

Heavy-flavour jets: A roadmap towards precision physics

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Jets and their substructure from LHC data

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Netherlands Organisation
for Scientific Research



Preliminaries

Identify the **flavour & kinematic** information of a hadronic jet



Use this information to probe underlying physics effects

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Identify the **flavour & kinematic** information of a hadronic jet



Use this information to probe underlying physics effects

Definition of a 'tag' and theory formalism depends on the goal

(i) **Absolute cross-sections of processes with flavoured jets**

This talk

(ii) **Jet fragmentation**

Bauer, et al.: [1312.5605](#), Bain et al.: [1603.06891](#), [1610.06508](#), Dai et al.: [1805.06014](#), [2104.14707](#), Markis et al.: [1807.09805](#)

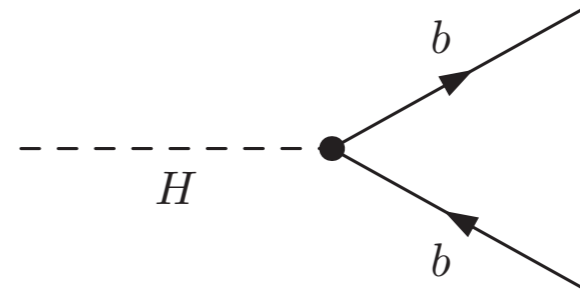
(iii) **Medium effects:**

Tao, Vitev: [1811.07905](#)

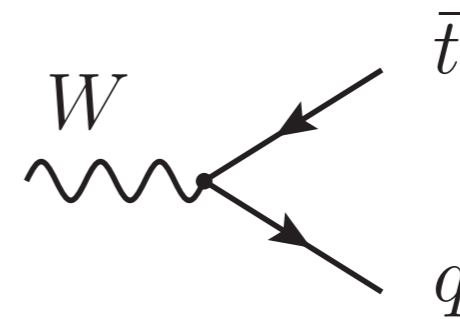
Preliminaries

Absolute cross-sections of processes with flavoured jets

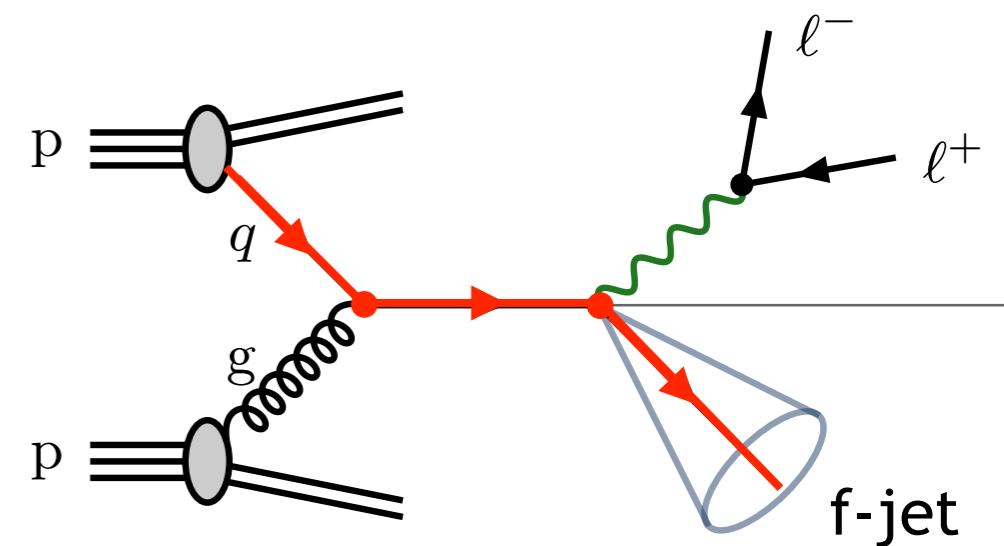
(i) Higgs physics (hadronic decays)



(ii) Top-quark physics ($|V_{tb}| \sim 1$)



(iii) New physics searches (f-jet + E_T^{miss})



(iv) Gauge-boson + heavy-flavour

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Absolute cross-sections of processes with flavoured jets

- (i) Higgs physics (hadronic decays) [Higgs couplings]
- (ii) Top-quark physics ($|V_{tb}| \sim 1$) [PDFs, α_s , BSM]
- (iii) New physics searches (f-jet + E_T^{miss}) [BSM]
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- in-direct (decays)**
direct
-

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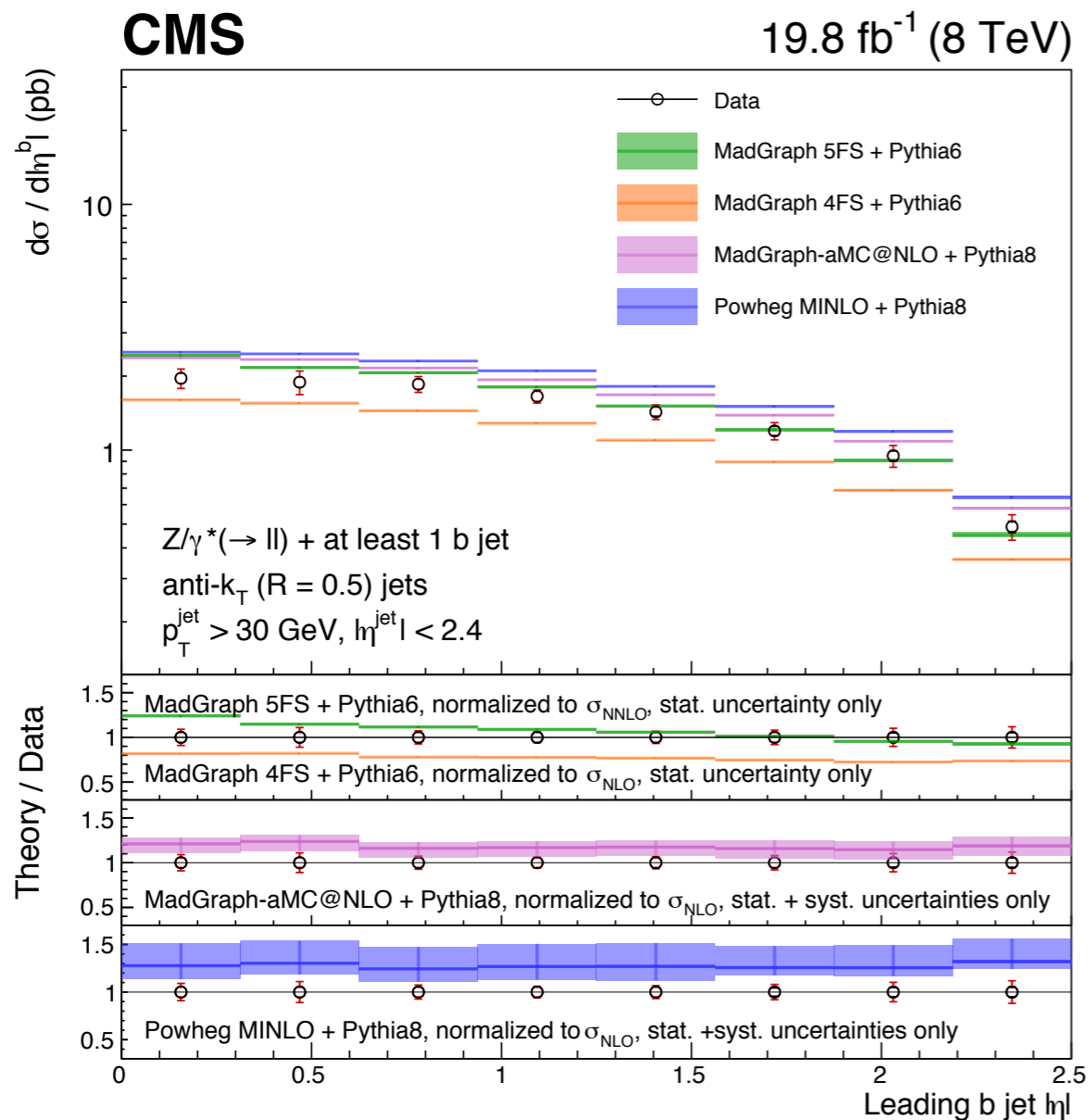
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(iv) Gauge-boson + heavy-flavour [PDFs, Theory framework]
[Dominant background for (i)-(iii)]

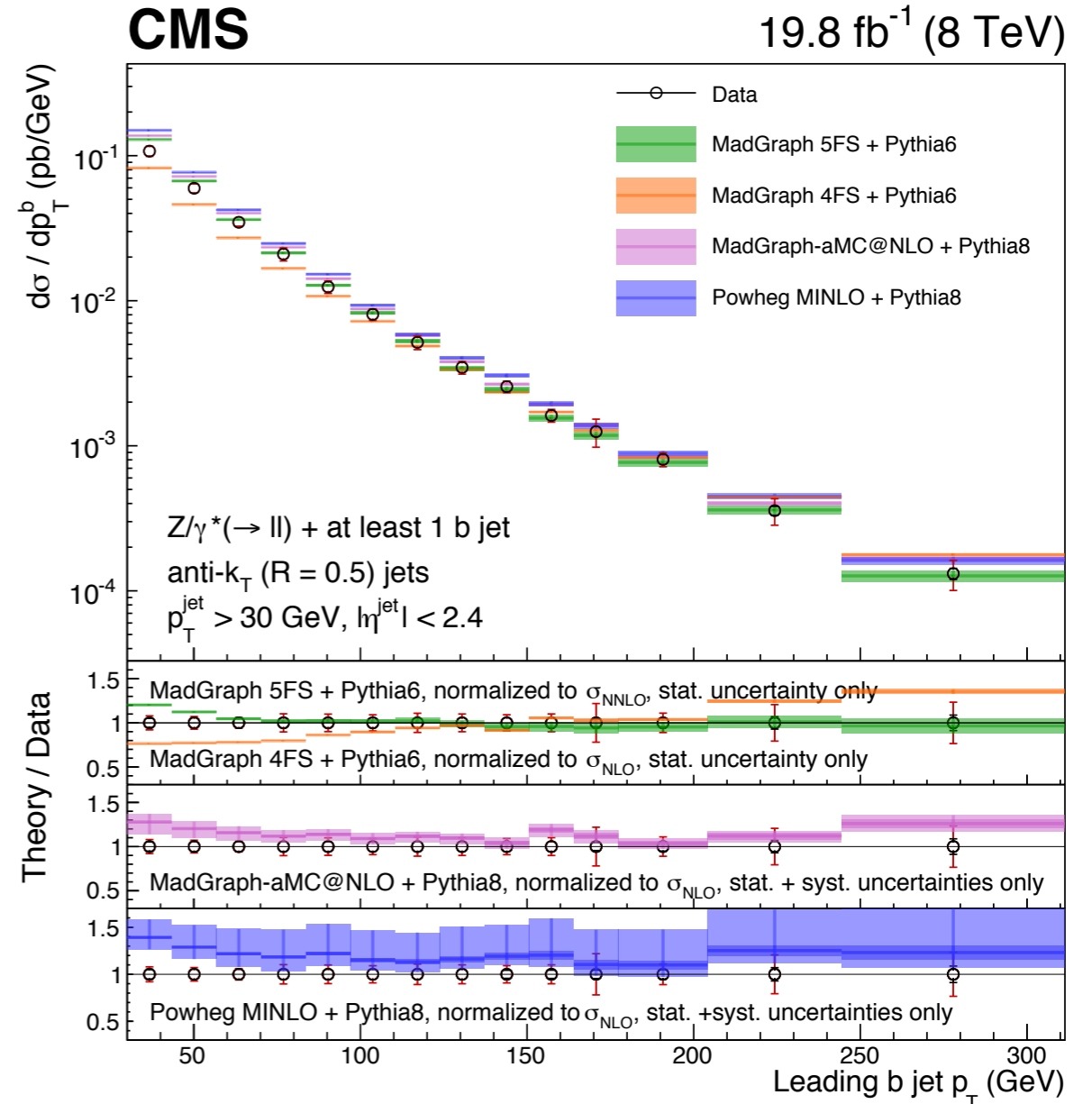
in-direct (decays)

