Indeterminism in Physics and Intuitionistic Mathematics

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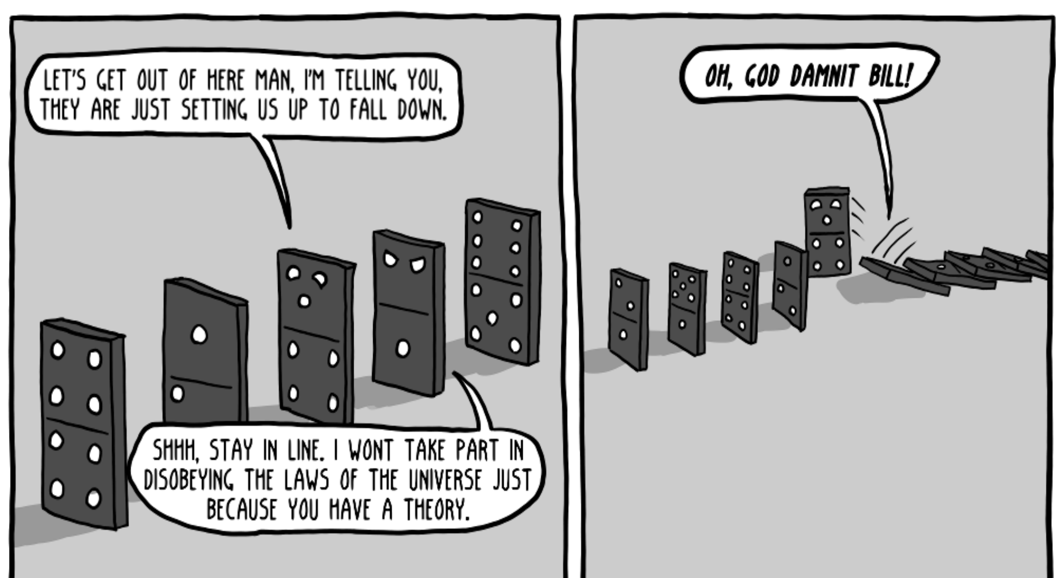
- Mathematical language**S**
- Motivations
- Classical chaos
- **Real numbers are not really real**
- Intuitionistic mathematics
- Intuitionistic logic, the continuum and arithmetic
- The relativity of indeterminacy



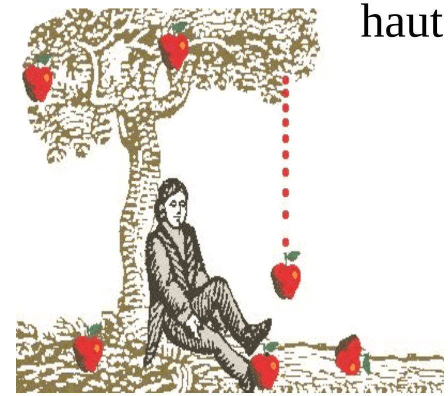
Motivations

- Physicists produce models of reality.
- The models should be as faithful as possible:

$\Rightarrow$ correct empirical predictions, and

$\Rightarrow$ **allow humans to tell stories about how nature does it.**

Stories



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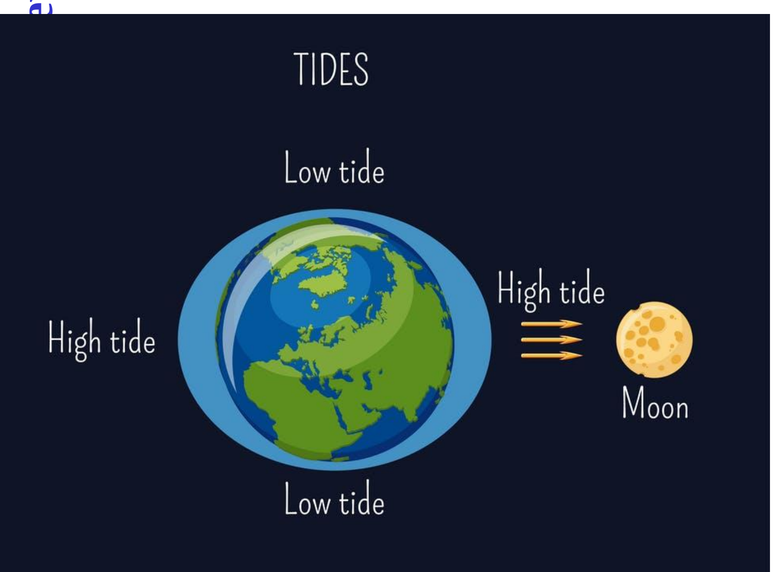
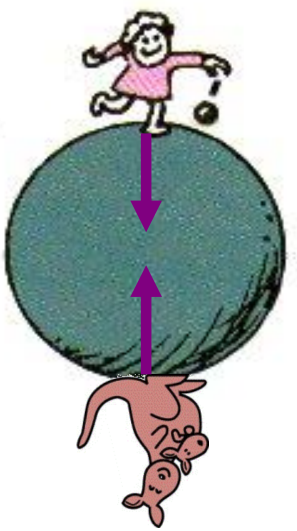
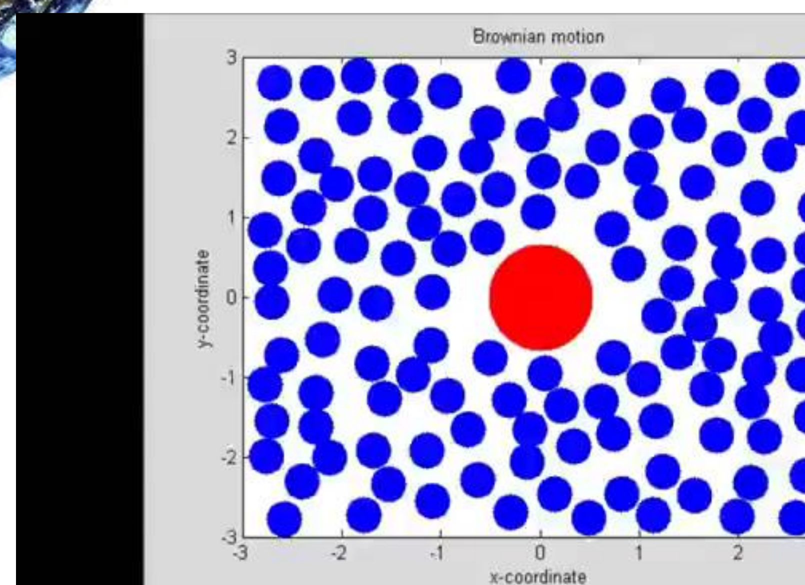
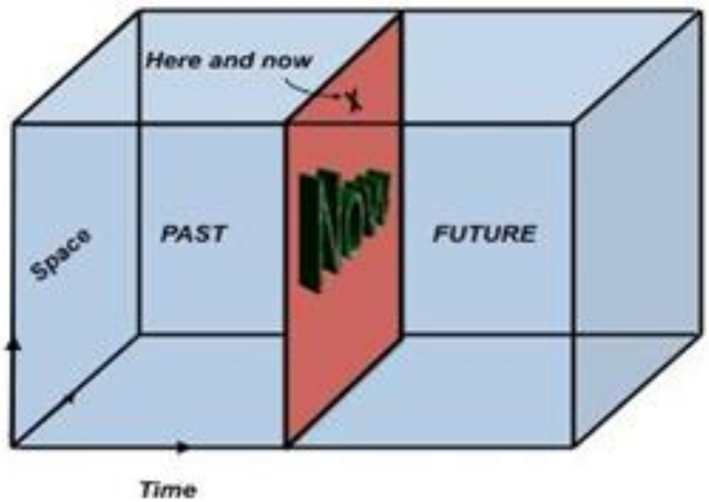
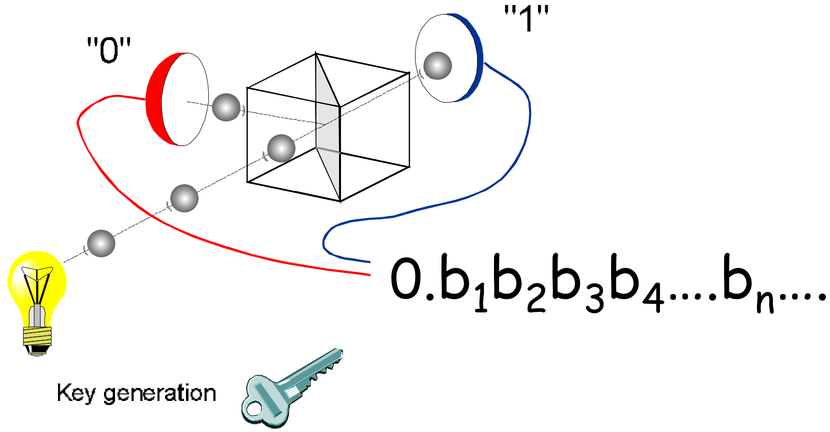






