) = \frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 32\rangle^2}$$

$$A(1_a^+ 2_b^+ 3_c^-) = \frac{[12]^3}{[13][32]} f_{abc} \qquad M(1^+ 2^+ 3^-) = \frac{[12]^6}{[13]^2[32]^2}$$

Simply replace f_{abc} with the kinematic structure.

Bern, Carrasco, and Johansson (BCJ) generalized color-kinematics duality to all tree amplitudes.

$$A_4 = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}$$

$$c_s = f_{abe} f_{cde}$$

$$c_t = f_{bce} f_{ade}$$

$$c_u = f_{cae} f_{bde}$$

Here n_s, n_t, n_u are non-unique functions of $p_i p_j$, $p_i e_j$, $e_i e_j$ that can be reshuffled to satisfy Jacobi.

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$$c_s + c_t + c_u = 0$$

(mathematical identity)

$$n_s + n_t + n_u = 0$$

(requires reshuffling)

Perhaps color and kinematics are interchangeable since they satisfy the same algebraic identities?

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4pt gluon
(*polarization* = e_μ)

Bern, Carrasco, Johansson (0805.3993)

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4pt gluon
(*polarization* $= e_\mu$)

$\downarrow\downarrow\downarrow$ " *double copy* "

$$M_4 = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}$$

4pt graviton
(*polarization* $= e_\mu e_\nu$)

Bern, Carrasco, Johansson (0805.3993)

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$$M_4 = \frac{n_s \tilde{n}_s}{s} + \frac{n_t \tilde{n}_t}{t} + \frac{n_u \tilde{n}_u}{u}$$

4pt graviton + two-form + dilaton
(*polarization* $= e_\mu \tilde{e}_{\bar{\mu}}$)

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4pt graviton +
two-form + dilaton
(*polarization* = $e_\mu \tilde{e}_{\tilde{\mu}}$)

Double copy is proven at tree and recycled to loop via unitarity methods for collider physics, SUGRA, and LIGO.

Bern, Carrasco, Johansson (0805.3993)

Double copy is weirdly ubiquitous among "nice" theories with very few coupling constants.

$\mathcal{N} > 4$ supergravity	• $\mathcal{N} = 4$ SYM theory • SYM theory ($\mathcal{N} = 1, 2, 4$)	[1, 2, 31, 291, 292]	
$\mathcal{N} = 4$ supergravity with vector multiplets	• $\mathcal{N} = 4$ SYM theory • YM-scalar theory from dim. reduction	[1, 2, 31, 293]	• $\mathcal{N} = 2 \times \mathcal{N} = 2$ construction is also possible
pure $\mathcal{N} < 4$ supergravity	• (S)YM theory with matter • (S)YM theory with ghosts	[188]	• ghost fields in fundamental rep
Einstein gravity	• YM theory with matter • YM theory with ghosts	[188]	• ghost/matter fields in fundamental rep
$\mathcal{N} = 2$ Maxwell-Einstein supergravities (generic family)	• $\mathcal{N} = 2$ SYM theory • YM-scalar theory from dim. reduction	[120]	• truncations to $\mathcal{N} = 1, 0$ • only adjoint fields
$\mathcal{N} = 2$ Maxwell-Einstein supergravities (homogeneous theories)	• $\mathcal{N} = 2$ SYM theory with half hypermultiplet • YM-scalar theory from dim. reduction with matter fermions	[121, 294]	• fields in pseudo-real reps • include Magical Supergravities
$\mathcal{N} = 2$ supergravities with hypermultiplets	• $\mathcal{N} = 2$ SYM theory with half hypermultiplet • YM-scalar theory from dim. red. with extra matter scalars	[121, 240]	• fields in matter representations • construction known in particular cases
$\mathcal{N} = 2$ supergravities with vector/ hypermultiplets	• $\mathcal{N} = 1$ SYM theory with chiral multiplets • $\mathcal{N} = 1$ SYM theory with chiral multiplets	[239, 241, 295]	• construction known in particular cases
$\mathcal{N} = 1$ supergravities with vector multiplets	• $\mathcal{N} = 1$ SYM theory with chiral multiplets • YM-scalar theory with fermions	[188, 239, 241, 295]	• fields in matter reps • construction known in particular cases
$\mathcal{N} = 1$ supergravities with chiral multiplets	• $\mathcal{N} = 1$ SYM theory with chiral multiplets • YM-scalar with extra matter scalars	[188, 239, 241, 295]	• fields in matter reps • construction known in particular cases
Einstein gravity with matter	• YM theory with matter • YM theory with matter	[1, 188]	• construction known in particular cases

$R + \phi R^2 + R^3$ gravity	• YM theory + F^3 + F^4 + ... • YM theory + F^3 + F^4 + ...	[296]	• extension to $\mathcal{N} \leq 4$ replacing one of the factors by undeformed SYM theory
Conformal (super)gravity	• DF^2 theory • (S)YM theory	[152, 153]	• $\mathcal{N} \leq 4$ • involves specific gauge theory with dimension-six operators
3D maximal supergravity	• BLG theory • BLG theory	[119, 243, 297]	• 3D only
YME supergravities	• SYM theory • YM + ϕ^3 theory	[120, 125, 133, 134, 140, 214, 216, 257, 283, 285, 289]	• trilinear scalar couplings • $\mathcal{N} = 0, 1, 2, 4$ possible
Higgsed supergravities	• SYM theory (Coulomb branch) • YM + ϕ^3 theory with extra massive scalars	[122]	• $\mathcal{N} = 0, 1, 2, 4$ possible • massive fields in supergravity
$U(1)_R$ gauged supergravities	• SYM theory (Coulomb branch) • YM theory with SUSY broken by fermion masses	[123]	• $0 \leq \mathcal{N} \leq 8$ possible • SUSY is spontaneously broken • only theories with Minkowski vacua
gauged supergravities (nonabelian)	• SYM theory (Coulomb branch) • YM + ϕ^3 theory with massive fermions	[284]	• SUSY is spontaneously broken • only theories with Minkowski vacua

DBI theory	• NLSM • (S)YM theory	[125, 126, 285, 298–301]	• $\mathcal{N} \leq 4$ possible • also obtained as $\alpha' \to 0$ limit of abelian Z-theory
Volkov-Akulov theory	• NLSM • SYM theory (external fermions)	[125, 302–308]	• restriction to external fermions from supersymmetric DBI
Special Galileon theory	• NLSM • NLSM	[125, 285, 301, 306, 309]	• theory is also characterized by its soft limits
DBI + (S)YM theory	• NLSM + ϕ^3 • (S)YM theory	[125, 126, 156, 285, 298–300, 306, 310]	• $\mathcal{N} \leq 4$ possible • also obtained as $\alpha' \to 0$ limit of semi-abelianized Z-theory
DBI + NLSM theory	• NLSM • YM + ϕ^3 theory	[125, 126, 156, 285, 298–300]	

Bern, Carrasco, Chiodaroli, Johansson, Roiban (1909.01358)

Is the double copy just a vestige of open/closed string duality or Kaluza-Klein or vierbiens?