direct

(Our) alternative motivation



η_b



$p_{T,b}$

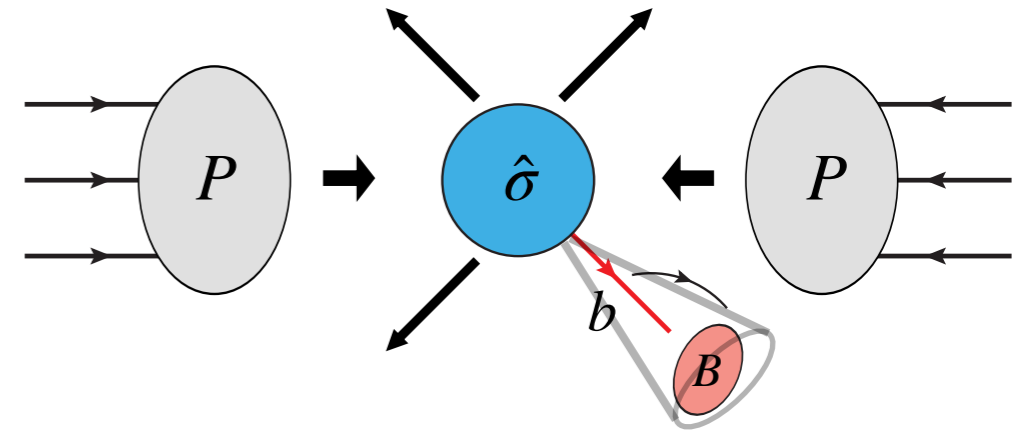
CMS Run I (8 TeV) measurement of $pp \rightarrow Z + b - \text{jet}(s)$ [Eur. Phys. J. C 77, 11 \(2017\) 751](#)

“Our” NNLOJET [Gauld, Gehrmann-De Ridder, Glover, Huss, Majer] [PRL 125 \(2020\) 22, 222002](#)

Overview of remainder of talk

Direct heavy-flavour production

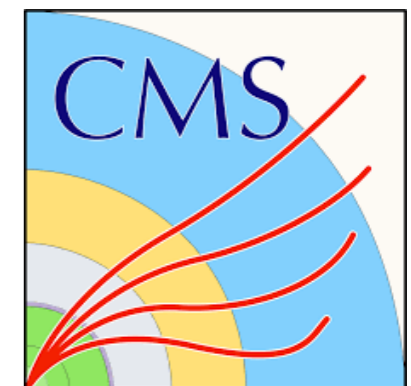
- ▶ The 'flavour scheme'
- ▶ Heavy-flavour PDFs



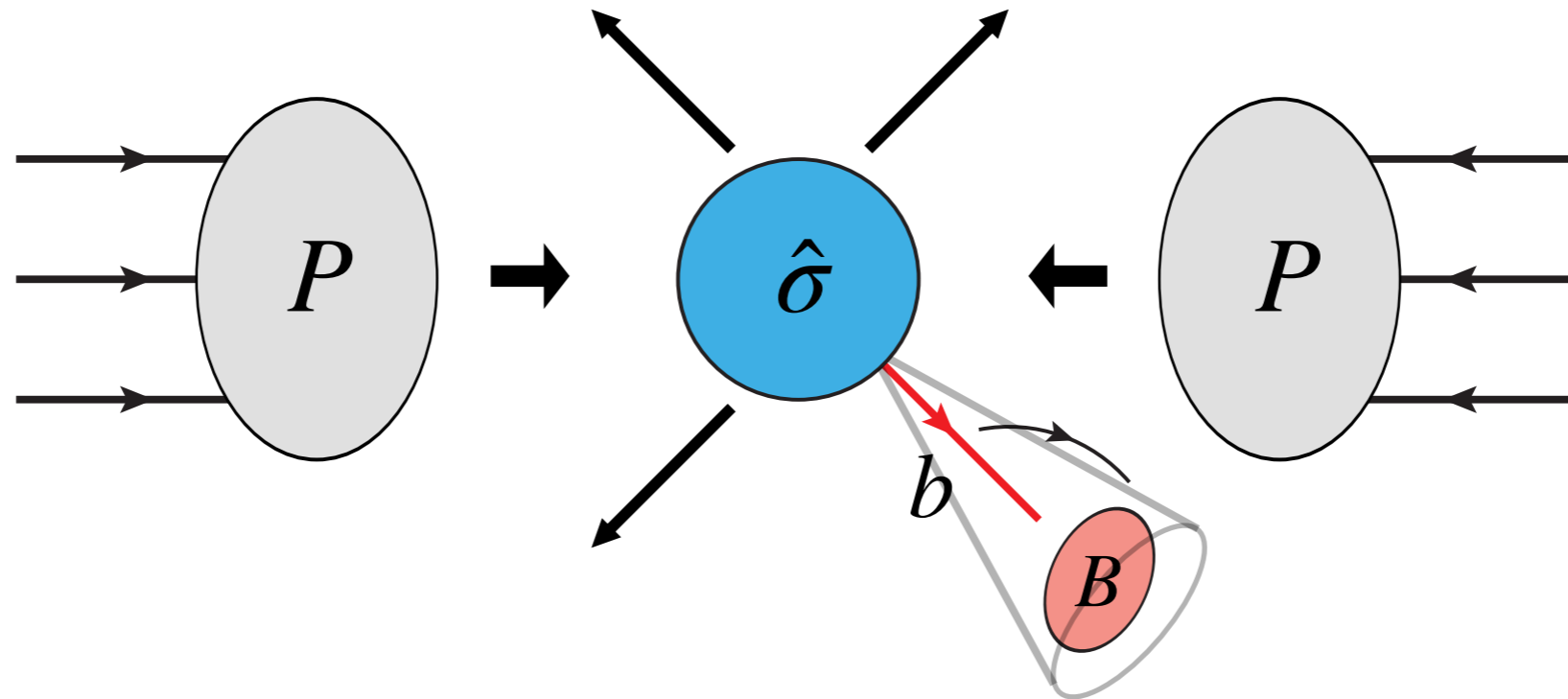
Flavoured jets & perturbative computations

- ▶ **InfraRed** and **C**ollinear safety
- ▶ Theory state-of-the-art

Theory meets experiment



Anatomy of heavy-flavour processes



Factorisation theorem

$$d\sigma_{pp \rightarrow X} = \sum_{i,j} \int dx_1 dx_2 f_i(x_1) f_j(x_2) d\hat{\sigma}_{ij \rightarrow \hat{X}}(\hat{s}, \dots) T(\hat{X} \rightarrow X)$$

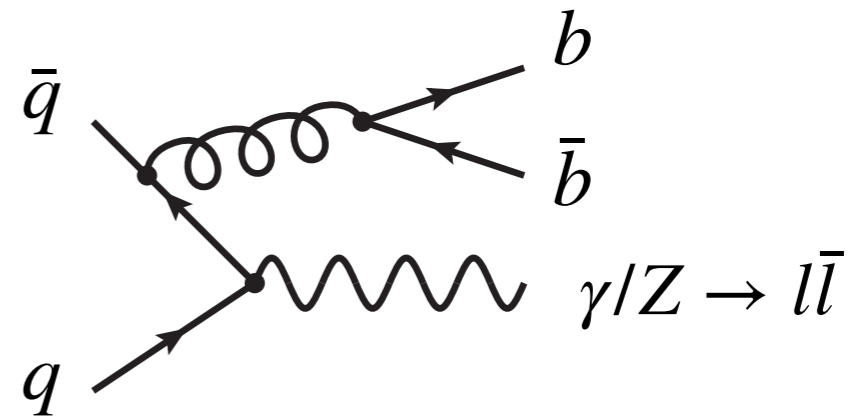
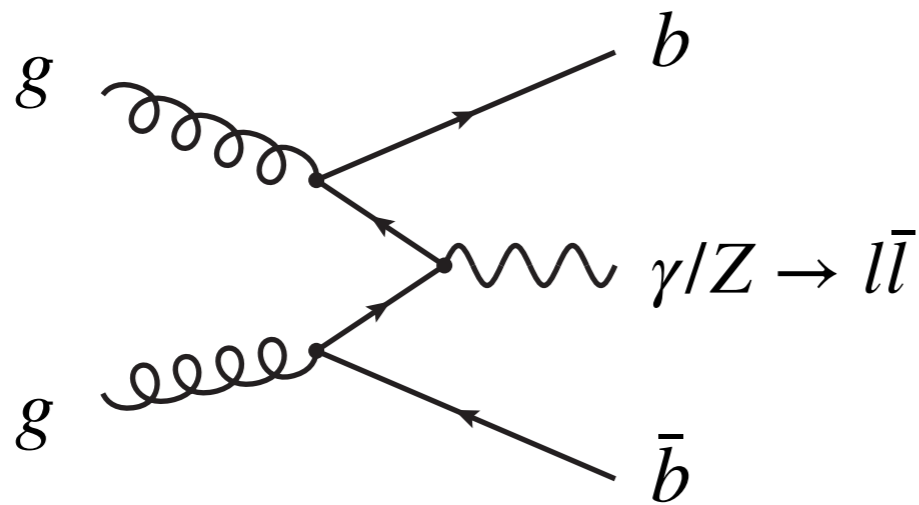
Partonic cross-section

$$d\hat{\sigma}_{ij \rightarrow \hat{X}} = d\hat{\sigma}_{ij \rightarrow \hat{X}}^{\text{LO}} + \alpha_s d\hat{\sigma}_{ij \rightarrow \hat{X}}^{\text{NLO}} + \alpha_s^2 d\hat{\sigma}_{ij \rightarrow \hat{X}}^{\text{NNLO}} + \dots$$

At fixed-order, define parton-level jets [corrected with $T(\hat{X} \rightarrow X)$]