Block Universe or Quantum Randomness





Everything is fixed since ever and for ever

New information gets created as time passes

Einstein:
"People like us, who believe in physics, know that the distinction between past, present, and future is only a stubbornly persistent illusion."

nature International weekly journal of science

LETTERS

Random numbers certified by Bell's theorem

S. Pironio[1,2]*, A. Acín[3,4]*, S. Massar[1]*, A. Boyer de la Giroday[5], D. N. Matsukevich[6], P. Maunz[6], S. Olmschenk[6], D. Hayes[6], L. Luo[6], T. A. Manning[6] & C. Monroe[6]

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Experimental fact:

Prob(a⊕b=x·y) > 3/4

x_i

a_i

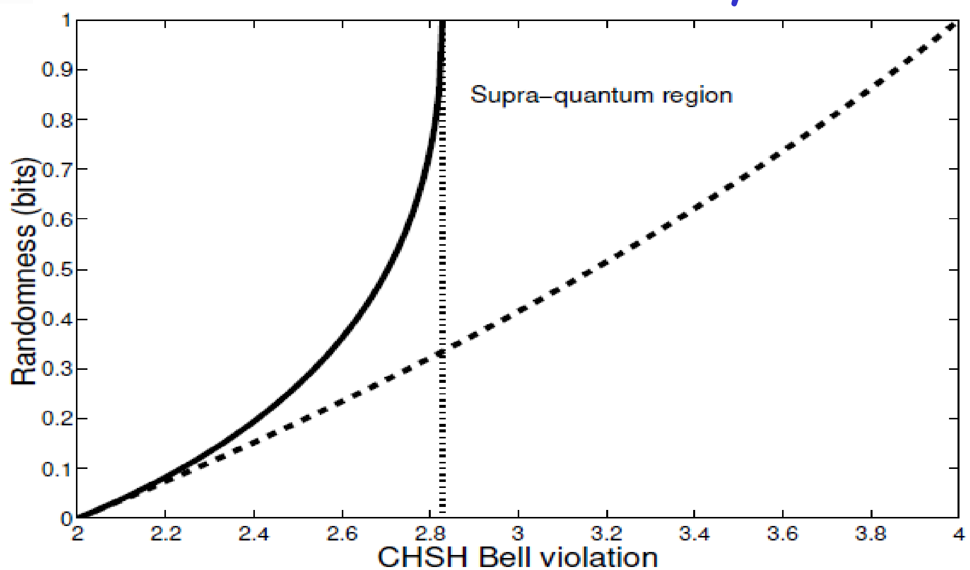
distance

y_i

b_i

Theorem:

⇒ The outcomes are truly random



N. Gisin
Quantum Chance
Springer 2014.
N. Brunner et al.,
RMP 86, 419
(2014).

Quantum Random Number Generator

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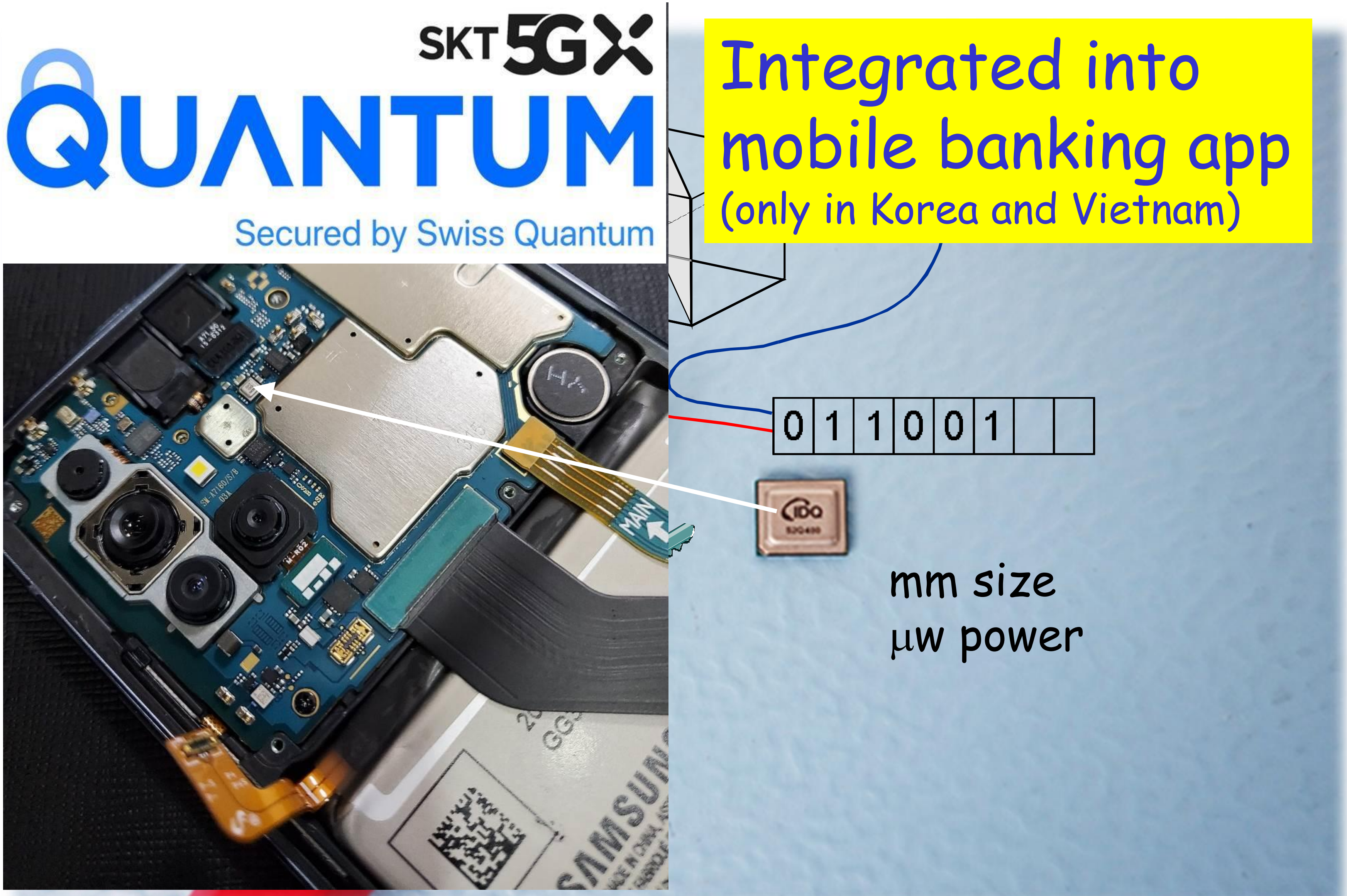
Integrated into mobile banking app
(only in Korea and Vietnam)



Motivations

- Physicists produce models of reality.
- The models should be as faithful as possible:
  $\Rightarrow$ correct empirical predictions, and
  $\Rightarrow$ allow humans to tell stories about how nature does it.

Main messages

1. *Whether Newtonian classical mechanics is deterministic or not, is not a scientific question; it depends on the physical significance one associates with mathematical real numbers.*
2. *The mathematical language we use when speaking physics has a huge influence on the world-view that physics presents to us.*
3. *We face a choice: either our stories present events as the consequence of long past initial conditions, or else, we understand the present as the result of indeterminate reality, and the future as open.*

Why does a physicist talk about intuitionistic mathematics?