- gluon $\otimes$ gluon = graviton
- pion $\otimes$ pion = special Galileon
- gluon $\otimes$ pion = Born-Infeld photon

In my view this is a QFT property that deserves an explanation with QFT.

Maybe these are all just games with amplitudes...

... or maybe amplitudes are revealing bona fide albeit hidden dualities and connections in QFT.

An analogy: imagine if the world was ϕ^4 theory, but we only had the field redefined action,

$$\mathscr{L} = \frac{1}{2}(\partial f)^2 - \frac{\lambda}{4!} f^4 \quad \text{where} \quad f = \begin{array}{l} Z_2\text{-violating} \\ \text{function of } \phi \end{array}$$

Physical amplitudes exhibit the Z_2. By reshuffling diagrams we can manifest it term by term.

E.g. the graviton + two-form + dilaton amplitude has separately contracted μ and $\tilde{\mu}$ indices.

$$M_4 = \frac{n_s \tilde{n}_s}{s} + \frac{n_t \tilde{n}_t}{t} + \frac{n_u \tilde{n}_u}{u}$$

$$n_s, n_t, n_u \;=\; \text{functions of } e_\mu, p_\mu$$

$$\tilde{n}_s, \tilde{n}_t, \tilde{n}_u \;=\; \text{functions of } \tilde{e}_{\tilde{\mu}}, p_{\tilde{\mu}}$$

With this foresight one can build an off-shell action with two independent Lorentz invariances.

CC, Remmen (1612.03927)

Can the double Lorentz invariance of gravity be made explicit in the off-shell action?

$$S_{\mathrm{EH}} = \int d^d x \sqrt{-g} \left(\frac{R}{16\pi G} + L_{\mathrm{GF}} \right)$$

arbitrary

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(h^2) + O(h^3) + \cdots$$

We can exploit the freedom of field basis and gauge fixing which leaves amplitudes unchanged.

CC, Remmen (1612.03927)

The resulting action is remarkably compact.

$$S_{\rm EH} = \frac{1}{16\pi G}\int d^d x\, \partial_A \Sigma_{CE} \partial_B \Sigma^{DE} \left(\frac{1}{16}\Sigma^{AB}\delta^C_D - \frac{1}{4}\Sigma^{CB}\delta^A_D \right)$$

$$\Sigma_{AB} = (e^H)_{AB}$$

$$\Sigma^{AB} = (e^{-H})^{AB}$$

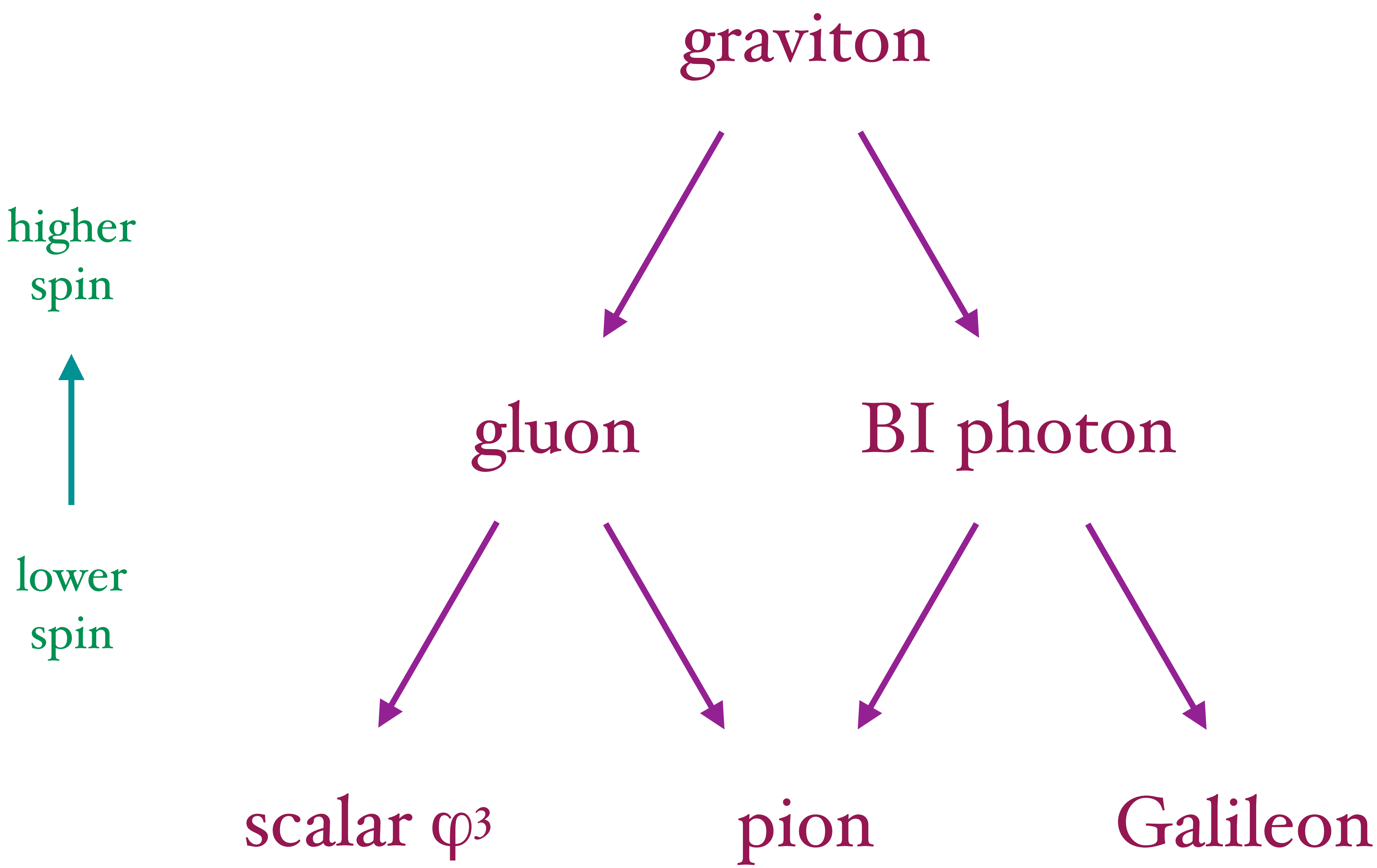
$$H_{AB} = \overset{\quad d \quad + \quad d}{\begin{pmatrix} 0 & h_{\mu\tilde{\nu}} \\ h_{\tilde{\mu}\nu} & 0 \end{pmatrix}}$$

$$\partial_A = (\overset{d}{\partial_\mu} \overset{+}{,} \overset{d}{\partial_{\tilde{\mu}}})$$

The off-shell Feynman diagrams have a doubled Lorentz invariance and the on-shell amplitudes have a doubled gauge symmetry.