A theoretical bypass: Z+b-jet

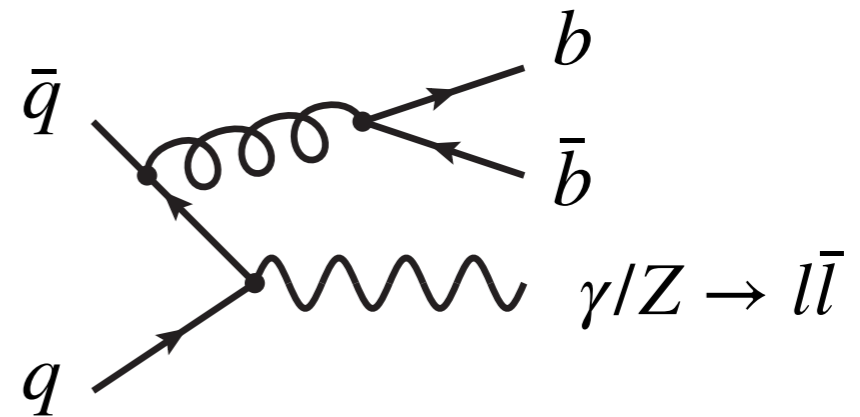
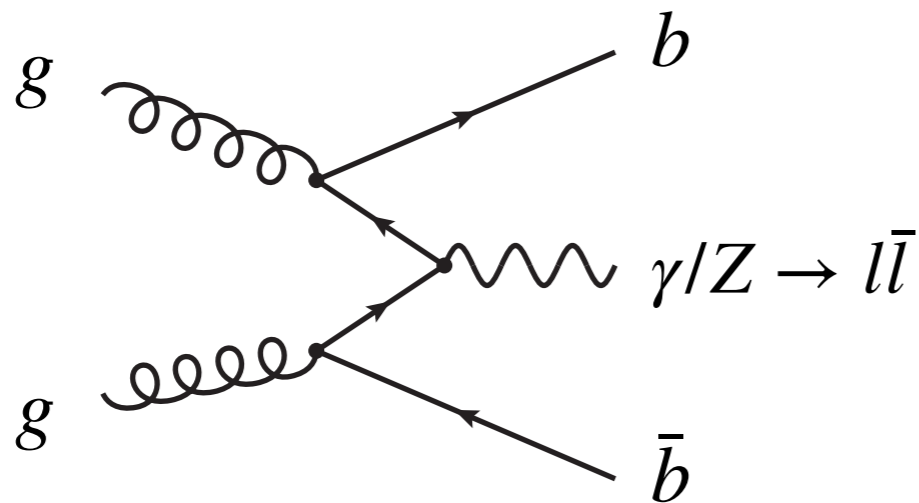
Computed in a scheme: e.g. “4fs” aka ZM-VFNS ($n_f^{\max} = 4$)



LO computation in 4fs $\mathcal{O}(\alpha_s^2)$, $d\hat{\sigma}_{ij}^{\text{LO}} \sim \int d\phi_{\hat{X}} |M_{ij \rightarrow \hat{X}}|^2$

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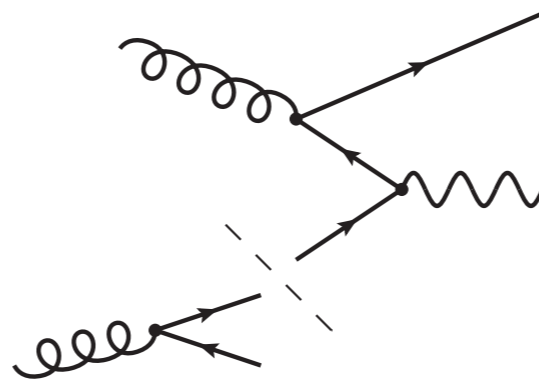
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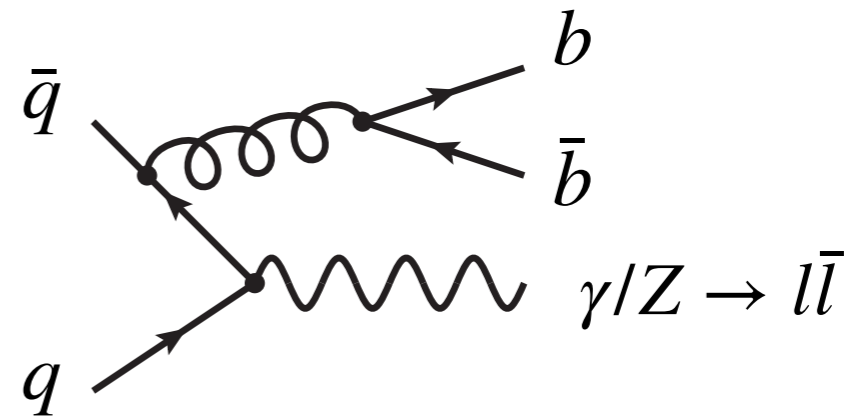
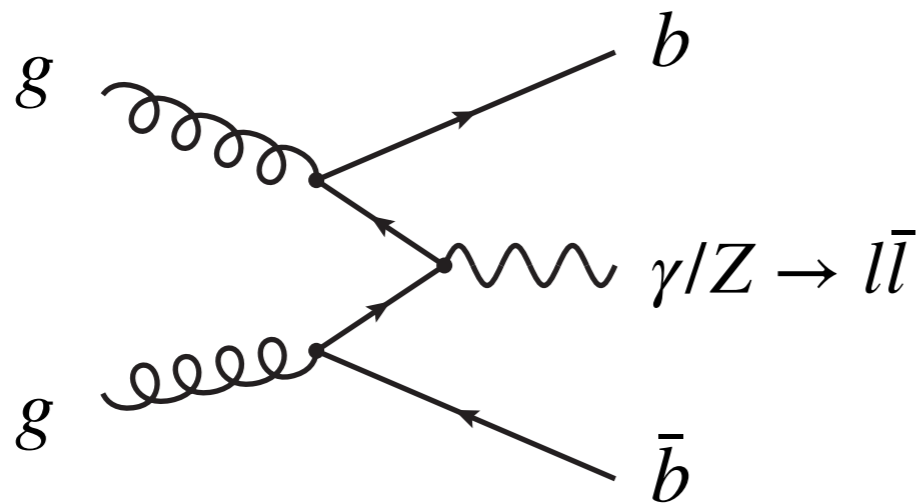
Massless component
 $\mathcal{O}(\alpha_s^2 n_f)$ in 5fs



$\mathcal{O}(m_b^2)$ effects
exact kinematics

A theoretical bypass: Z+b-jet

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100% = -32%

+135%

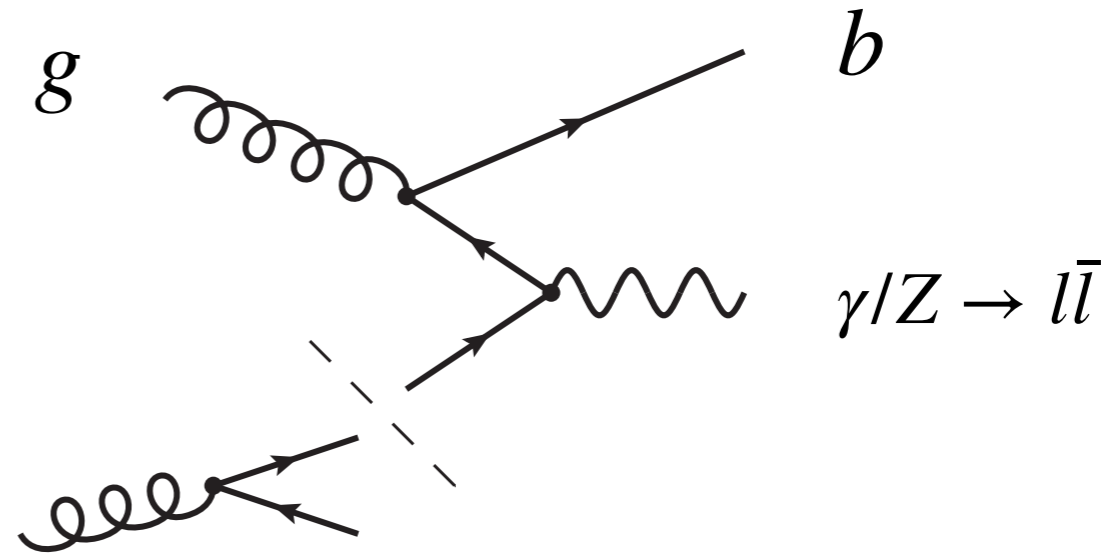
-3%

The b-quark PDF

Clearly, we wish to understand the logarithmic part (it is the largest)

$$d\sigma^{\ln[m_b]} \sim$$

$(g \rightarrow b)_{||}$



$$d\sigma^{\ln[m_b]} \sim f_g(x_1, \mu_F^2) \otimes f_b^{(1)}(x_2, \mu_F^2) \otimes d\hat{\sigma}_{bg \rightarrow Zb}^{m_b=0}$$