- To do physics, we need physical concepts and appropriate mathematical tools and concepts.
- But which mathematics?
- The language we speak conditions the ideas we express.
- Brouwer, the father of intuitionism, introduced into his mathematics the concept of an ideal mathematician – the creating subject – who continually produces new information by solving mathematical conjectures.
- I will present intuitionism without any ideal mathematician and motivate it by the physical concept of indeterminism. Brouwer would not have liked my presentation. I am a physicist, hence a (naive?) realist.

- My main claim is that intuitionistic mathematics is the natural mathematical tool to describe indeterminism in physics,
like derivative is the natural tool to describe velocity.

Numbers are there to compute

- How can it be that most of the so-called real numbers are not computable?

Let's define some integers:

1. n1=0 if every even integer bwt 4 and 10^4 is the sum of 2 primes, and n1=1 otherwise.
2. n2=0 if every even integer bwt 4 and 10^{100} is the sum of 2 primes, and n2=1 otherwise.
3. n3=0 if every even integer larger than 4 is the sum of 2 primes, and n3=1 otherwise (Goldbach conjecture).

- There is no known finite method to compute n3.
- Nevertheless, every student knows that in order not to fail an exam he has to claim that n3 has a determined value.
- Except if it is an exam in intuitionistic mathematics. Then the student should answer that the value of n3 is indeterminate and that the law of the excluded middle is not valid.
- And the value of n3 may evolve over time ...

… from indeterminate to determinate.

Intuitionistic stand-point

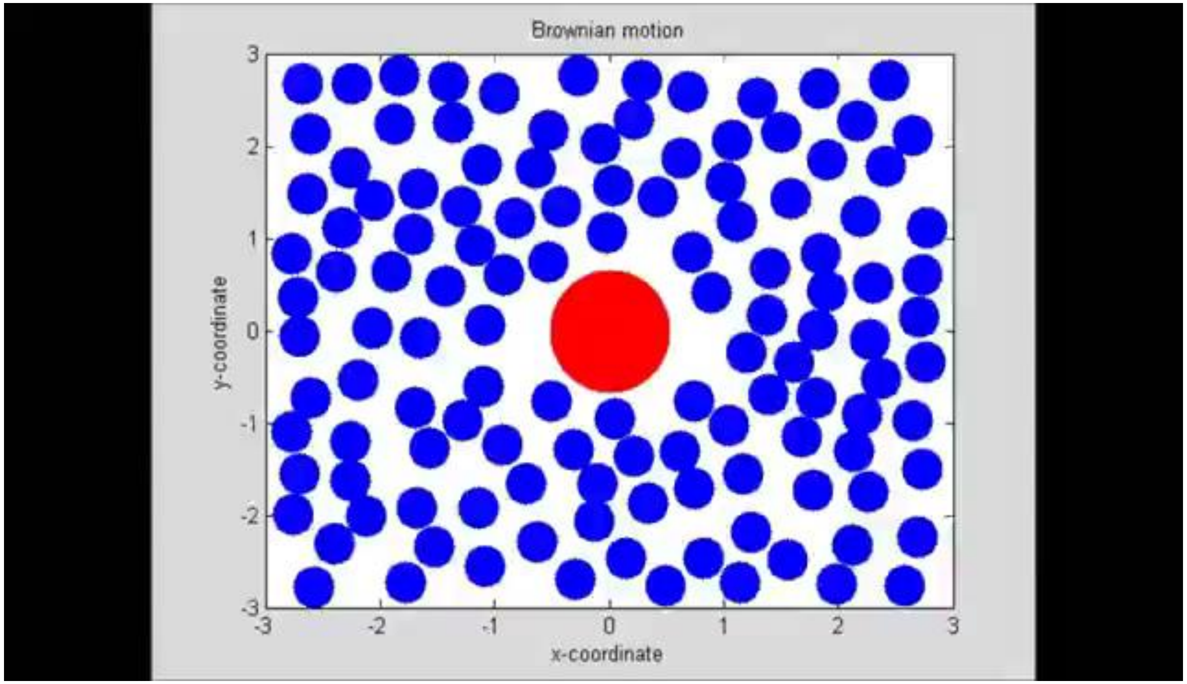
- Carl Posy: “We humans have finite memories, finite attention spans and finite lives. So we can fully grasp only finitely many finite sized pieces of a compound thing. There’s no infinite helicopter allowing us to survey the whole terrain or to tell how things will look at the end of time.”
- Erret Bishop: “The classicist wishes to describe God's mathematics; the constructivist, to describe the mathematics of finite beings, man's mathematics for short … Constructive mathematics does not postulate a pre-existent universe, with objects lying around waiting to be collected and grouped into sets, like shells on a beach.”
- Brouwer: **“Nature simply has not yet fully determined all objects”**. This can be compared to the “uncertainty” principle used in quantum mechanics.
- That’s the essence of intuitionism. But I see that you get worried: how could there be things, including mathematical objects, not fully determined?
- To calm a bit your worries, let me state that with intuitionistic mathematics you can compute and prove theorems, though not always the same theorems as in classical mathematics or not following the same proofs. For sure, everything one can do on a (classical) computer can be done with intuitionistic mathematics. Hence, all of physics can be done.
- Climate physics uses truncated numbers and stochastic remainders.

Palmer, T. N. *Nat. Rev. Phys.* **1**, 463–471 (2019).

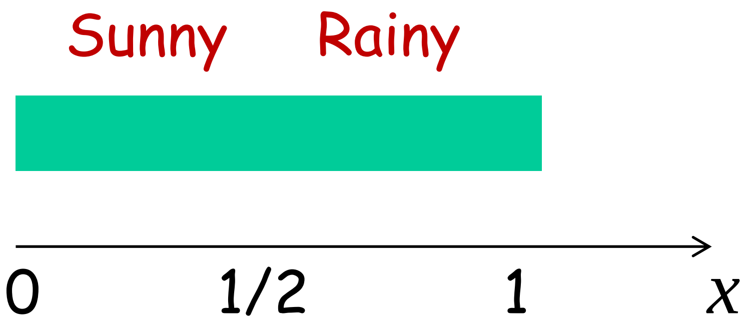
“Deterministic” chaos

- Chaotic classical dynamical systems are usually considered deterministic.
- But is this really so? Or is it the consequence of using the language of classical mathematics? i.e. assuming a God’s eye view, i.e. a view from the end of time?

- Yuval Dolev (a philosopher in Jerusalem): ”tense and passage are not, never were, and probably cannot be part of physics and its language.”



Example of a chaotic system



$$x \mapsto \begin{cases} 2x & \text{if } x < \tfrac{1}{2} \\ 2x-1 & \text{if } x \geq \tfrac{1}{2} \end{cases}$$

X binary = $0.b_1b_2b_3b_4....b_n.....$

$\rightarrow$ X binary = $0.b_2b_3b_4....b_n.....$

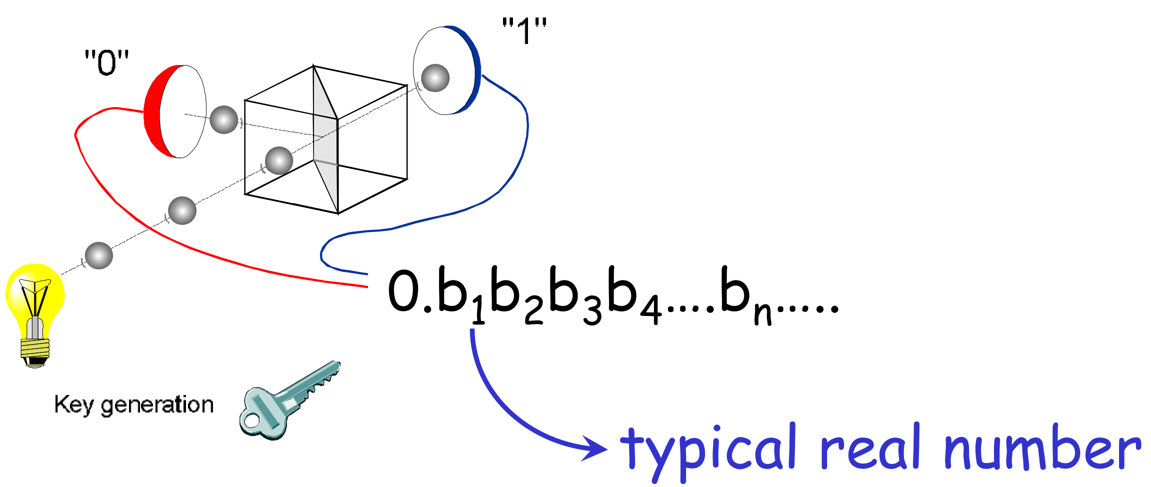
Whether x lies in the left half or the right half after n steps depends on the n^{th} bit, b_n, of the initial condition x.