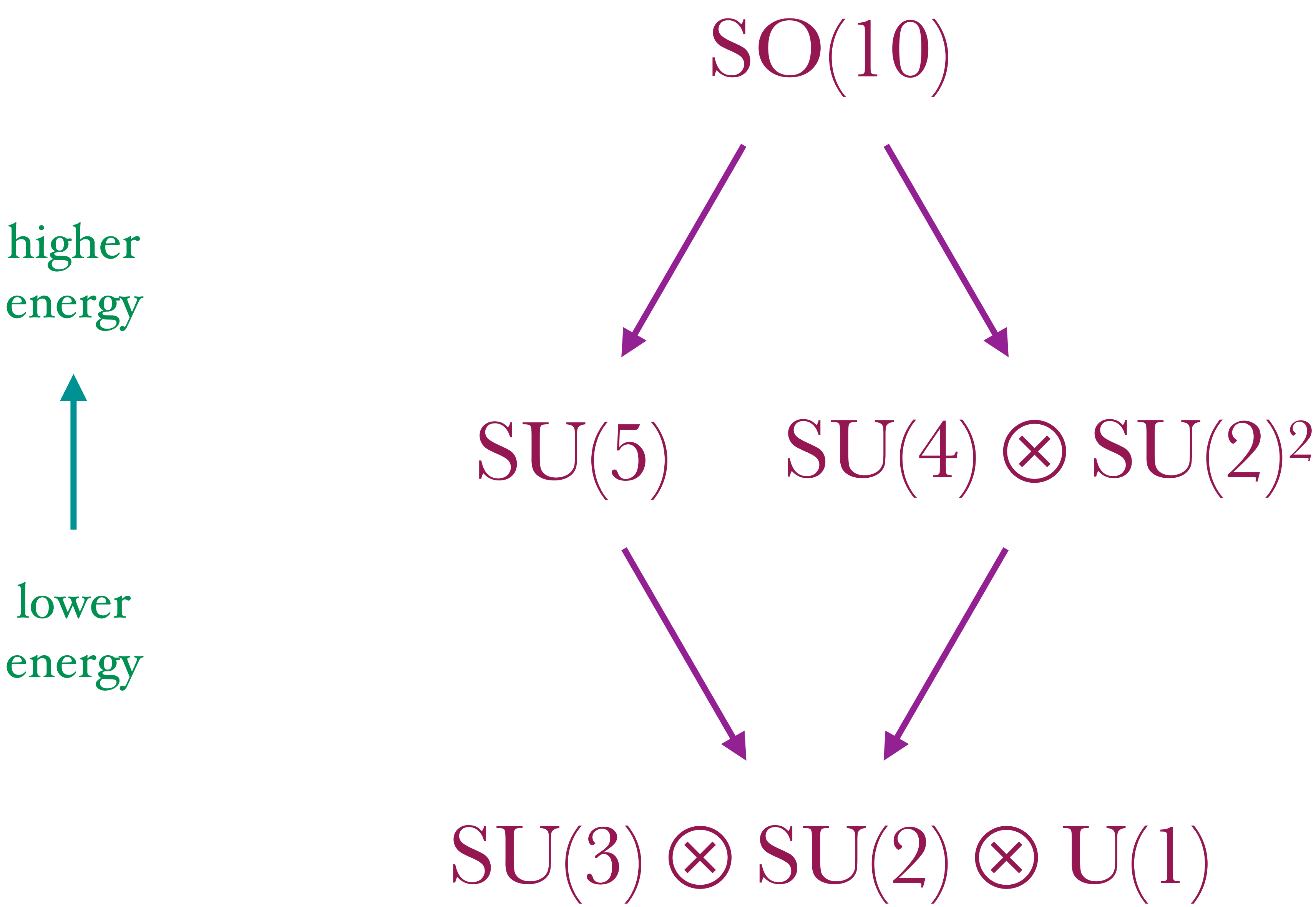
CC, Remmen (1612.03927)

New fact: gravity is the mother of theories!



CC, Shen, Wen (1705.03025)

This is distinct from textbook grand unification.



To begin, we think of gluon tree amplitudes as abstract functions of kinematic invariants.

$$A = e_1^{\mu_1} e_2^{\mu_2} \cdots e_n^{\mu_n} A_{\mu_1 \mu_2 \cdots \mu_n}$$

$$= \text{scalar function of } p_i p_j, p_i e_j, e_i e_j$$

Crucially, we maintain the on-shell conditions.

massless — helicity basis

$$p_i p_i = p_i e_i = e_i e_i = 0$$

transverse

A physical on-shell amplitude satisfies several constraints. The first is the Ward identity.

$$A\Big|_{e_i=p_i} \quad = \quad W_i\, A \quad = \quad 0$$

Here we recast the Ward identity as a differential operator that annihilates the amplitude.

$$W_i \quad = \quad \sum_{v=p_j,\, e_j} (p_i v) \frac{\partial}{\partial\, (v e_i)}$$

The second constraint is typically trivial: total momentum conservation.

$$P_v A = 0$$

As before, we can define an operator for this property of the amplitudes.

$$P_v = \sum_i p_i v$$

Now let us construct an operator T that acts on the amplitude A to produce a new one $T \cdot A$.

If the operator satisfies the conditions,

$$[\, W_i \,, T \,] \sim 0 \qquad\qquad [\, P_\nu \,, T \,] \sim 0$$

then if A is gauge invariant and momentum-conserving then so too is $T \cdot A$.

$$W_i \cdot (\, T \cdot A \,) = 0 \qquad\qquad P_\nu \cdot (\, T \cdot A \,) = 0$$

One can derive simple "transmutation operators" which alter particle type.

$$T_{ij} = \frac{\partial}{\partial(e_i e_j)}$$ 2 gluon → 2 scalar

$$T_{ijk} = \frac{\partial}{\partial(p_i e_j)} - \frac{\partial}{\partial(p_k e_j)}$$ 1 gluon → 1 scalar

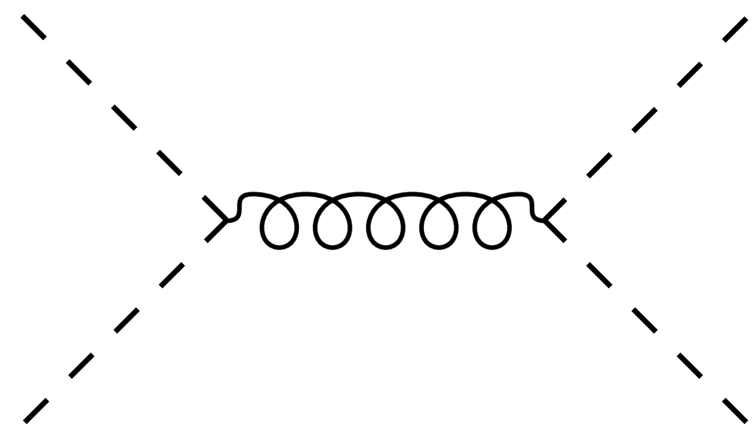
$$T_i = \sum_j p_i p_j \frac{\partial}{\partial(p_j e_i)}$$ 1 gluon → 1 pion

We proved transmutation for all graviton, gluon, pion tree amplitudes + explicit checks up to 8pt.