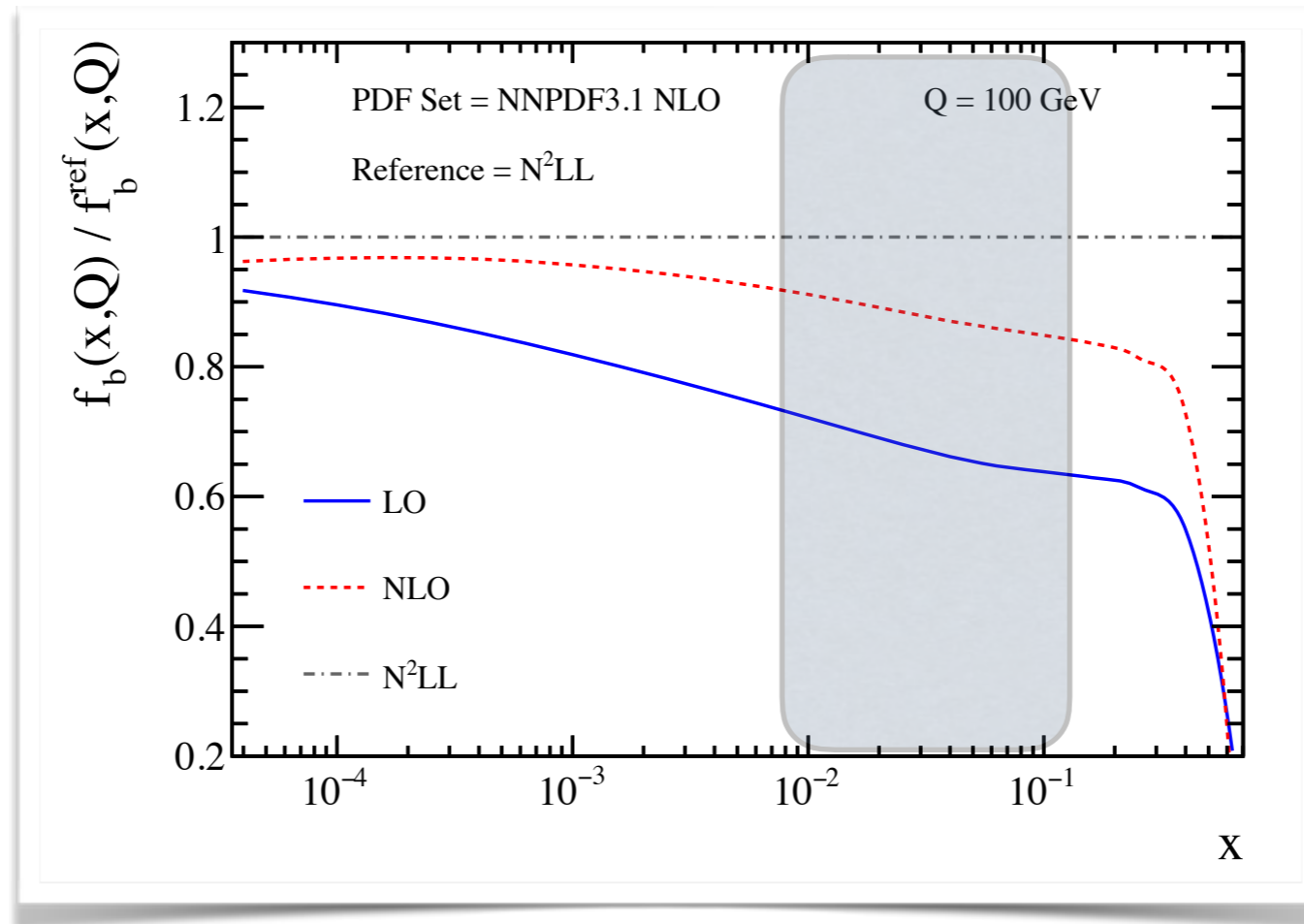
the fixed-order b-quark pdf

$$f_b^{(1)}(x, \mu_F^2) = \frac{\alpha_s}{2\pi} \left(P_{g \rightarrow q}^{(0)} \otimes f_g \right) [x, m_b^2] \ln \left[\frac{\mu_F^2}{m_b^2} \right]$$

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Clearly, we wish to understand the logarithmic part (it is the largest)

We do know it well (use the renormalisation group, “PDF evolution”)



I am showing fixed-order pdf versus a resummed one (PDF evolution)

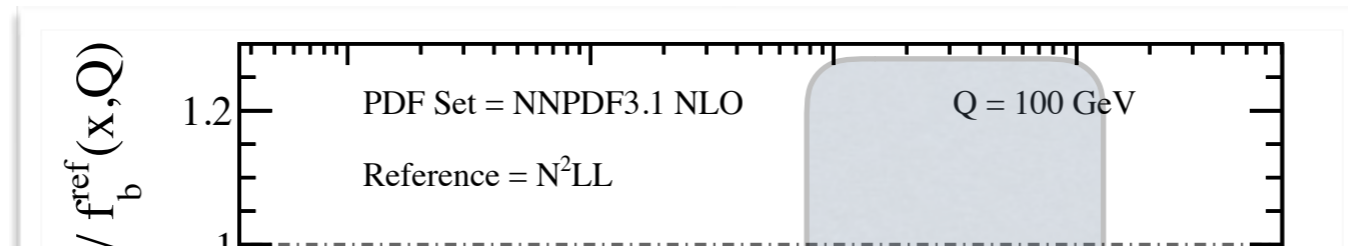
$$\alpha_s^m \ln^n[\mu_F^2/m_b^2], \quad m \geq n$$

$$\text{Note! } \alpha_s \ln[m_Z^2/m_b^2] \approx 0.7$$

The b-quark PDF

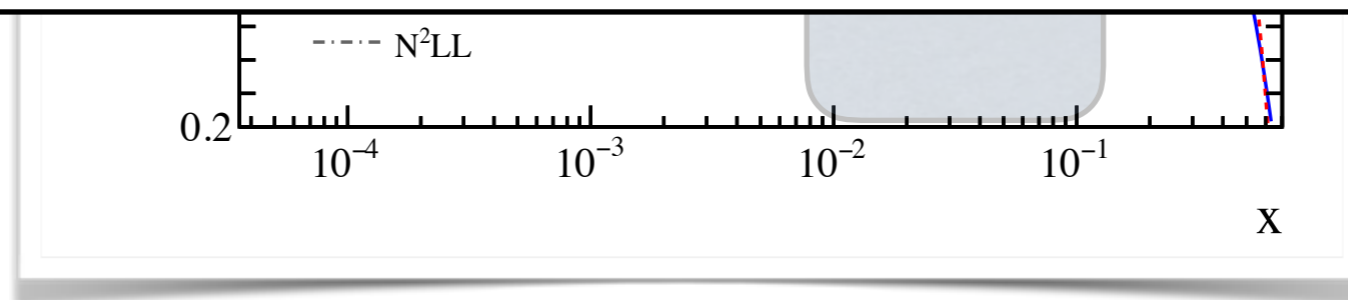
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Resumming the PDF **highly** desirable (essential?)

aka: the massless computation (w/ QCD corrections to $\ln[m_b]$ part)



I am showing fixed-order pdf versus a resummed one (PDF evolution)

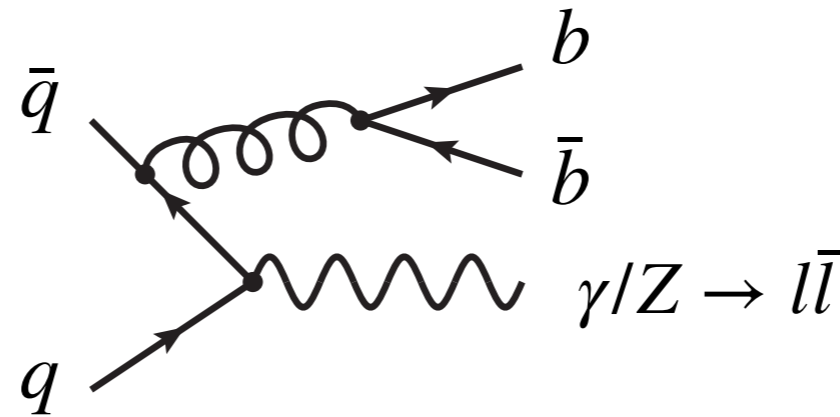
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InfraRed and Collinear safety

What happens if we apply anti- k_T alg. as in an experimental set-up

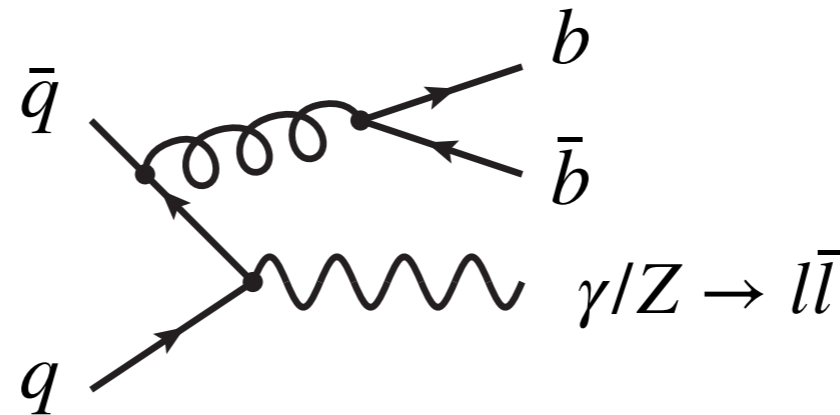
Collinear safety



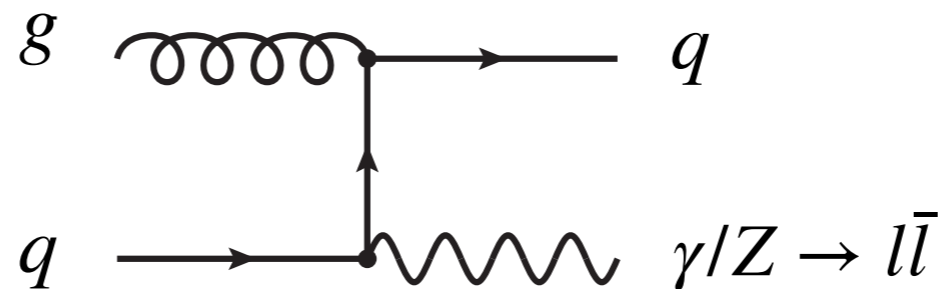
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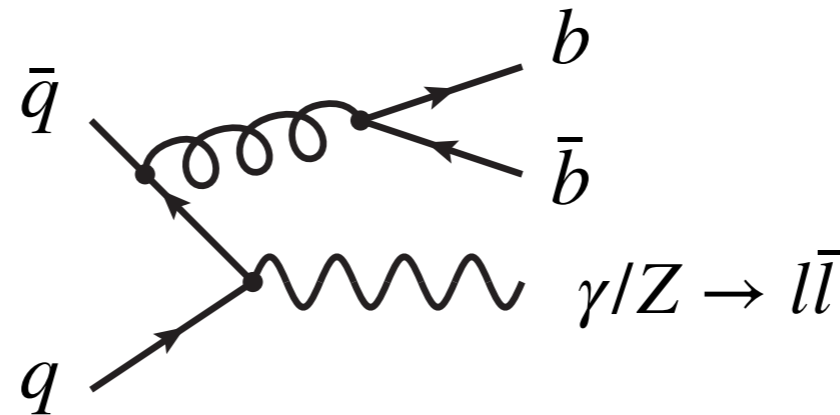
Soft safety



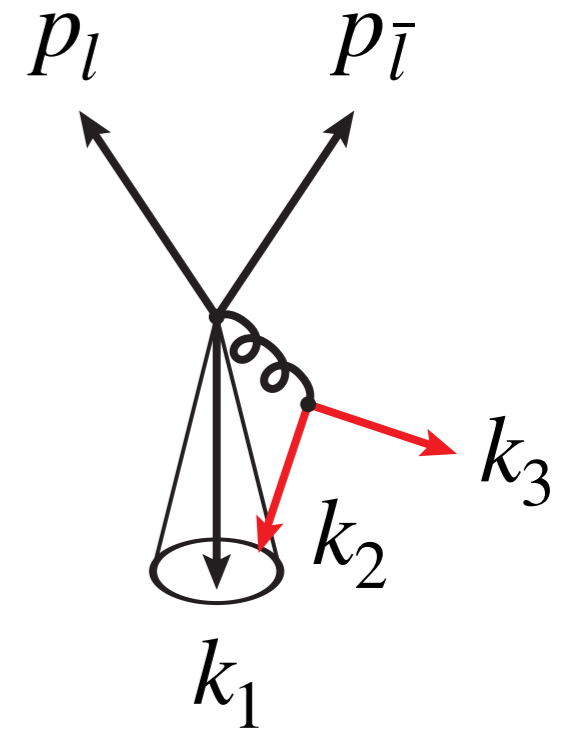
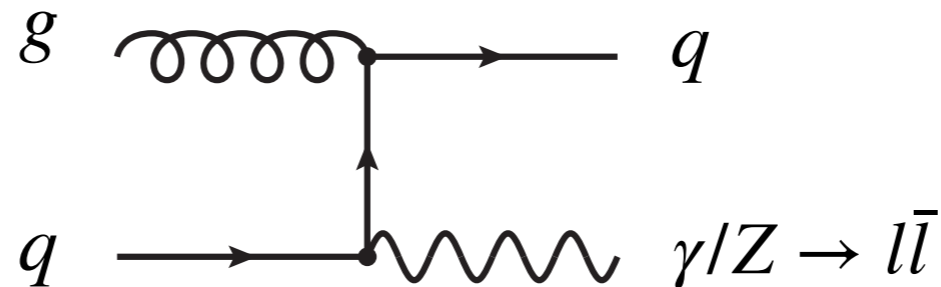
InfraRed and Collinear safety