Is the millionth bits "physically real"? The question is not whether the millionth bit can be measured, but whether it corresponds to something physical?

N. Gisin, *Indeterminism in Physics, Classical Chaos and Bohmian Mechanics. Are Real Numbers Really Real?*, Erkenntnis (2019).

Typical real numbers

- All real numbers one encounters are exceptional:
  they have a name and are defined by a (finite) algorithm: $\sqrt{2}$, π, 77/125, etc.
- The bits of typical real numbers have no structure: the bits are random, as random as quantum measurement outcomes:



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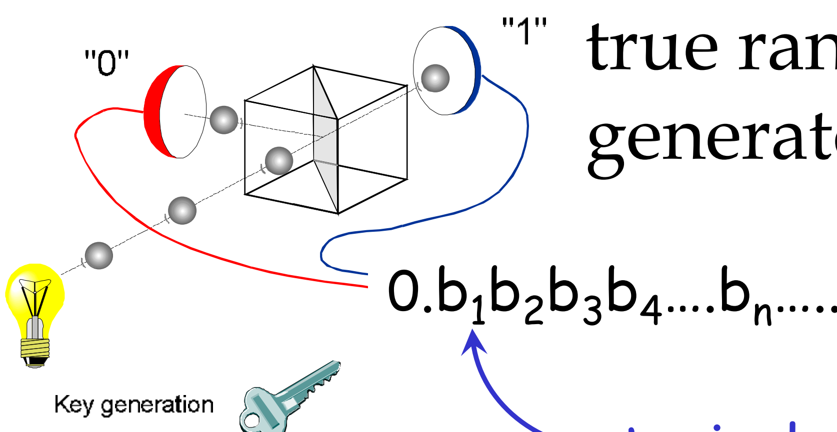
Typical real numbers

- Sinc … names and algo … ain an infi … tion.
- One … s to all que … uman lang …

When we say:
Let $x_0 \in \Re$
what we effectively say is:
let x_0 denote an infinite amount of information.

- The only good way of thinking of a typical real number is the unlimited string of outcomes of a true random number generator.

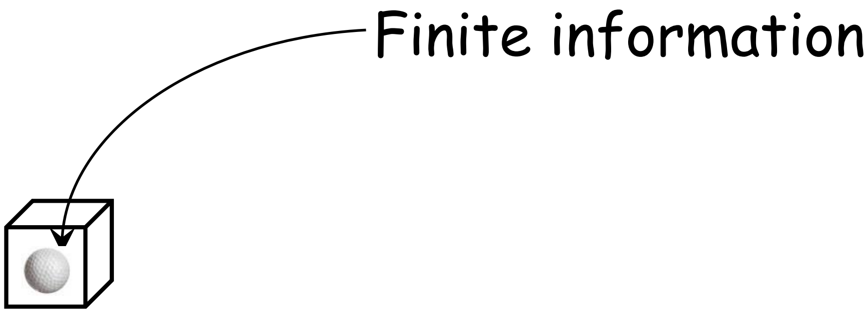


typical real number

G. Chaitin, The Labyrinth of the Continuum, in Meta Math!, Vintage 2008

Finite volume ⇒ finite information

- A finite volume of space can not contain infinitely many bits of information.
- Hence, the position of a classical particle is not a real number.



Beckenstein bound? Yes, but one doesn't need quantum field theory and black holes. Finite information density also applies to Newtonian mechanics and to special relativity.

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Mathematical real numbers are not Physically real

Mathematical real numbers are Physically random

And these random numbers should be at the basis of scientific determinism?

0.2867342976490001644822966639633430739550O

- The precise location at which bits cease to be real is unimportant and <u>independent of technology</u>. Eventually, it depends on quantum theory.

Whether Newtonian classical mechanics is deterministic or not, is not a scientific question; it depends on the physical significance one associates with mathematical real numbers

- The high success of Newton's astronomy was in one way an intellectual disaster: it produced an illusion… for this gave the impression that we had here an ideal of scientific explanation; whereas the truth was, it was mere obligingness on the part of the solar system, by having had so peaceful a history in recorded time, to provide such a model. (Elizabeth Anscombe, 1971)

Supplementary variables

- Instead of "God playing dice" when potentialities become actual, God played all dice at the initial time and coded all results in the initial condition.
- *We face a choice: either the fact that at present certain things happen and others do not is interpreted as revealing, retroactively, information about long past initial conditions, or else, we understand the present as the result of indeterminate reality, and the future as open.*

Real Numbers are the Hidden Variables of Classical Mechanics,
(Quantum Studies: Mathematics and Foundations (2019))

- *The fact is that almost all physicists accept real numbers – without noticing that they are supplementary variables – while simultaneously reject Bohmian positions as unnecessary.*

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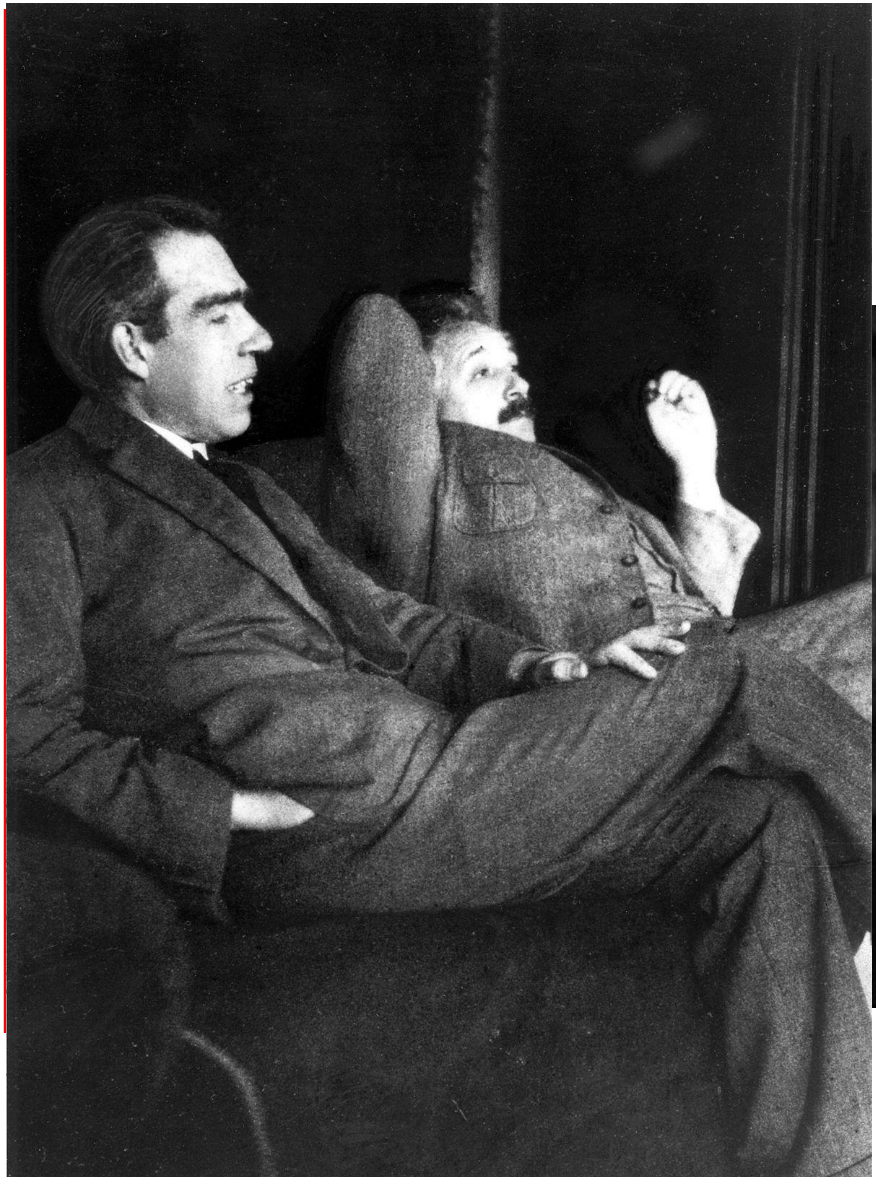
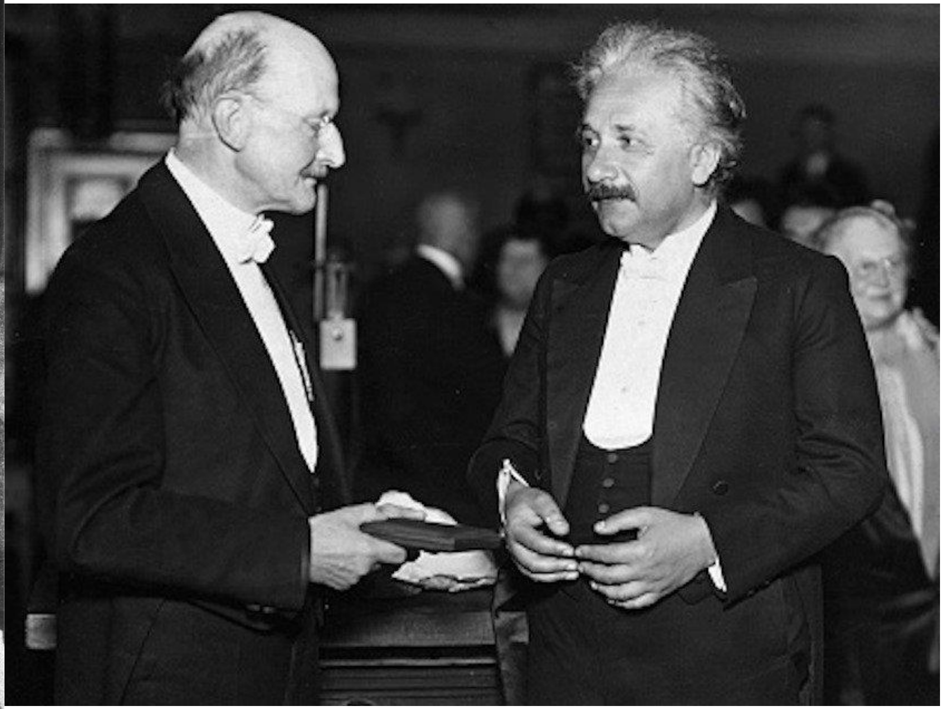
Einstein – Bohr