Example #1: YM to SQED

$$T_{12} \cdot T_{34} \cdot A(g_1, g_2, g_3, g_4) = \left[\frac{\partial}{\partial\,(e_1 e_2)} \frac{\partial}{\partial\,(e_3 e_4)}\right] A(g_1, g_2, g_3, g_4)$$

$$= \frac{p_1 p_3}{p_1 p_2} = A(\phi_1, \phi_2, \phi_3, \phi_4) =$$



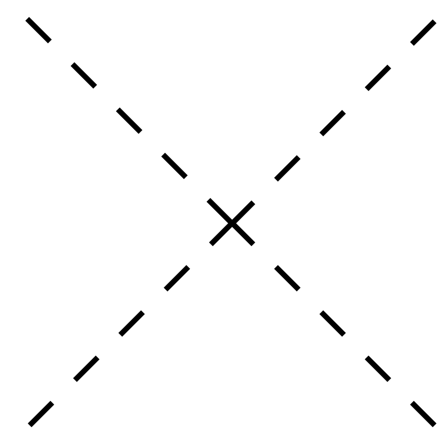
Extracting the $(e_1 e_2)(e_3 e_4)$ term is dimensional reduction to two new flavors of charged scalars.

$$e_1^\mu = e_2^\mu = \overset{d+1+1}{(0,1,0)} \qquad e_3^\mu = e_4^\mu = \overset{d+1+1}{(0,0,1)}$$

Example #2: YM to NLSM

$$T_{14}\cdot T_2\cdot T_3\cdot A(g_1,g_2,g_3,g_4) = \left[\frac{\partial}{\partial\,(e_1e_4)}\cdots\right]A(g_1,g_2,g_3,g_4)$$

$$= p_1p_3 = A(\pi_1,\pi_2,\pi_3,\pi_4) =$$



We can recast pions as oddly polarized gluons under dimensional reduction.

$$e_1^\mu = e_2^\mu = \overset{d+1+d}{(0,1,0)} \qquad e_2^\mu = \overset{d+1+d}{(p_2^\alpha, 0, ip_2^\beta)} \qquad e_3^\mu = \overset{d+1+d}{(p_3^\alpha, 0, ip_3^\beta)}$$

As it turns out, d-dim NLSM is a very exotic dimensional reduction of $(2d+1)$-dim YM.

$$A_\mu = \left(\frac{X_\alpha + Z_\alpha}{\sqrt{2}}, Y, \frac{X_\beta - Z_\beta}{i\sqrt{2}} \right)$$

$(p^\alpha, 0, ip^\beta)$

$(0,1,0)$

In terms of the X, Y, Z fields, the pion amplitude is

$$A(\pi_1, \pi_2, \cdots, \pi_n) = A(Y_1, Z_2, \cdots, Z_{n-1}, Y_n)$$

(Bose symmetry is not manifest!)

From YM we derive manifestly color-kinematic dual actions realizing (NLSM)² = Galileon.

$$L_{\mathrm{YM}} = -\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + L_{\mathrm{GF}}$$

(transmute the action)

$$L_{\mathrm{NLSM}} = \mathrm{Tr}\left(X_\alpha \Box Z^\alpha + \frac{1}{2}Y \Box Y + iX_{\alpha\beta}[Z^\alpha, Z^\beta] + iZ^\alpha[Y, \partial_\alpha Y]\right)$$

(literally square the action)

$$L_{\mathrm{Gal}} = X_{\alpha\tilde{\alpha}} \Box Z^{\alpha\tilde{\alpha}} + \frac{1}{2}Y \Box Y + X_{\alpha\beta\tilde{\alpha}\tilde{\beta}}Z^{\alpha\tilde{\alpha}}Z^{\beta\tilde{\beta}} + Z^{\alpha\tilde{\alpha}}Y\partial_\alpha\partial_{\tilde{\alpha}}Y$$

CC, Remmen, Shen, Wen (1709.04932)

This new NLSM action has peculiar features:

- Interactions are purely cubic, in sharp contrast with the typical formulation.
- Bose symmetry and pion parity are obscured.
- Kinematic algebra of pions comes from the higher-dimensional Poincare algebra.
- Weinberg gluon soft theorem maps onto the Adler zero condition.

11) Black Hole Binary Dynamics

Remarkably, hints of a double copy have appeared in classical solutions. The Schwarzschild metric in Kerr-Schild gauge is

(monopole)^2

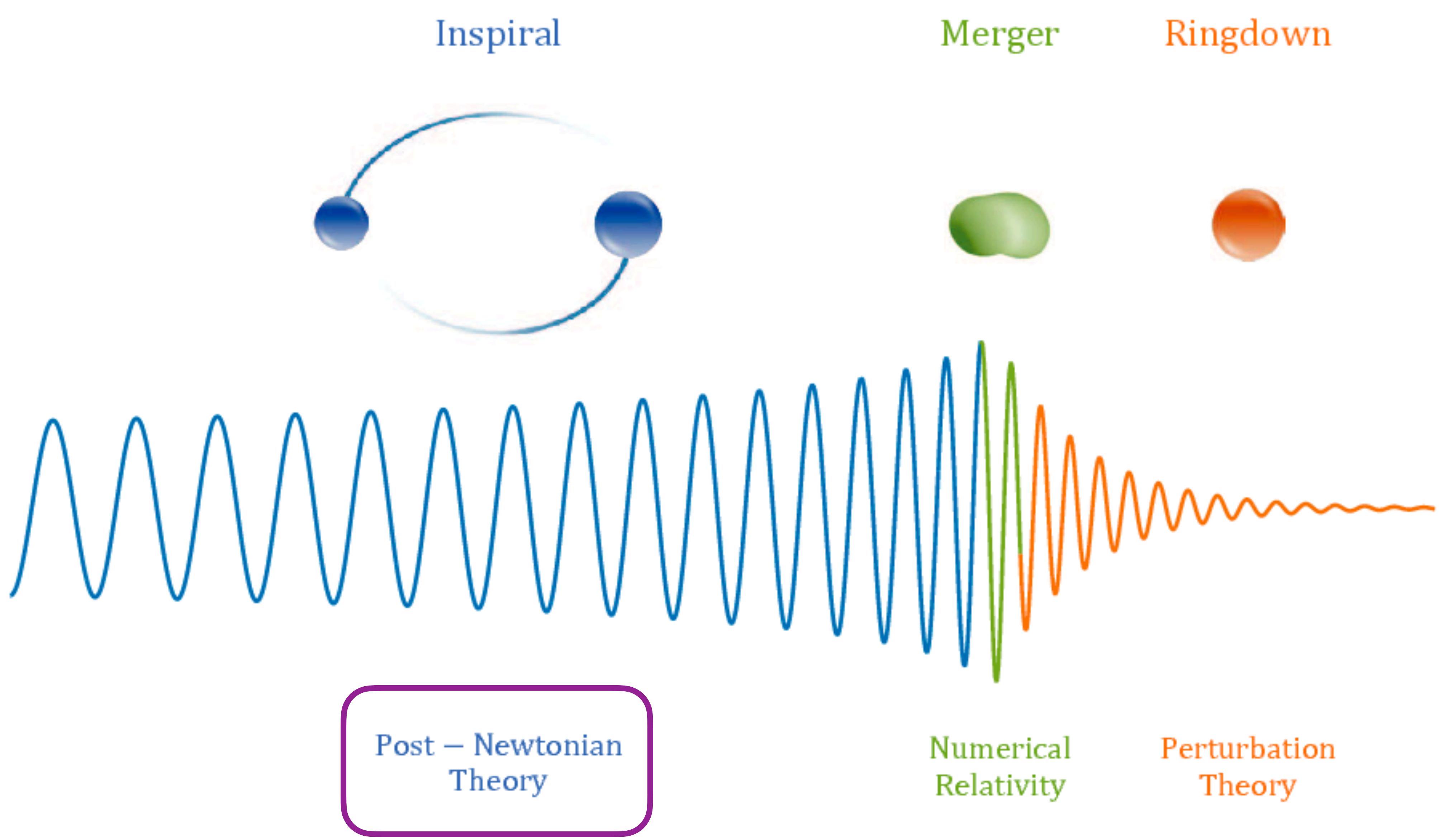
black hole → $$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2GM}{r} k_\mu k_\nu$$

At present it is not known how this leverage this observation to the general inspiral problem.