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Collinear safety



Soft safety



Massless(5fs): infinite

Massive(4fs): finite, but contains large corrections like

$$C \propto \alpha_s \ln[Q^2/m_b^2]$$

$$S \propto \alpha_s^2 \ln^2[Q^2/m_b^2]$$

The flavour- k_T algorithm

Target: resummed heavy-flavour PDFs and avoid large “C, S” corrections

Theoretical Physics | Published: 19 May 2006

Infrared-safe definition of jet flavour

[A. Banfi](#) , [G.P. Salam](#) & [G. Zanderighi](#)

[The European Physical Journal C - Particles and Fields](#) **47**, 113–124(2006) | [Cite this article](#)

109 Accesses | **71** Citations | [Metrics](#)

(1) Quantum flavour assignment:

$$b = +1, \bar{b} = -1$$

(2) Flavour specific clustering

$$d_{ij} = \frac{\Delta y_{ij}^2 + \Delta \phi_{ij}^2}{R^2} \begin{cases} \max(k_{ti}, k_{tj})^\alpha \min(k_{ti}, k_{tj})^{2-\alpha} & \text{softer of } i, j \text{ is flavoured,} \\ \min(k_{ti}, k_{tj})^\alpha & \text{softer of } i, j \text{ is unflavoured.} \end{cases}$$

Note! anti- k_T clustering not well suited (preference of soft-hard clustering)

Theory state-of-the-art (f-jets @ NNLO)

$$d\hat{\sigma}_{ij \rightarrow \hat{X}} = d\hat{\sigma}_{ij \rightarrow \hat{X}}^{\text{LO}} + \alpha_s d\hat{\sigma}_{ij \rightarrow \hat{X}}^{\text{NLO}} + \alpha_s^2 d\hat{\sigma}_{ij \rightarrow \hat{X}}^{\text{NNLO}} + \dots$$

$V + (H \rightarrow b\bar{b})$	Ferrera et al. (1705.10304), Caola et al. (1712.06974), Gauld et al. (1907.05836)
$Z + b - \text{jet}$	Gauld et al. (2005.03016)
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flavoured-jet algorithm applied

anti- k_T algorithm applied (regulated by m_b , a tech. cut, or 'prescription')

$t\bar{t}$ with decay	Behring et al. (1901.05407), Czakon et al. (2008.11133)
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Theory meets experiment

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Theory: parton-level flavour- k_T jets

Data: hadron-level anti- k_T jets

A correction (which may be small) **is** required for these calculations

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Unfold data with RooUnfold

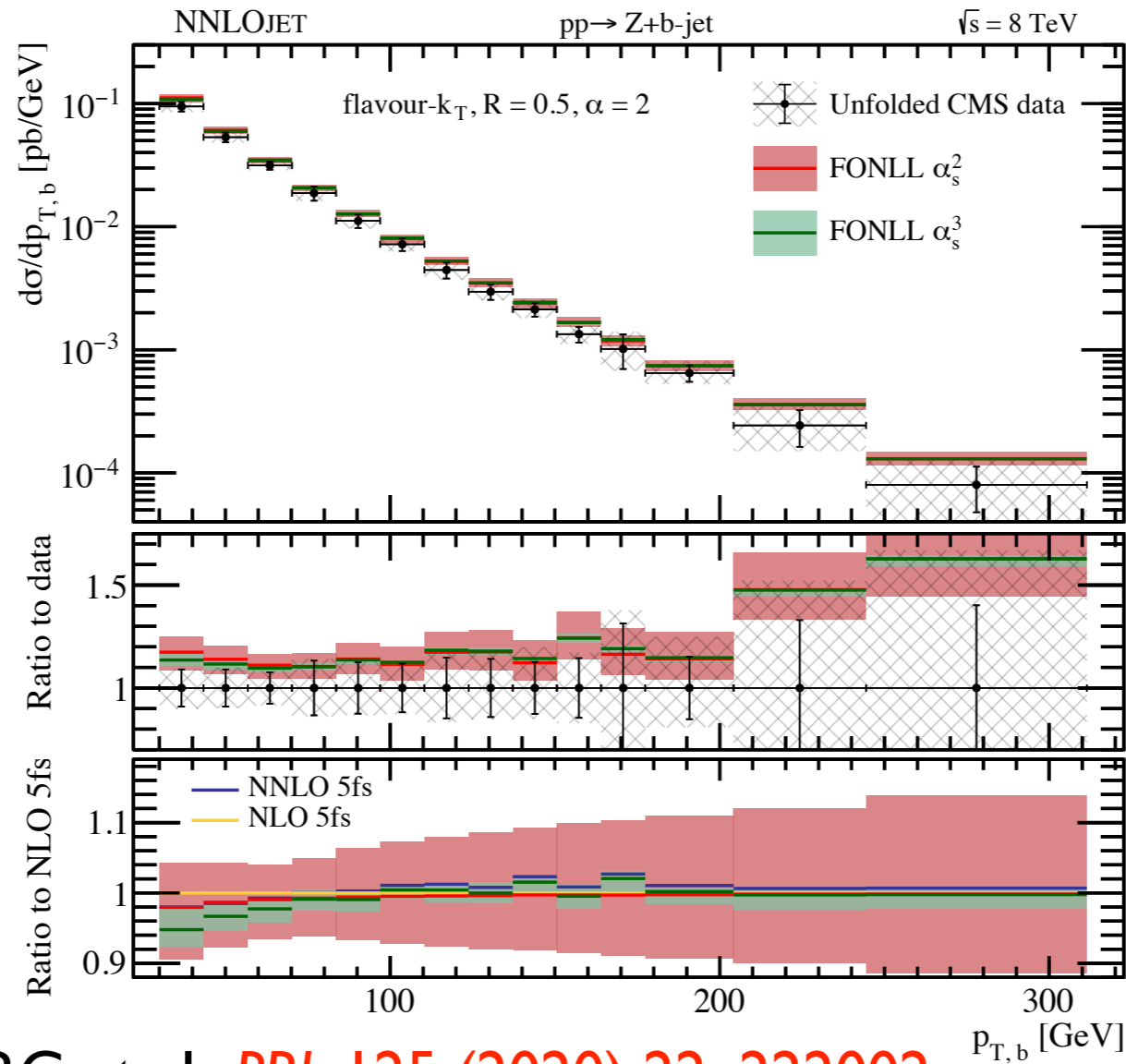
Includes:
 $T(\hat{X} \rightarrow X)$

Data: hadron-level anti- k_T jets

Model: NLO+PS
(4fs and 5fs)

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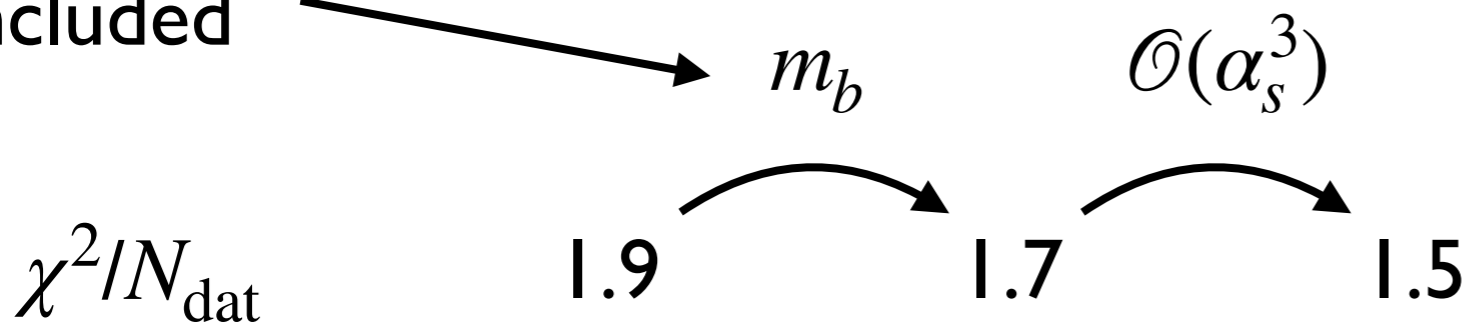


← (2-3)% unc.