Einstein – Bergson (Paris 1922)



The Classical and Intuitionistic continuum

D. Hilbert

L.E.J. Brouwer

Classical mathematics:
Every real number is an individual completed entity.
All digits are given at once.
The continuum is a collection of individual points.

Real numbers exists outside of time in some ideal Platonistic world.

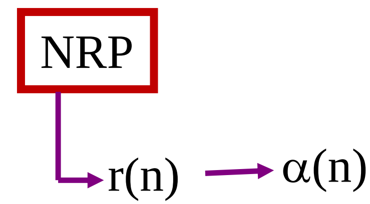
Intuitionistic mathematics:
Real numbers are processes that develop in time.
Digits are not all given at once (except computable numbers).
The continuum is a “viscous collection” of processes.

Time is essential to intuitionism: at any instant only finite information exists.

Intuitionistic Mathematics

Choice sequences $\alpha(n)$, choices made by an idealized mathematician

Nature has the power to produce true randomness, i.e. to produce new information
(what else?)



At each time step (instant) n a Natural Random Process (NRP) outputs some random number r(n).

$\alpha(n)=fct(\alpha(n-1),n,r(1),\ldots,r(n)) \in$ computable number,

where fct is a computable function and the series $\alpha(n)$ converges (with probability 1).

e.g. $|\alpha(n)-\alpha(n-1)| \leq 2^{-n}$

My definition is close to what Brouwer called «projections of choice sequences» or «Intentional sequences».
The function fct plays the role of Brouwer's spreads & fans.

Examples of intuitionistic numbers

1. Totally random numbers

$\alpha(n)=\alpha(n-1)+r(n)\cdot 2^{-n} = 0.b_1 b_2 b_3 b_4 \ldots . b_n$

2. Computable numbers

$\alpha(n)=fct(\alpha(n-1),n,r(1)..r(k)) \Rightarrow \alpha(n)$, $n>>k$, is a pseudo-random series, where r(1)..r(k) is the seed.

π typical computable number

- All bit[...]ully determ[...]) algorithm.
- It take[...]t every bit can be
- The bi[...] *look random*, [...]den in the algorithm. For example the following algorithm allows one to *compute the n^{th} hexadecimal of π without computing the previous bits*:

The bits of π don't come one after the other: they are already all there !

$$\pi = \sum_{k\geq 0} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)$$

$$= 3.1415926...9821480865...$$

https://en.wikipedia.org/wiki/Bailey-Borwein-Plouffe_formula

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Examples of intuitionistic numbers

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3. Finite Information Quantities (FIQs) PRA 100.6 (2019)

$$\alpha(n) = \alpha(n-1) + \text{majority}\{r(n), r(n-1), \ldots, r(n-k+1)\} \cdot 2^{-n}$$

$$= 0.\underbrace{q_1 q_2 q_3 \ldots q_n}_{=\,0,1\ \text{fully determined}}\ \underbrace{q_{n+1} \ldots q_{n+k-1}}_{=\,\text{fraction propensities}}\ \underbrace{q_{n+k} q_{n+k+1} \ldots}_{=\,1/2\ \text{totally undetermined}}$$

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Natural RP

k=5

r(n)	α(n)
1	0.0
1	0.00
0	0.000
1	0.0000
1	0.00001(1,3/4,7/8,11/16)
0	0.000011(1/2,3/4,1/2,5/16)
0	0.0000110(1/2,1/4,1/8,5/16)
…	…

$r(n) \rightarrow \alpha(n)$

1

1

0

1

1 $\quad 0.q_1q_2q_3q_4\,1$

…

Examples of intuitionistic numbers

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…	…

$r(n)$	$\alpha(n)$
1	
1	
0	
1	
1	$0.q_1q_2q_3q_4\,11$
…	

Examples of intuitionistic numbers

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$r(n) \rightarrow \alpha(n)$

1

1

0

1

1 $\quad 0.q_1q_2q_3q_4\,11\tfrac{3}{4}$

...

Examples of intuitionistic numbers

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1

1

0

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…

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...	...

r(n) → α(n)

1

1

0

1

1 $\quad 0.q_1q_2q_3q_4\,1\,1_{\frac{3}{4}\,\frac{7}{8}\,\frac{11}{16}}$

...

Examples of intuitionistic numbers

4. Mortal numbers

$$\alpha(n)=\frac{1}{2}+(-1)^{r(n)}\cdot 10^{-n}$$

$r(n)=0 \Rightarrow \alpha(n)>1/2$
$r(n)=1 \Rightarrow \alpha(n)<1/2$

until, by chance, the last n/2 random bits r(j) all happen to have the same value, and n≥4 is even, then the series terminates - dies:

$\alpha(n)=\alpha(n-1)$ for all future n.

Because the probability of termination decreases exponentially, there is an a priori probability that the sequence goes on for ever.

In such a case, the proposition $\alpha<1/2$ is neither true nor false, but indeterminate.