Nevertheless, in recent years, the amplitudes field has mobilized to make real bonafide progress relevant to gravitational wave physics at LIGO.

Montiero, O'Connell, White (1410.0239)

The binary black hole merger has three phases.



perturbation theory
is applicable here

figure from 1610.03567

State-of-the-art perturbative computations in gravitational wave physics center on the "post-Newtonian" expansion, based on

(virial theorem)

$$v^2 \sim \frac{GM}{r} \ll 1$$

which is tiny and perturbatively calculable during the inspired phase of the merger.

The so-called "post-Minkowskian" expansion parameter is G, and we call it perturbation theory.

Map of Perturbation Theory

	0PN	1PN	2PN	3PN	4PN	5PN	6PN	7PN		
1PM	$(1$	$+v^2$	$+v^4$	$+v^6$	$+v^8$	$+v^{10}$	$+v^{12}$	$+v^{14}$	$+\cdots)$	G
2PM		$(1$	$+v^2$	$+v^4$	$+v^6$	$+v^8$	$+v^{10}$	$+v^{12}$	$+\cdots)$	G^2
3PM			$(1$	$+v^2$	$+v^4$	$+v^6$	$+v^8$	$+v^{10}$	$+\cdots)$	G^3
4PM				$(1$	$+v^2$	$+v^4$	$+v^6$	$+v^8$	$+\cdots)$	G^4
5PM					$(1$	$+v^2$	$+v^4$	$+v^6$	$+\cdots)$	G^5

(*circa 2018*)

Map of Perturbation Theory

0PN 1PN 2PN 3PN 4PN 5PN 6PN 7PN

1PM $\left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + v^{14} + \cdots \right) G$

2PM $\left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots \right) G^2$

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5PM $\left(1 + v^2 + v^4 + v^6 + \cdots \right) G^5$

(*circa 2018*)

Can amplitudes give an efficient and scaleable path to higher PN? Naively, there are issues.

- black holes $\neq$ SYM gluons

  (double copy, recursion, etc. all apply to masses)

- LIGO does not observe scattering

  (NRQCD solved the amplitudes - potentials map)

All these puzzles have been surmounted. New results on conservative dynamics, radiation, spin, finite size effects are appearing swiftly.

full theory

amplitudes methods

BH / graviton tree amplitudes: A_{tree}

generalized unitarity

integral representation: $$A = \sum_i d^{(i)} I^{(i)}$$

multi-loop integration

full loop amplitude: $A(p,q)$

effective theory

build ansatz

effective BH Lagrangian: $V(p,q)$

Feynman diagrams

integral representation: $$A_{\text{EFT}} = \sum_i d^{(i)}_{\text{EFT}} I^{(i)}$$

multi-loop integration

EFT loop amplitude: $A_{\text{EFT}}(p,q)$

identical physics

$$A(p,q) = A_{\text{EFT}}(p,q)$$

gluons

(double copy) ↓ $A_{\text{grav}} = A_{\text{YM}} \otimes A_{\text{YM}}$

gravitons

(transmute) ↓ $A_{\phi_1\phi_2+\text{grav}} = \dfrac{\partial A_{\text{grav}}}{\partial(e_1 e_2)}$

massless scalars + gravitons

(add mass) ↓ $A_{\phi_1'\phi_2'+\text{grav}} = A_{\phi_1\phi_2+\text{grav}}\Big|_{p_1p_2 \to p_1p_2 - m_1m_2}$

massive scalars + gravitons

Map of Perturbation Theory

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CC, Solon,
Rothstein: 1808.02489

Map of Perturbation Theory

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CC, Solon, Rothstein: 1808.02489

Bern, CC, Roiban, Shen, Solon, Zeng: 1901.04424, 2003.08351, 2006.06665

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CC, Solon, Rothstein: 1808.02489

Bern, CC, Roiban, Shen, Solon, Zeng: 1901.04424, 2003.08351, 2006.06665

Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, Zeng: 2101.07254

Our result is now the state-of-the-art in PM, and matches all known results at overlapping orders.

$$V(\boldsymbol{p},\boldsymbol{r}) = \sum_{i=n}^{\infty} \frac{G^n c_n(\boldsymbol{p}^2)}{|\boldsymbol{r}|^n}$$

$$c_1 = \frac{\nu^2 m^2}{\gamma^2 \xi}\left(1-2\sigma^2\right),$$

$$c_2 = \frac{\nu^2 m^3}{\gamma^2 \xi}\left[\frac{3}{4}\left(1-5\sigma^2\right) - \frac{4\nu\sigma\left(1-2\sigma^2\right)}{\gamma\xi} - \frac{\nu^2(1-\xi)\left(1-2\sigma^2\right)^2}{2\gamma^3\xi^2}\right],$$