$p_{T,b}$

RG et al., *PRL* 125 (2020) 22, 222002

Exact m_b effects included

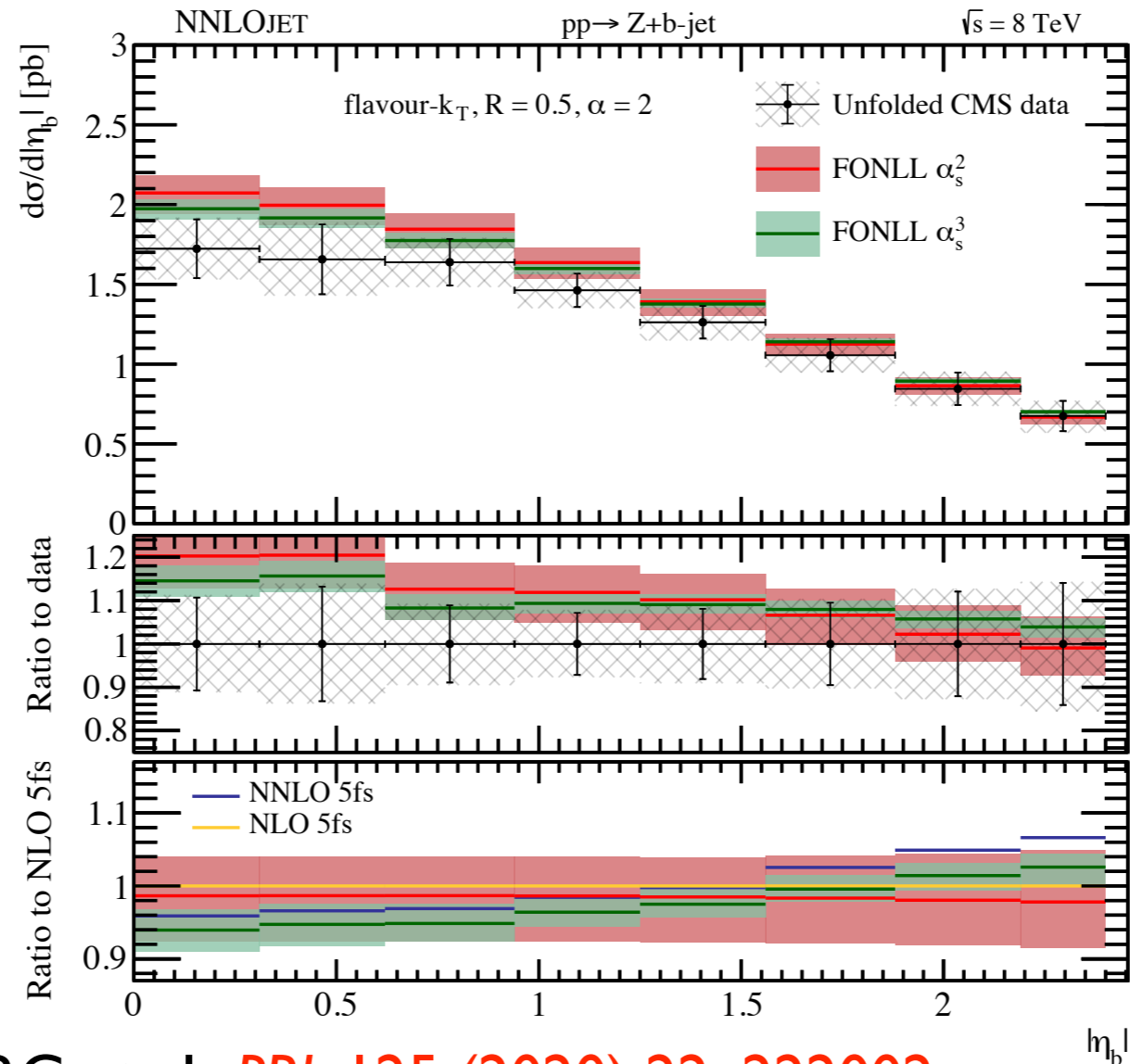


χ^2/N_{dat}

NLO (5fs)

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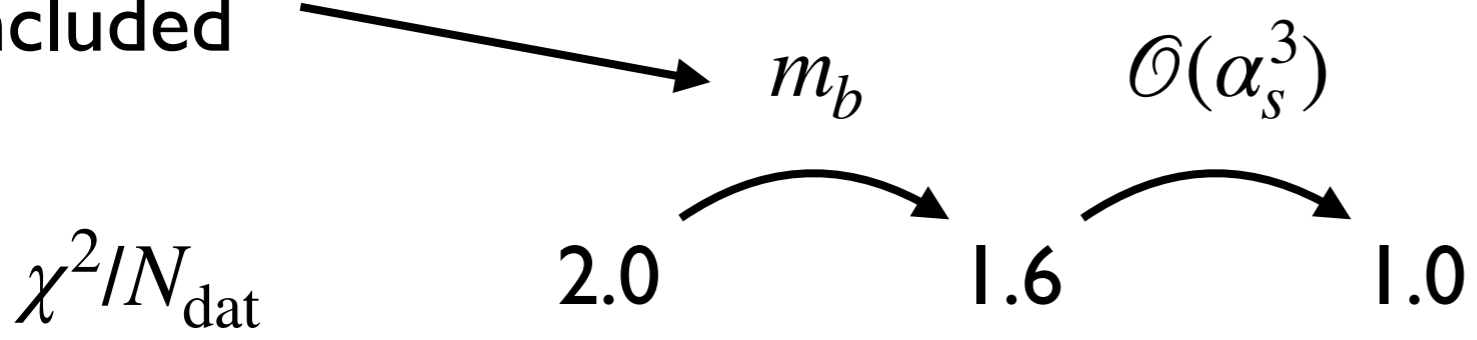


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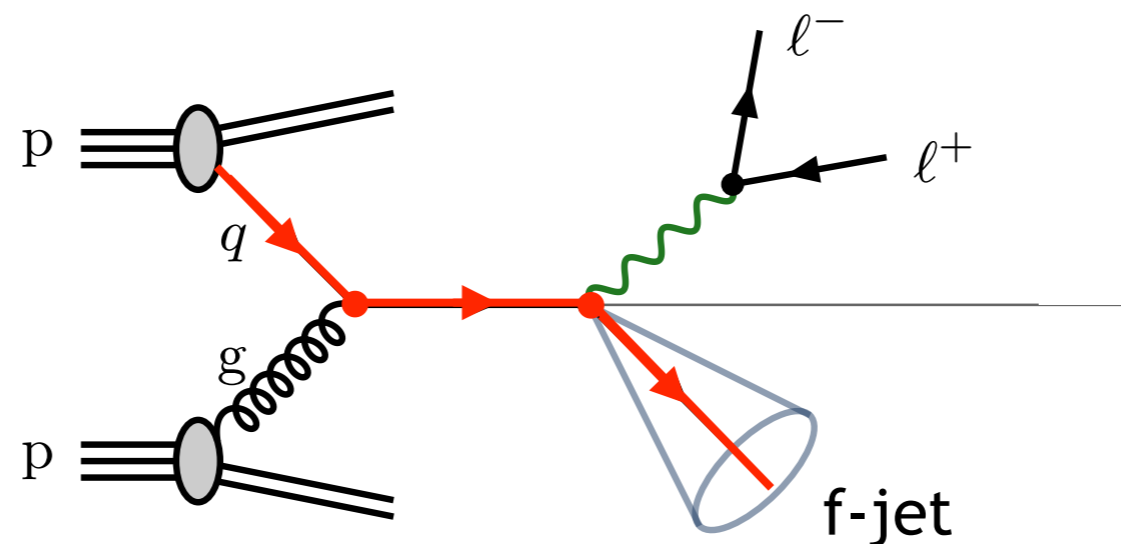
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Summary: Part 1

Direct heavy-flavour production (V+f-jet)

- (i) Resumming heavy-flavour PDF essential (computation in VFNS)
- (ii) Demands the use of a flavoured-jet algorithm (or ad hoc prescription)
- (iii) Theoretically well motivated (avoiding the collinear even-tags)

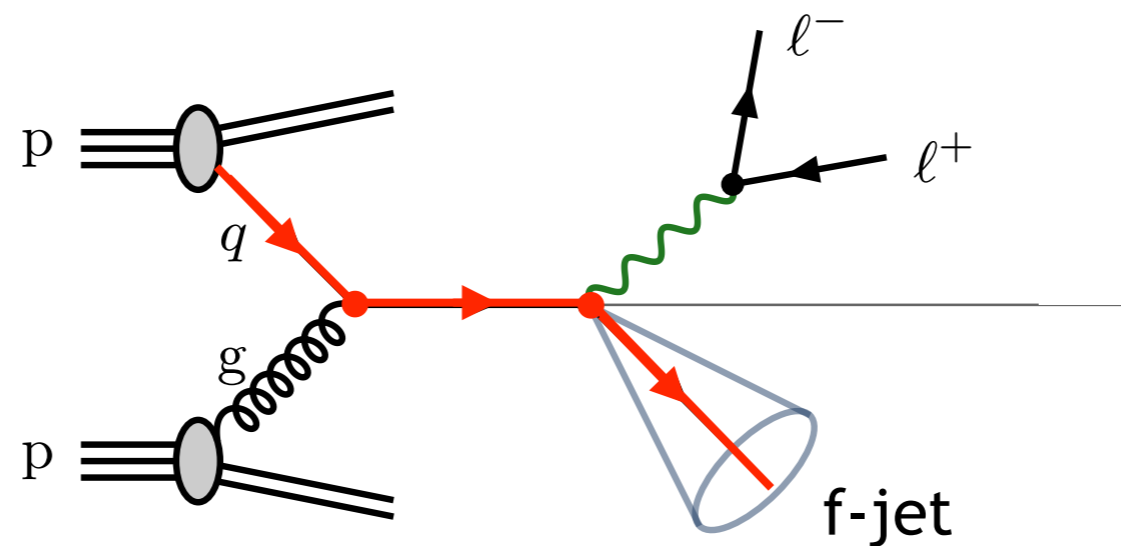


Critical for probing the proton flavour via: $V + f - \text{jet}$, $V = W^\pm, \gamma, Z$

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Critical for probing the proton flavour via: $V + f - \text{jet}$, $V = W^\pm, \gamma, Z$

This alone motivates (i.e. the Roadmap to precision)

- (a) [Exp.] Feasibility of applying a flavour-dependent jet alg. in data
- (b) [Cont.] Will we have to rely on unfolding? How reliable is NLO+PS

Summary: Part 2

Collinear safety (even-tags)

- (i) Unique assignment of $b = 1$ and $\bar{b} = -1$ to avoid collinear log or $1/\epsilon$
- (ii) Manifests as double/even-tagged jets in practice
- (iii) Critical issue for all fixed-order (anti- k_T or flavour- k_T)
- (iv) Even if its impact is small for 'signal' (VH) its large for background (Vj)