Examples of intuitionistic numbers

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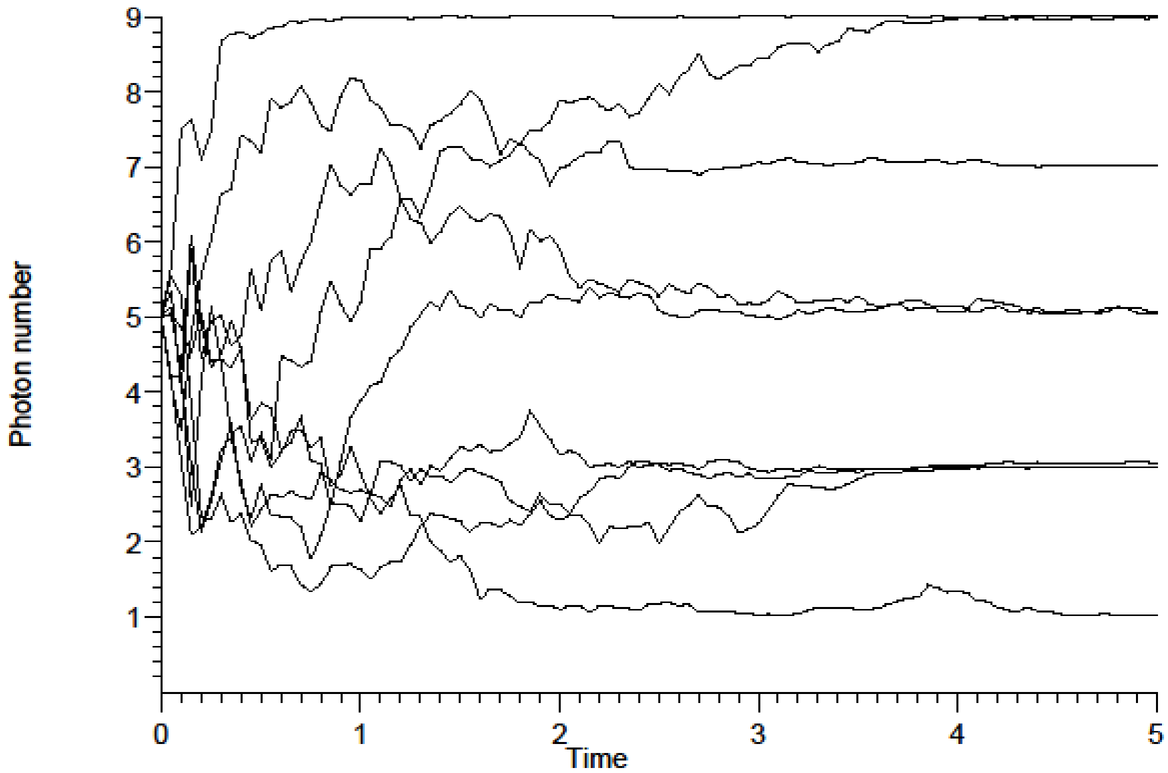
5. Autonomous numbers (fct independent of n)

$$\alpha(n+1)=\alpha(n)+(-1)^{r(n)}\cdot\mu\cdot\alpha(n)\cdot(1-\alpha(n)) \qquad \mu\in[0,1] \text{ rational}$$

1 and 0 are fixed points.

$\alpha(n)$ converges to 1 (0) with probability $\alpha(0)$ $(1-\alpha(0))$

Stochastic Schrödinger equations GRW - QSD



- Each solution converges to an eigenstate, with the standard quantum probabilities.
- Intuitionism brings classical mechanics closer to quantum

Intuitionistic Logic

- Time is essential in intuitionistic mathematics, Stanford Encyclopaedia of Philosophy, *Intuitionism in the Philosophy of Mathematics.*
- The statement r(n)=0 is indeterminate before the n^{th} time step, the n^{th} instant.
- The proposition $\alpha < ½$ is indeterminate as long as the mortal numbers didn't die.
- The proposition «It will rain in exactly one year from now at Piccadilly Circus» is, at present, indeterminate.

$\Rightarrow$ **The law of the excluded middle is not valid.**

$\Rightarrow$ No non-constructive existence proofs.

Surprising !

… but makes plenty of sense in an indeterministic world, i.e. in a world in which the future is open.

- Brouwer: excluded middle fails because the world is not-determinate, so truths about it are indeterminate.

The intuitionistic continuum

- Elements of the continuum (intuitionistic «real numbers») are evolving sequences of computable numbers (Brouwer's choice sequences).
- For some sequences it is never determined (at any finite time) whether they are larger or smaller than ½.
- Hence, one can't cut the continuum into larger and smaller than ½.
- Brouwer's theorem:

  All total functions are continuous (no step fct).
- The intuitionistic continuum is viscous: one can't pick out an individual point.
- The intermediate value theorem does not hold. However,

Theorem

If f is a continuous function such $f(a) < 0$ and $f(b) > 0$ (where $a < b$), then for any $\varepsilon > 0$ one can construct a value of x between a and b, for which $|f(x)| < \varepsilon$.

- The standard proof of Gleason's theorem does not hold intuitionistically. However, F. Richman proved that Gleason's original formulation is in fact constructively provable.

Intuitionistic arithmetic

- Given $\alpha(n)$ & $\beta(n)$, define $\alpha(n)+\beta(n)$ & $\sin(\alpha(n))$, …
- $\alpha>\beta$ iff one can <u>construct</u> two integers k and n s.t. for all $m>0$ $\alpha(n+m) - \beta(n+m) > 2^{-k}$
- The relation > defines only a partial order :

  $(\alpha=\beta) \vee (\alpha<\beta) \vee (\alpha>\beta)$ does not always hold.
- α,β are apart, $\alpha\#\beta$, iff one can <u>construct</u> two integers k and n such that for all $m>0$

  $|\alpha(n+m) - \beta(n+m)| > 2^{-k}$
- Intuitionism: not($\alpha\#\beta$) implies ($\alpha=\beta$),

  but: not($\alpha=\beta$) does not imply ($\alpha\#\beta$)

The mathematical language we use when speaking physics has a huge influence on the world-view that physics presents to us.

Indeterministic Physics	Intuitionistic Mathematics
Past, present and future are **not** all given at once	Digits of real numbers are **not** all given at once
Time passes	Numbers are processes
Indeterminism	Numbers contain finite information
The present is thick	The continuum is viscous
The future is open	No law of the excluded middle
Becoming	Choice sequences
Experiencing	Intuitionism

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Sabine Hossenfelder (youtube)