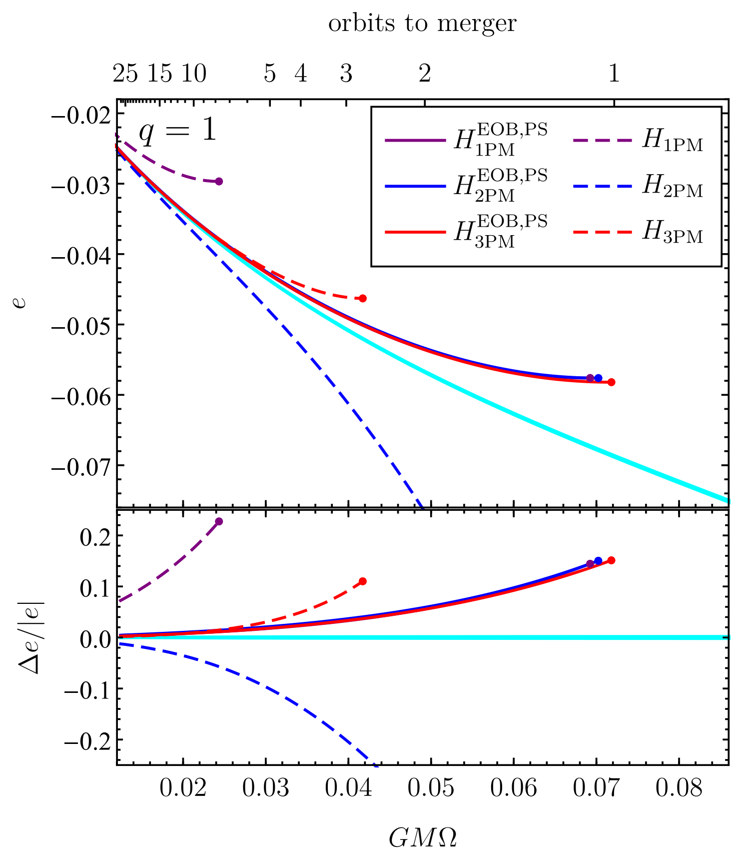
$$c_3 = \frac{\nu^2 m^4}{\gamma^2 \xi}\left[\frac{1}{12}\left(3-6\nu+206\nu\sigma-54\sigma^2+108\nu\sigma^2+4\nu\sigma^3\right) - \frac{4\nu\left(3+12\sigma^2-4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}}\right.$$
$$- \frac{3\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{2(1+\gamma)(1+\sigma)} - \frac{3\nu\sigma\left(7-20\sigma^2\right)}{2\gamma\xi} + \frac{2\nu^3(3-4\xi)\sigma\left(1-2\sigma^2\right)^2}{\gamma^4\xi^3}$$
$$\left. - \frac{\nu^2\left(3+8\gamma-3\xi-15\sigma^2-80\gamma\sigma^2+15\xi\sigma^2\right)\left(1-2\sigma^2\right)}{4\gamma^3\xi^2} + \frac{\nu^4(1-2\xi)\left(1-2\sigma^2\right)^3}{2\gamma^6\xi^4}\right].$$

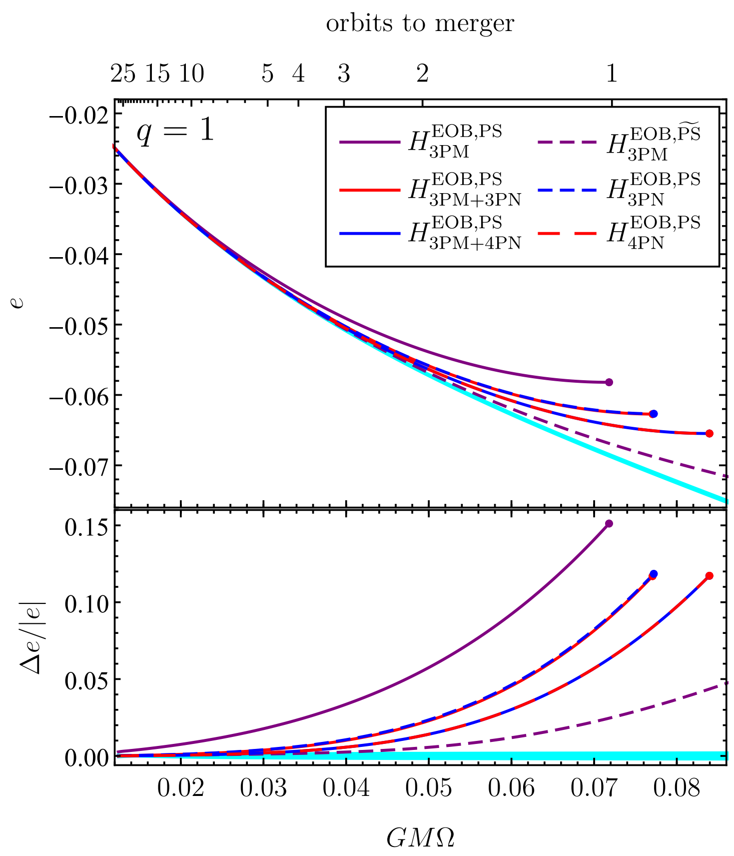
Bern, CC, Roiban, Shen, Solon, Zeng (1901.04424, 1908.01493, 2003.08351)

Folks in the gravitational wave community have taken new results seriously and want more.

Energetics of two-body Hamiltonians in post-Minkowskian gravity

Andrea Antonelli,[1] Alessandra Buonanno,[1,2] Jan Steinhoff,[1] Maarten van de Meent,[1] and Justin Vines[1]
[1] *Max Planck Institute for Gravitational Physics (Albert Einstein Institute), Am Mühlenberg 1, Potsdam 14476, Germany*
[2] *Department of Physics, University of Maryland, College Park, MD 20742, USA*
(Dated: January 23, 2019)





Finite size corrections are crucial for neutron star mergers. We can also include tidal effects from the mass and current quadrupole moments.

$$\Delta S = \sum_{A=1,2} \int d\tau_A \left(\frac{\mu_A^{(2)}}{4} E^2 + \frac{2\sigma_A^{(2)}}{3} B^2 \right)$$

↕ (worldline vs QFT)

$$\Delta S = \int \sqrt{-g}\, C_{\mu\alpha\nu\beta} C^{\rho\alpha\sigma\beta} \sum_{A=1,2} \left(\lambda_A \phi_A^2 \delta_\rho^\mu \delta_\sigma^\nu + \frac{\eta_A}{m_A^4} \nabla^\mu \nabla^\nu \phi_A \nabla_\rho \nabla_\sigma \phi_A \right)$$

Then compute the tidal corrections to scattering.

Applying the same tools as before yields state-of-the-art results for leading tidal effects in PM.

$$\Delta V(\boldsymbol{p}, \boldsymbol{r}) = \sum_{n=2}^{\infty} \frac{G^n \Delta c_n(\boldsymbol{p}^2)}{|\boldsymbol{r}|^{n+4}}$$

$$\Delta c_2 = -\frac{3m_2^3}{E^2\xi}\left[4\lambda_1 + \frac{\eta_1}{32}(11 - 30\sigma^2 + 35\sigma^4)\right],$$

$$\begin{aligned}\Delta c_3 = \frac{15m_2^3}{2E^2\xi}\Bigg[&4\lambda_1\left(\frac{8m_2}{5} - \frac{m_1\sigma(5-2\sigma^2)}{(\sigma^2-1)^2} + \frac{6m_1\sinh^{-1}\sqrt{\frac{\sigma-1}{2}}}{(\sigma^2-1)^{5/2}}\right) + \eta_1\Bigg(\frac{m_2(305-363\sigma^2-110\sigma^4)}{560}\\ &- \frac{m_1\sigma(5401-195\sigma^2-94\sigma^4)}{80} - \frac{m_1\sigma(673+2168\sigma^2)}{2(\sigma^2-1)^2} + \frac{3m_1\left(33+474\sigma^2+440\sigma^4\right)\sinh^{-1}\sqrt{\frac{\sigma-1}{2}}}{(\sigma^2-1)^{5/2}}\Bigg)\\ &+ 2(1-2\sigma^2)\left[4\lambda_1 + \frac{\eta_1}{32}(11-30\sigma^2+35\sigma^4)\right]\frac{E(E_2-m_2)}{m_2(\sigma^2-1)}\Bigg]\\ &+ \frac{3\nu m_2^3}{m\gamma^5\xi^3}\left[\nu(1-\xi)(1-2\sigma^2)\left[4\lambda_1 + \frac{\eta_1}{32}(11-30\sigma^2+35\sigma^4)\right] + 4\gamma^2\xi\sigma\left[4\lambda_1 + \frac{\eta_1}{32}(26-95\sigma^2+105\sigma^4)\right]\right]\end{aligned}$$

CC, Solon (2006.06665)

Fluid Dynamics

The amplitudes approach is broadly applicable.