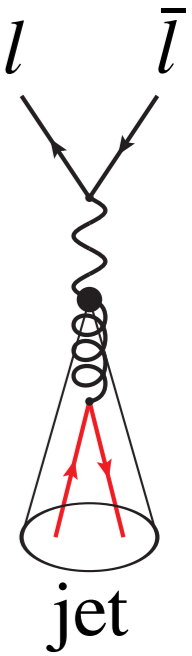
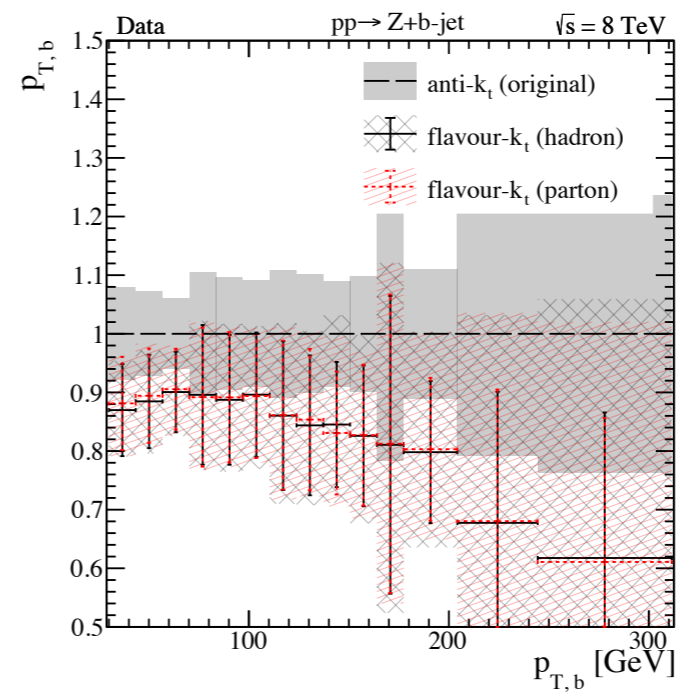
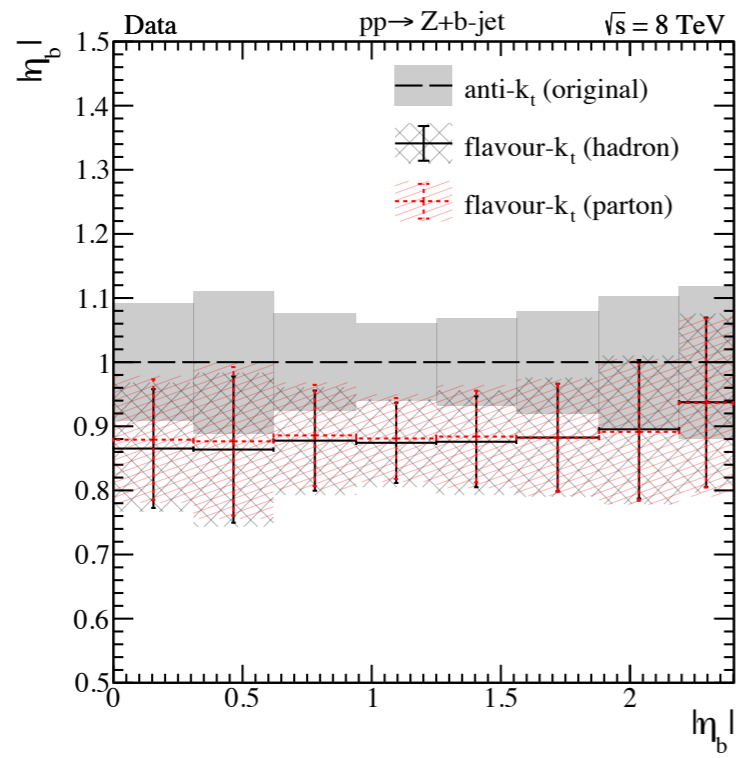
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My questions (also related to Part 1), and part of the Roadmap:

- (a) How well can we isolate even-tagged anti- k_T jets in data (ϵ_{tag}^2)
- (b) To what extent does NLO+PS describe the double-tag enriched regions
- (c) Are V+heavy-flavour control regions double-tag enriched? No I guess
- (d) Is higher (fixed / logarithmic) accuracy needed — e.g. NNLO+(NLL PS)



Unfolding correction $\sim 10\%$ for Z+b-jet (grows for large $p_{T,b}$)

Questions/comments and discussion encouraged!

Whiteboard

Whiteboard

Size of unfolding correction

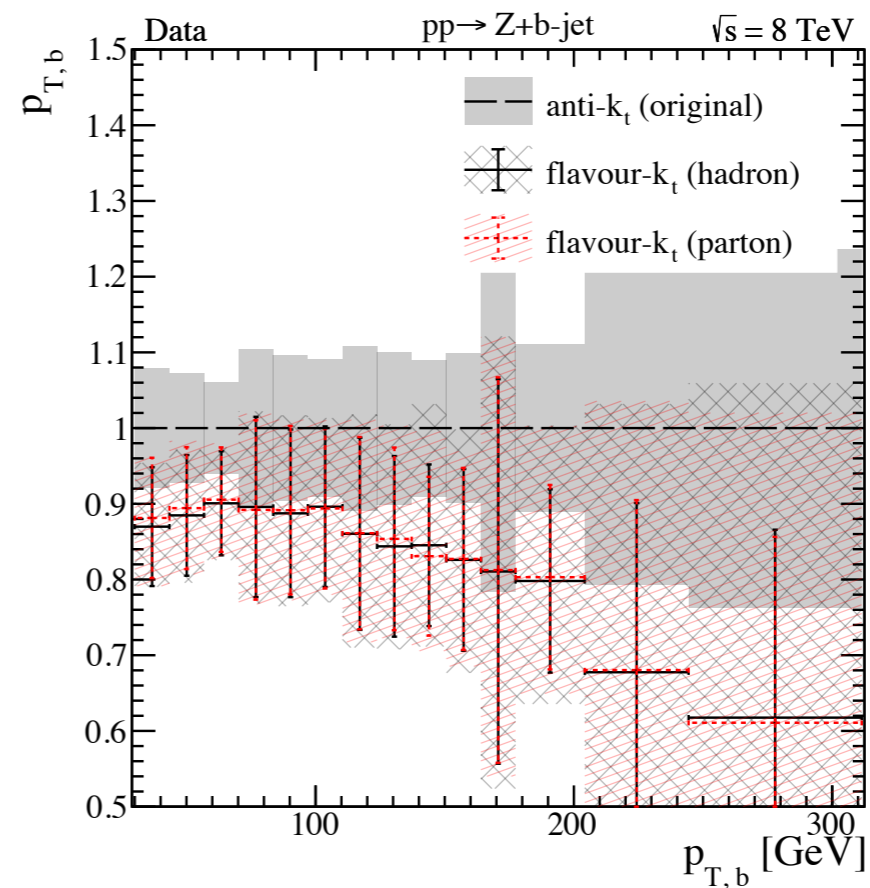
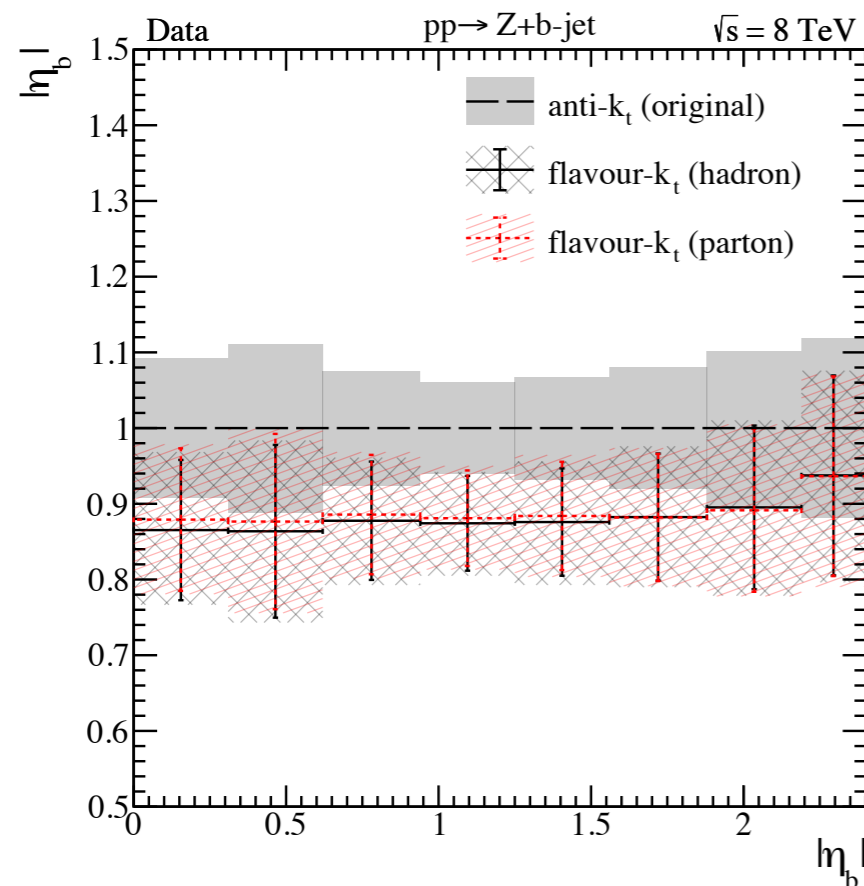
How to account for theory-experiment mismatch?

Use an NLO + Parton Shower prediction (which can evaluate both)

1) Prediction at parton-level, flavour- k_T algorithm (**Theory**)

2) Prediction at hadron-level, anti- k_T algorithm (**Experiment**)

Calculate an “Unfolding” correction from 2) Experiment \rightarrow 1) Theory



We use RooUnfold (following the procedure used in the exp. analyses)

NNLO flavoured jet cross-sections

Primarily challenge: dealing with flavour

$$d\hat{\sigma}_{ij,\text{NLO}} = \int_{n+1} [d\hat{\sigma}_{ij,\text{NLO}}^R - d\hat{\sigma}_{ij,\text{NLO}}^S] + \int_n [d\hat{\sigma}_{ij,\text{NLO}}^V - d\hat{\sigma}_{ij,\text{NLO}}^T], \quad (2.1)$$

Generic structure of higher-order terms (see [arXiv: 1907.05836](https://arxiv.org/abs/1907.05836))

$$d\hat{\sigma}_{ij,\text{NLO}}^R = \mathcal{N}_{\text{NLO}}^R d\Phi_{n+1}(\{p_3, \dots, p_{n+3}\}; p_1, p_2) \frac{1}{S_{n+1}} \\ \times \left[M_{n+3}^0(\{p_{n+3}\}, \{f_{n+3}\}) J_n^{(n+1)}(\{p_{n+1}\}, \{f_{n+1}\}) \right]. \quad (2.2)$$

Jet function acts on flavour and momenta of reduced MEs. In general $(i, j, k) \rightarrow (I, K)$
flavour
momentum

$$d\hat{\sigma}_{ij,\text{NLO}}^S = \mathcal{N}_{\text{NLO}}^R \sum_k d\Phi_{n+1}(\{p_3, \dots, p_{n+3}\}; p_1, p_2) \frac{1}{S_{n+1}} \\ \times \left[X_3^0(\cdot, k, \cdot) M_{n+2}^0(\{\tilde{p}_{n+2}\}, \{\tilde{f}_{n+2}\}) J_n^{(n)}(\{\tilde{p}_n\}, \{\tilde{f}_n\}) \right], \quad (2.3)$$

The \sim functions denoted mapped (in soft/collinear limits) momenta/flavour sets

Flavour- k_T Jet algorithm

Original work: Banfi, Salam, Zanderighi et al. hep-ph/0601139

These details from — ([arXiv: 1907.05836](https://arxiv.org/abs/1907.05836))

$$d_{ij} = \frac{\Delta y_{ij}^2 + \Delta \phi_{ij}^2}{R^2} \begin{cases} \max(k_{ti}, k_{tj})^\alpha \min(k_{ti}, k_{tj})^{2-\alpha} & \text{softer of } i, j \text{ is flavoured,} \\ \min(k_{ti}, k_{tj})^\alpha & \text{softer of } i, j \text{ is unflavoured,} \end{cases} \quad (2.4)$$

and

$$d_{i\bar{B}} = \begin{cases} \max(k_{ti}, k_{t\bar{B}}(y_i))^\alpha \min(k_{ti}, k_{t\bar{B}}(y_i))^{2-\alpha} & \text{softer of } i, j \text{ is flavoured,} \\ \min(k_{ti}, k_{t\bar{B}}(y_i))^\alpha & \text{softer of } i, j \text{ is unflavoured.} \end{cases} \quad (2.5)$$