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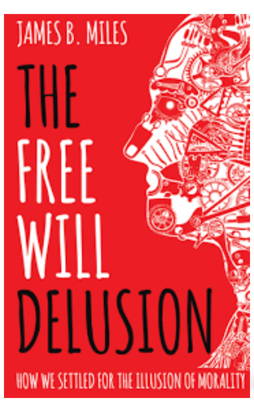
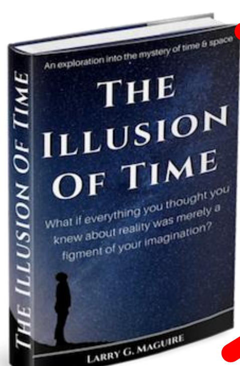
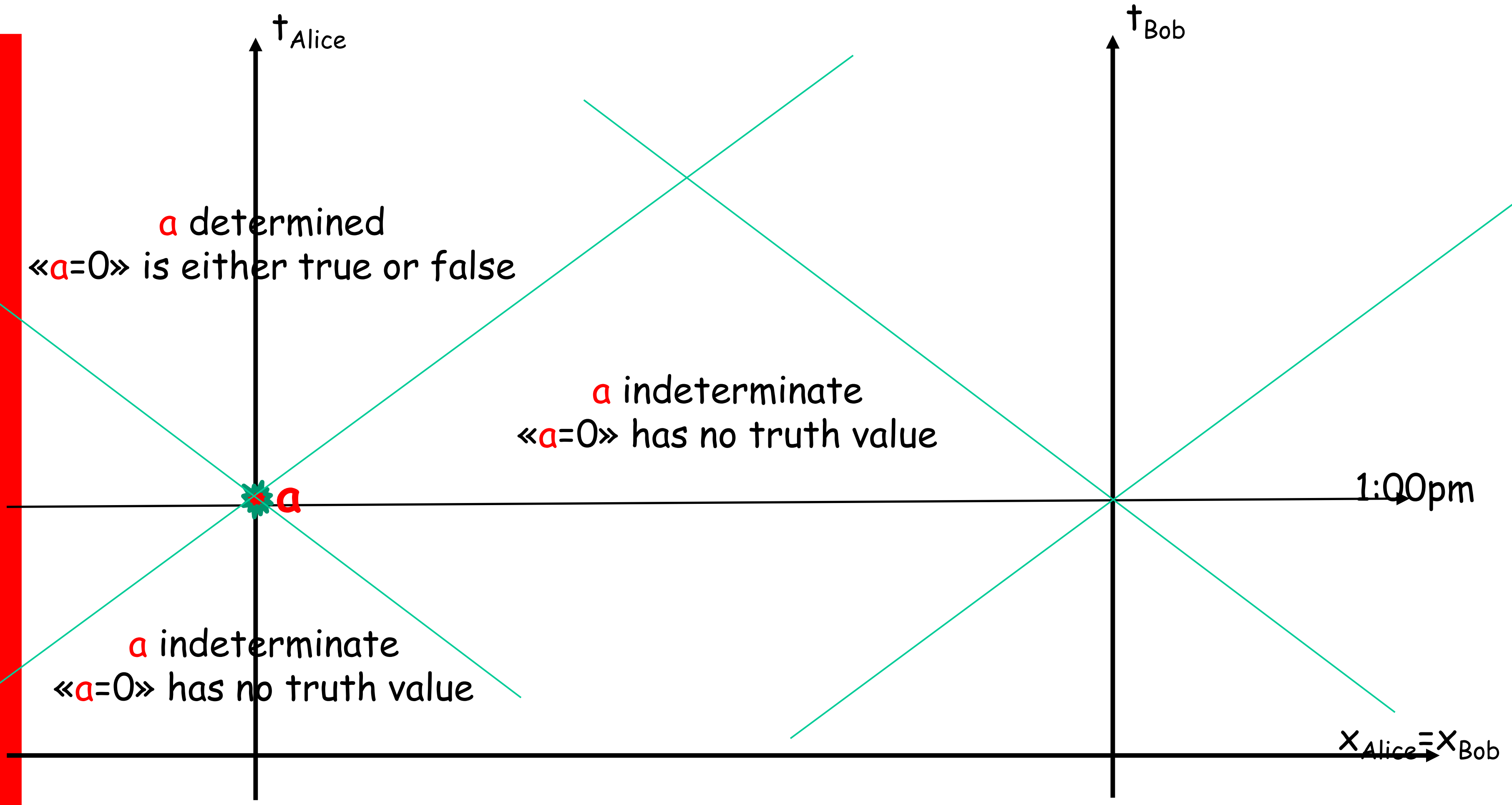
- **You are here to hear what Science says.**
- Initial conditions determine all events, including chaotic systems.
- **… the entire story of the universe was already determined at the big bang.**
- *~~We~~ are just watching it playing out.*
- **We know the future is determined by the present !**
- If you think otherwise, you are denying scientific evidence.





The Relativity of Indeterminacy



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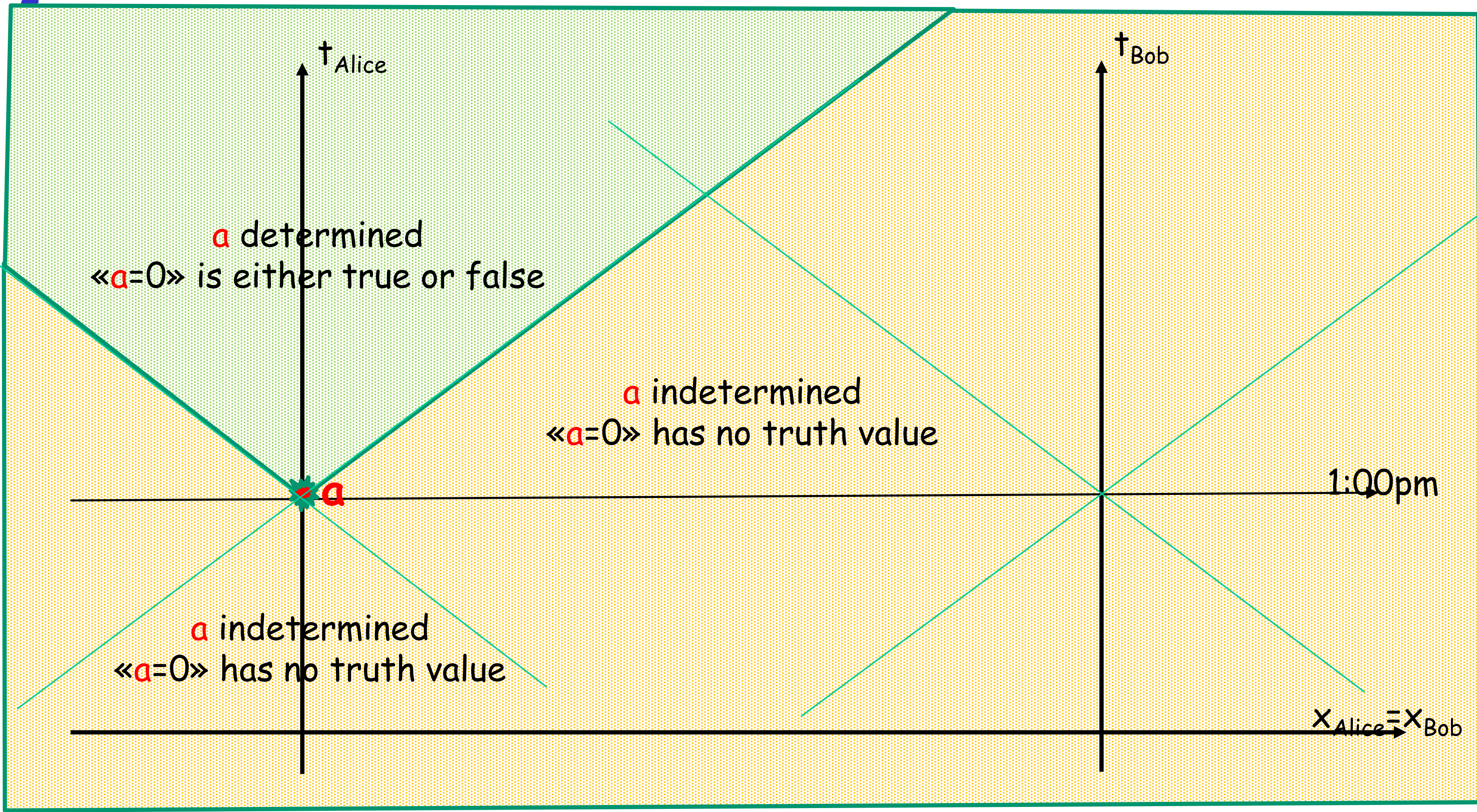
Flavio Del Santo and N. Gisin, The Relativity of Indeterminacy, arXiv:2101.04134