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A potential area for progress is fluid dynamics.

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Notably, an incompressible velocity field is highly reminiscent of a gauge field.

$$\partial_i u_i = 0$$

solenoidal 3-vector

$$\partial^\mu A_\mu = 0$$

transverse 4-vector

It is then natural to (re)apply everything we know about gauge theory to fluids and see what sticks.

An incompressible fluid velocity field satisfies the Navier-Stokes equation (NSE).

fluid velocity field — source

$$\left(\partial_0 - \nu\partial^2\right) u_i + u_j\partial_j u_i + \partial_i \left(p/\rho\right) = J_i$$

viscosity — pressure / density

NSE ≠ Euler-Lagrange equations of some simple canonical action. Dissipation commonly entails complicated Schwinger-Keldysh doubling.

No action, no problem!

"Color" is missing, but we can just generalize to a non-Abelian Navier-Stokes equation (NNSE).

$$\left(\partial_0 - \nu\partial^2\right) u_i^a + f^{abc} u_j^b \partial_j u_i^c + \partial_i \left(p^a/\rho\right) = J_i^a$$

The solenoidal condition $\partial_i u_i^a = 0$ allows us to eliminate the pressure term.

$$\left(\partial_0 - \nu\partial^2\right) u_i^a + f^{abc} u_j^b \partial_j u_i^c = J_i^a$$

↕ similar to Yang-Mills !

$$\partial^\mu F_{\mu\nu}^a + g f^{abc} A^{b\mu} F_{\mu\nu}^c = J_\nu^a$$

Equations of motion define a tree-level S-matrix.

equations of motion	=	Berends-Giele recursion relations

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Consider a toy model, $(\Box + m^2)\phi + g\phi^2 = J$, and solve it perturbatively in the coupling.

$$\phi(x, J) = \phi_0 + \phi_1 + \phi_2 + \cdots$$

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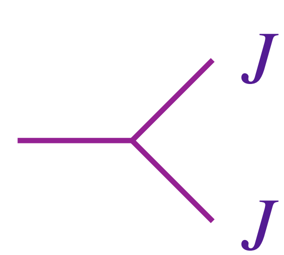
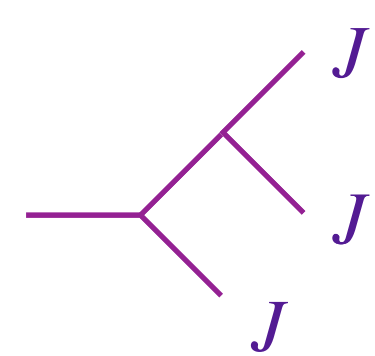
$\phi_1 =$ 

$\phi_2 =$ 

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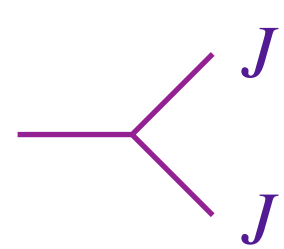
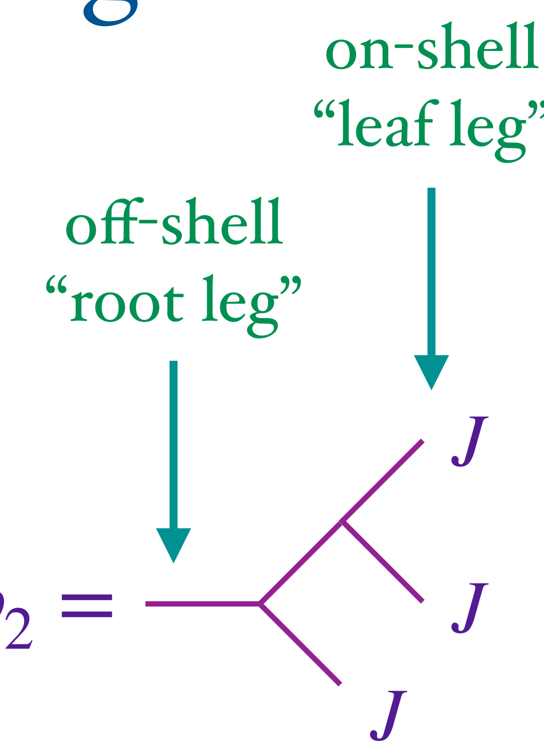
$\phi_1 =$ 

$\phi_2 =$ 

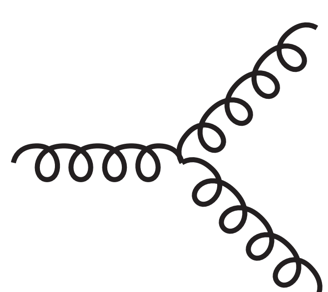
Ditto for the NSE (Wyld, 1961). Linearized NSE and incompressibility give "on-shell conditions".

$$i\omega - \nu p_i^2 = 0 \qquad\qquad p_i \varepsilon_i = 0$$

Feynman rules for NNSE are simple.

$u^{a_1}_{i_1}(\omega_1, p_1)$  $u^{a_2}_{i_2}(\omega_2, p_2)$ $= \dfrac{\delta^{a_1 a_2}\delta_{i_1 i_2}}{i\omega_1 - \nu p_1^2}$

$u^{a_2}_{i_2}(p_2)$

$u^{a_3}_{i_3}(p_3)$  $= f^{a_1 a_2 a_3}(p_{3 i_1}\delta_{i_2 i_3} - p_{3 i_2}\delta_{i_1 i_3})$

$u^{a_1}_{i_1}(p_1)$

Cheung, Mangan (2010.15970)

These semi-on-shell NNSE amplitudes are rational functions of momenta and polarizations.

$$A(123) = f^{a_1 a_2 a_3} \left[(p_1 \varepsilon_2)(\varepsilon_1 \varepsilon_3) - \{1 \leftrightarrow 2\} \right]$$

$$A(1234) = f^{a_1 a_2 b} f^{b a_3 a_4} \times \frac{1}{p_1 p_2} \left[(p_1 \varepsilon_2) \left[(p_3 \varepsilon_1)(\varepsilon_3 \varepsilon_4) + (p_4 \varepsilon_3)(\varepsilon_1 \varepsilon_4) \right] - \{1 \leftrightarrow 2\} \right]$$

$$+ \ t\text{-channel} \ + \ u\text{-channel}$$

Energies drop out of these amplitudes, and we learn that the coupling constant is $\sim 1/\nu$.

$$\frac{\delta^{a_1 a_2} \delta_{i_1 i_2}}{i \sum_A \omega_A - \nu (\sum_A p_A)^2} = -\frac{1}{\nu} \frac{\delta^{a_1 a_2} \delta_{i_1 i_2}}{\sum_{A \neq B} p_A p_B}$$

Cheung, Mangan (2010.15970)

Hence, the turbulent regime is strongly coupled.

$$\frac{1}{\nu} \sim Re \gg 1$$

Turbulence entails tree amplitudes at arbitrarily high multiplicity. Of course, some such objects are actually computable in gauge theory.

$$A(1^+2^+3^+\cdots i^-\cdots j^-\cdots n^+) \quad = \quad \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

Resummation/renormalization are also options.