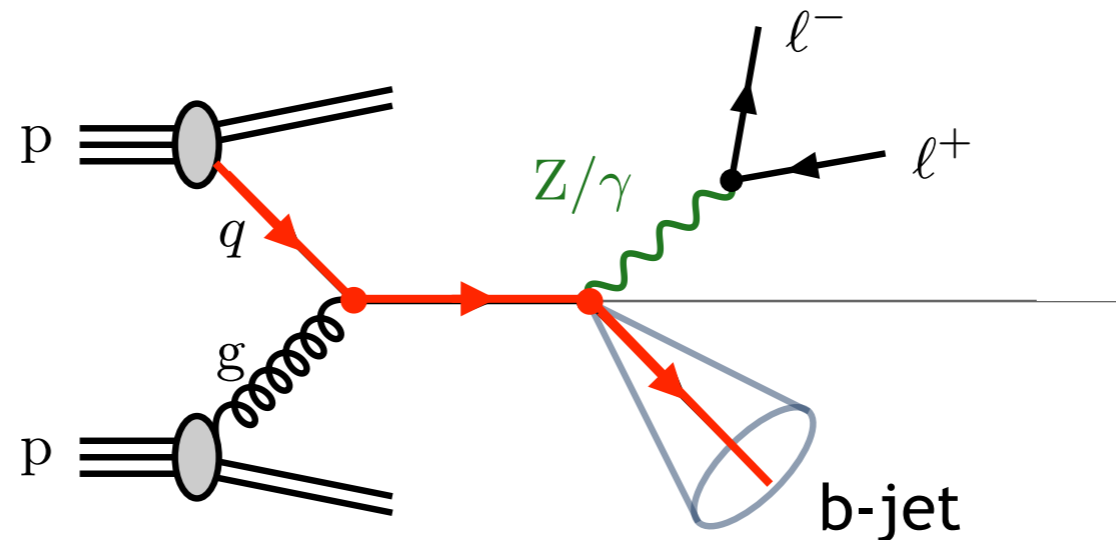
Introduction of a beam momentum, controls clusterings

$$k_{tB}(y) = \sum_i k_{ti} (\Theta(y_i - y) + \Theta(y - y_i) e^{y_i - y}), \quad (2.6)$$

$$k_{t\bar{B}}(y) = \sum_i k_{ti} (\Theta(y - y_i) + \Theta(y_i - y) e^{y - y_i}), \quad (2.7)$$

The massless NNLO calculation

$$d\hat{\sigma}_{ij \rightarrow \hat{X}} = d\hat{\sigma}_{ij \rightarrow \hat{X}}^{\text{LO}} + \alpha_s d\hat{\sigma}_{ij \rightarrow \hat{X}}^{\text{NLO}} + \alpha_s^2 d\hat{\sigma}_{ij \rightarrow \hat{X}}^{\text{NNLO}} + \dots$$



At LO: 4 distinct channels, $d\hat{\sigma}_{ij}^{\text{LO}} \sim \int d\phi_{\hat{X}} |M_{ij \rightarrow \hat{X}}|^2$

A small Feynman diagram showing a b quark and a gluon meeting at a vertex, with a Z boson and a b quark as outgoing particles.

At NNLO: 1000 distinct channels (amplitudes also become very complicated)

Use a “flavoured dressed” version of the computation of $Z + 1j$ @ NNLO

Gehrmann-De Ridder et al., <https://arxiv.org/abs/1507.02850>

PRL 117, 022001 (2016)

The massless NNLO calculation

$$d\hat{\sigma}_{ij \rightarrow \hat{X}} = d\hat{\sigma}_{ij \rightarrow \hat{X}}^{\text{LO}} + \alpha_s d\hat{\sigma}_{ij \rightarrow \hat{X}}^{\text{NLO}} + \alpha_s^2 d\hat{\sigma}_{ij \rightarrow \hat{X}}^{\text{NNLO}} + \dots$$



Theory collaboration between CERN, Durham, Karlsruhe, Lisbon, Nikhef, Zurich

A (parton level) Monte Carlo generator, antenna subtraction formalism
(Gehrmann et al. 2005-2013)

My collaborators for this work, i.e. $pp \rightarrow Z + b - \text{jet}$, *PRL* 125 (2020) 22, 222002



RG



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The mass of the b-quark

$$m_b^{\text{pole}} \sim 5 \text{ GeV}$$

The NNLO QCD computation assumes $m_b = 0$, what does this mean

The b-quark is **active** in the running of α_s and PDF. Meaning?

$$\frac{d\alpha_s}{d \ln \mu} = -\beta_0 \frac{\alpha_s^2}{2\pi} + \mathcal{O}(\alpha_s^3)$$

$$\alpha_s(\mu_R) = \frac{\alpha_s(\mu_0)}{1 + \frac{\beta_0}{2\pi} \alpha_s(\mu_0) \ln[\mu_r^2 / \mu_0^2]}$$

$$\beta_0 = \left(\frac{11}{6} c_a - \frac{2}{3} t_r n_f \right)$$

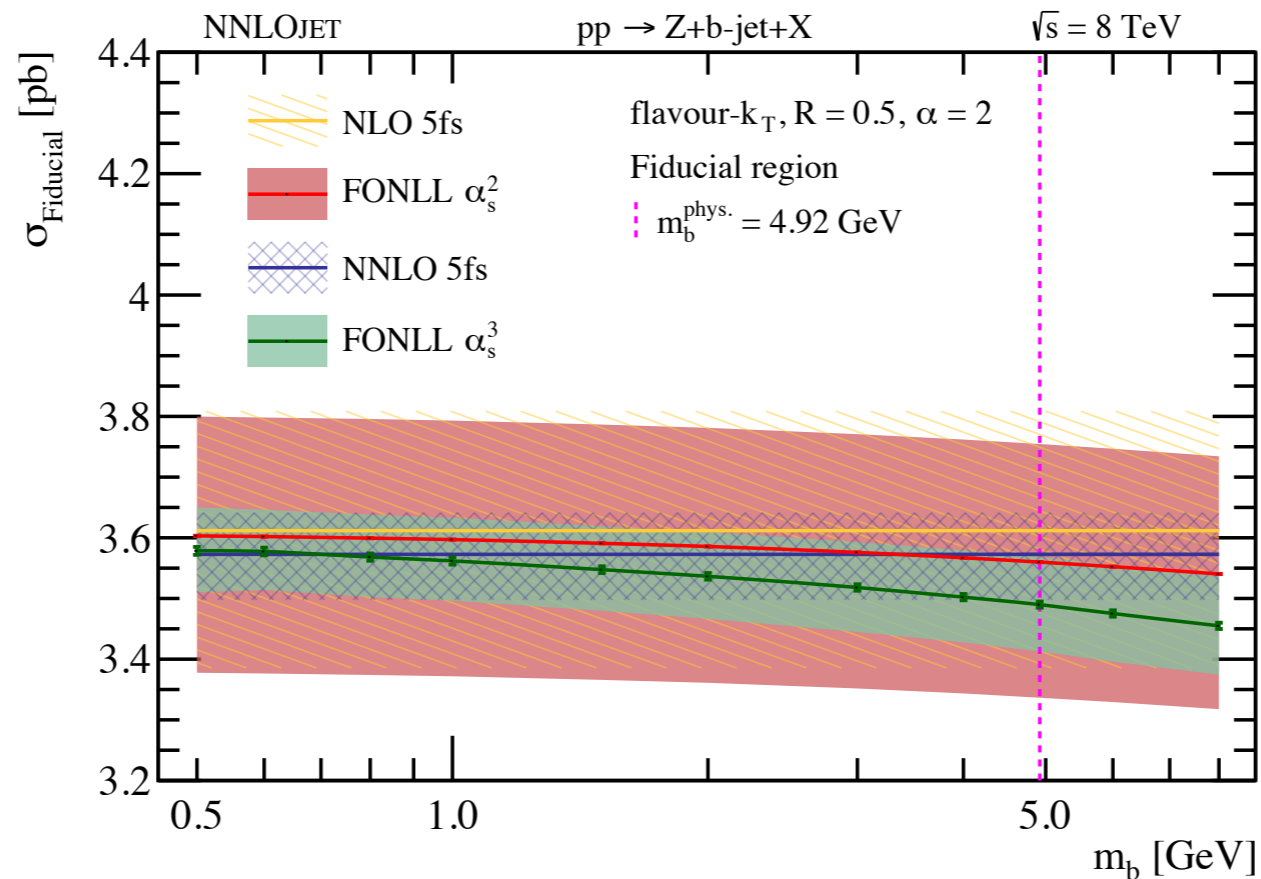
$$\frac{x}{1 + Bx} \approx x (1 - Bx + B^2 x^2) + \mathcal{O}(x^4)$$

Use of 'RG' improved perturbation theory to (re)sum $\ln[\mu_r/m_b]$ terms

Including mass corrections

Construct a massive variable flavour number scheme (M-VFNS)

$$d\sigma^{\text{M-VFNS}} = d\sigma^{5fs} + \left(d\sigma^{4fs} - d\sigma^{4fs}_{m_b \rightarrow 0} \right)$$



$$d\sigma^{4fs} = d\sigma^{m_b=0} + d\sigma^{\ln[m_b]} + \left(d\sigma^{m_b} \right) \approx -3\%$$

aMC@NLO

dedicated computation

*M-VFNS = FONLL

The zero mass limit

$$d\sigma^{GMVFNS} = d\sigma^{m=0} + \left(\underline{d\sigma^m} - d\sigma^{m \rightarrow 0} \right)$$

$$\underline{d\sigma^m} = d\sigma^{m=0, n_f} + d\sigma^{L[m]} + d\sigma^{\mathcal{O}(m^2)}$$

$d\sigma^{L[m]}$ is built from:

- 1) convolutions of a massless partonic cross section and OME
- 2) explicit virtual corrections, implicit via $\alpha_s^{n_f}$

Example: gg-channel at $\mathcal{O}(\alpha_s^2)$.

$$d\sigma_{gg}^{L[m]} = \int dx_1 dx_2 g(x_1, \mu_F^2) \left[\hat{A}_{g \rightarrow b}(z, \mu_F^2/m_b^2) \otimes g(x_2/z, \mu_F^2) \right] \hat{\sigma}_{bg \rightarrow Zb}^{m=0}(\alpha_s(\mu_r), \mu_r, \mu_F, \hat{S})$$

$$f_b^{(1)}(x, \mu_F^2) = \int_x^1 g(x/z, \mu_F^2) \hat{A}_{gb} + \mathcal{O}(\alpha_s^2), \quad \hat{A}_{gb} = \frac{\alpha_s(\mu_F^2)}{2\pi} P_{g \rightarrow q}(z) \ln[\mu_F^2/m_b^2]$$