The Relativity of Indeterminacy



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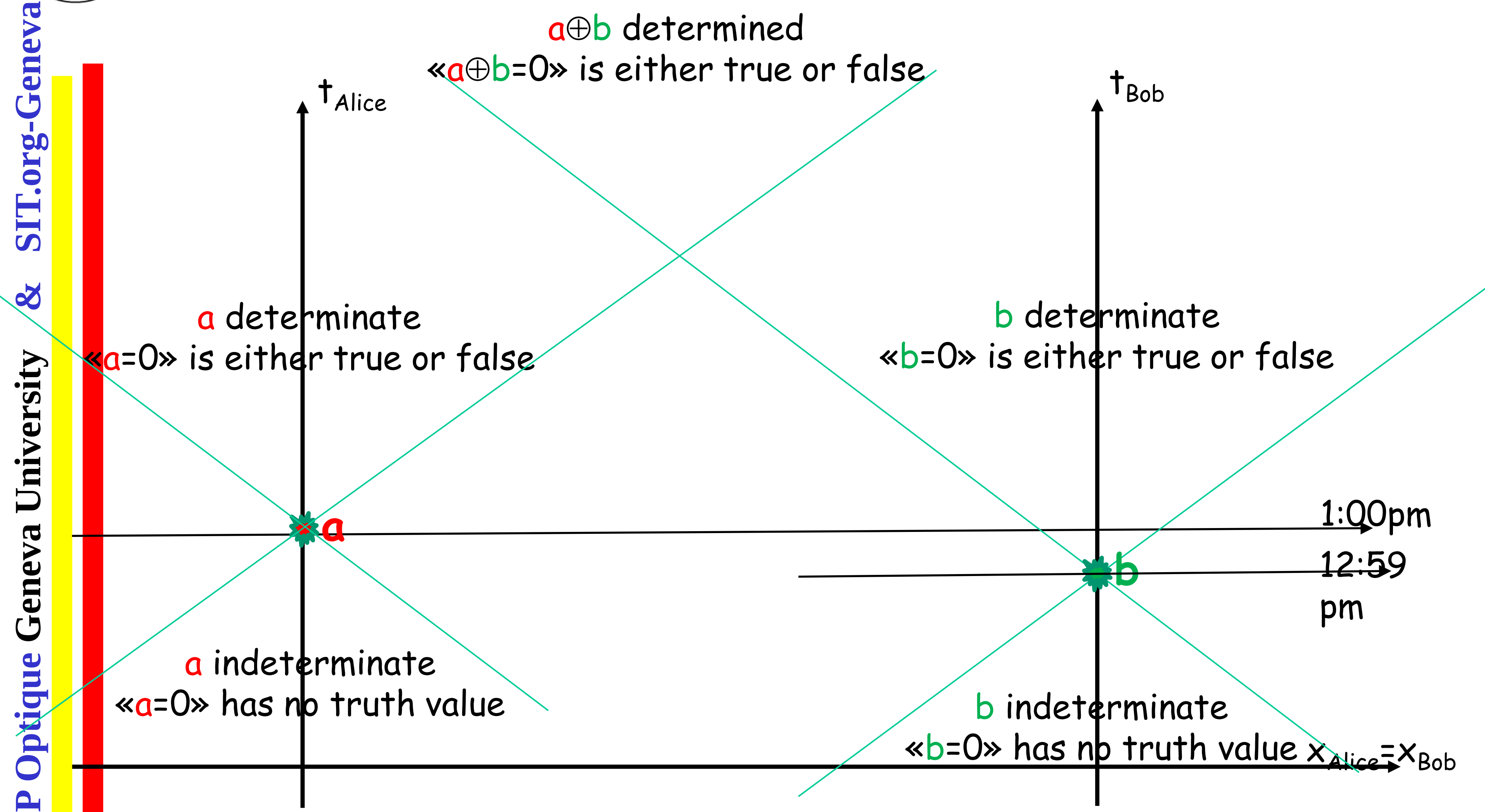

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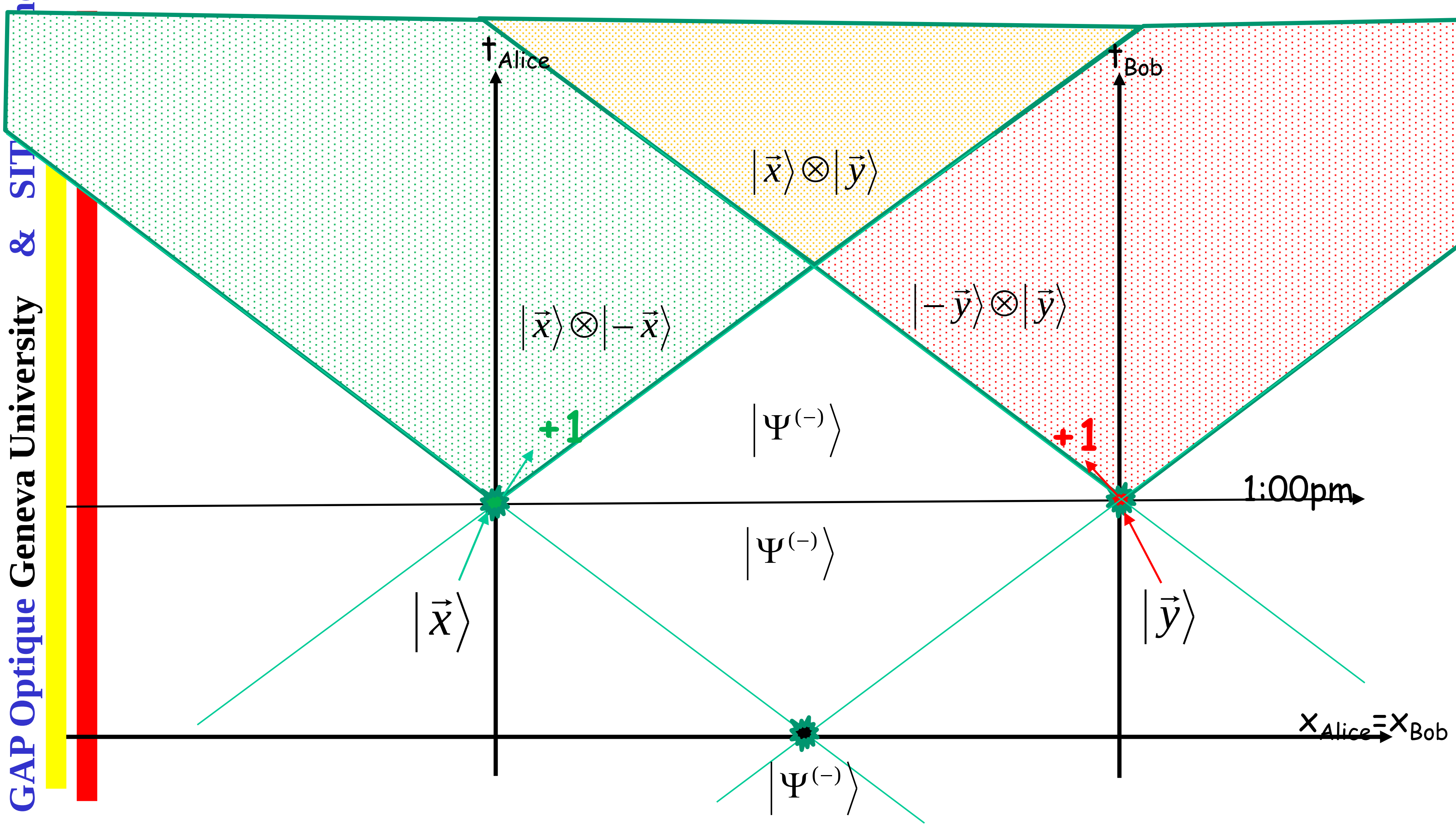

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Conclusion

- *The mathematical language we speak when "talking physics" impacts on our world-view:*
  - with classical mathematics classical mechanics is deterministic,
  - with intuitionistic mathematics it is indeterministic.
- *- Classical real numbers are the hidden variables of classical mechanics.*
  - Classical mathematics assumes a view from the end of time.
- *Intuitionism brings classical closer to quantum.*
- *Indeterminacy is relative.*
- **E. Bishop: Classical mathematics can be regarded as a branch of constructive mathematics, whereas to regard constructive mathematics as a branch of classical mathematics is not possible.**

- C. Posy, Mathematical Intuitionism, Cambridge Univ. Press, 2020.
- N. Gisin, *Indeterminism in Physics, Classical Chaos and Bohmian Mechanics. Are Real Numbers Really Real?*, Erkenntnis (2019).
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- Flavio Del Santo and N. Gisin, *Physics without determinism: Alternative interpretations of classical physics*, Physical Review A 100.6 (2019).
- N. Gisin, *Mathematical languages shape our understanding of time in physics*, Nature Physics 16, 114-119 (2020).
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Axiom of choice

- For all collections of non-empty sets $\{A_x\}_x$ there exist a function F such that
$$F(A_x) \in A_x \quad \text{for all } x$$
- Looks obvious as the sets A_x are non-empty for all x.
- Looks absurd as it implies the possibility to make a continuous infinity of choices in a finite time (actually in no time).

- The axiom of choice is intuitionistically absurd.

Axiom of choice

- The axiom of choice gives us a superb feeling of power: in a single stroke one can make a continuous infinity of choices.
- Very nice feeling, indeed.
- But when applied to the real world absurd consequences follow: there is a function that predicts with probability 1 all future events, including outcomes of a true random number generator.
- $\Rightarrow$ keep the AoC for Platonistic math, but reject it for mathematical physics.

Ch. Hardin and A. Taylor, *A peculiar connection between the axiom of choice and predicting the future*, Am. Math. Monthly **115**, 91-96 (2008)