Let us now run through the amplitudes checklist.

A) spinor helicity

B) soft theorems

C) on-shell recursion relations

D) color-kinematics duality

E) double copy

Cheung, Mangan (2010.15970)

A) Fluids have nice spinor helicity amplitudes.

$$A(1^-2^-3^-) = k\,\langle 12\rangle\langle 23\rangle\langle 31\rangle$$

$$A(1^+2^-3^-) = k\,[12\rangle\langle 23\rangle\langle 31]$$

$$A(1^-2^-3^+) = k\,[31\rangle\langle 12\rangle\langle 23]$$

$$k = \frac{f^{a_1a_2a_3}}{\langle 11]\langle 22]}$$

The little group constrains the amplitudes but does not uniquely specify them.

$$A(1^-2^-3^+) = f^{a_1a_2a_3}\frac{\langle 12\rangle^3}{\langle 13\rangle\langle 32\rangle} \times \left[1 - \frac{\langle 11]}{\langle 22]} - \frac{\langle 22]}{\langle 11]}\right]$$

Maldacena, Pimentel (1104.2846)

B) Fluids exhibit universal leading soft theorems.

The root leg satisfies an "Adler zero".

$$\lim_{p \to 0} A_{n+1}(p_1, \cdots, p_n) = 0$$

The leaf legs satisfy a "Weinberg soft theorem".

$$\lim_{p_n \to 0} A_{n+1}(p_1, \cdots, p_n) = \left[\sum_{A=1}^{n-1} \frac{p_A e_n}{p_A p_n} \right] A_n(p_1, \cdots, p_{n-1})$$

C) Fluids satisfy on-shell recursion relations.

standard recursion:

$$A_{n+1}(0) = \oint \frac{dz}{z} A_{n+1}(z)$$

soft recursion:

$$A_{n+1}(0) = \oint \frac{dz}{z} \prod_i \frac{1}{1 - z/z_{i*}} A_{n+1}(z)$$

$$A_{n+1}(z_{i*}) = 0$$

The existence of on-shell recursion relations for the (N)NSE implies an S-matrix definition of incompressible fluid mechanics.

The energy independence of (N)NSE amplitudes allows for immense freedom in momentum shifts.

- shift by polarization vectors

$$p_A \to p_A + z\tau_A\varepsilon_A$$

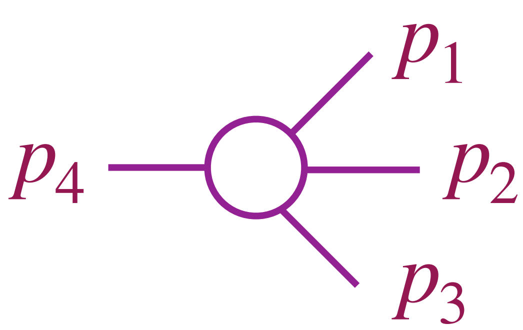
$$\longrightarrow \quad pp \sim z^2, p\varepsilon \sim z, \varepsilon\varepsilon \sim 1 \quad \longrightarrow \quad A_{n+1} \sim z^{-n+3}$$

- shift by a reference vector

$$p_A \to p_A + z(p_A\eta)\eta \qquad \varepsilon_A \to \varepsilon_A - z(\varepsilon_A\eta)\eta$$

$$\longrightarrow \quad pp \sim z, p\varepsilon \sim 1, \varepsilon\varepsilon \sim 1 \quad \longrightarrow \quad A_{n+1} \sim z^{-n+2}$$

D) Fluids exhibit color-kinematics duality.



$$= \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}$$

$$n_s = \left[p_{1i_2}(p_{1i_3} + p_{2i_3})\delta_{i_1 i_4} + p_{2i_1} p_{3i_2} \delta_{i_3 i_4} - \{1 \leftrightarrow 2\}\right]$$

$$c_s = f^{a_1 a_2 b} f^{b a_3 a_4}$$

The kinematic Jacobi identity holds off-shell!

$$n_s + n_t + n_u = 0$$

The NNSE can be recast in a simpler form by defining a kinematic structure constant.

$$\left(\partial_0 - \nu\partial^2\right) u_i^a + \frac{1}{2} f^{abc} f_{ijk} u_j^b u_k^c = J_i^a$$

$$f_{ijk} v_j w_k = v_j \partial_j w_i - w_j \partial_j v_i$$

These structure constants literally define the algebra of spatial diffeomorphisms.

$$\left[v_j \partial_j, w_k \partial_k\right] = f_{ijk} v_j w_k \partial_i$$

The NNSE is invariant under an off-shell color and kinematic symmetry.

color symmetry: $u_i^a \to u_i^a + f^{abc}\theta^b u_i^c$

kinematic symmetry: $u_i^a \to u_i^a + f_{ijk}\theta_j u_k^a$

From the kinematic symmetry we can derive an additional conservation equation for the NNSE.

$$0 = \partial_0 Q_i = \int d^3x\ \partial_l J_{li} = \frac{1}{\nu}\int d^3x\ f_{ijk} u_j^a \overleftrightarrow{\partial_0} u_k^a$$

E) Fluids double copy.

We can literally square the NNSE to double copy to the tensor Navier-Stokes equation (TNSE).

$$\left(\partial_0 - \nu\partial^2\right) u_i^a + \frac{1}{2} f^{abc} f_{ijk} u_j^b u_k^c = J_i^a$$

↕ substitute f^{abc} with $f_{\bar{i}\bar{j}\bar{k}}$

$$\left(\partial_0 - \nu\partial^2\right) u_{i\bar{i}} + \frac{1}{2} f_{\bar{i}\bar{j}\bar{k}} f_{ijk} u_{j\bar{j}} u_{k\bar{k}} = J_{i\bar{i}}$$

There exist simple monopole solutions to the NNSE (for SU(2) gauge group) and the TNSE.

NNSE monopole $$u_i^a = -\frac{2\nu\epsilon_{aij}x_j}{r^2}$$

TNSE monopole $$u_{i\bar{i}} = 2\nu\delta_{i\bar{i}} + \frac{Cx_i x_{\bar{i}}}{r^4}$$

This exactly parallels the Kerr-Schild double copy relating monopoles and black holes. Notably, the fluid monopole solutions critically balance the linear and nonlinear interactions.

Conclusions

- Scattering amplitudes have uncovered hidden structures lurking inside real-world theories like gravitons, gluons, and pions.

- Double copy, generalized unitarity and EFT have together led to state-of-the-art results for gravitational wave physics, with more to come!

- The Navier-Stokes equations define an S-matrix for fluid quanta which exhibit soft theorems, recursion relations, and the double copy.

Thank you!