Eliminating single-band dominance in dual-band pulsar light curve fitting

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Overview

1 Background
   - Statistical LC Fitting
   - Real-world Fits

2 Finding a Better Statistic
   - The Cause of the Failure
   - The Scaled-flux Standardised $\Psi^2_{\Phi,c}$ Statistic

3 Synthetic and Real-world Tests
   - Characterising Dual-band Statistics
   - Results—Synthetic Test
   - Results—Real-world Test

4 Conclusion
Background
Minimising $\chi^2_c = \chi^2_r + \chi^2_\gamma$ to find $M^{\text{PCS}}_c$ (yellow LC pair) yields

$$\vartheta^{\text{PCS}} = \left( (65^{+1}_{-1})^\circ, (56^{+1}_{-2})^\circ \right)$$

(yellow diamond); a point estimate of $\vartheta_{\text{true}}$ (yellow dot)
Background Real-world Fits

LC pair $\chi^2_r \chi^2_\gamma \chi^2_c$

$M_c^{\text{pcs}} \quad 175508 \quad 17960 \quad 193467$

$M_c^{\text{eye}} \quad 295657 \quad 524 \quad 296181$

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Finding a Better Statistic
The Cause of the Failure

Stated simply

*Equal decreases in* $\chi^2_r$ and $\chi^2_\gamma$ *do not necessarily correspond to equal increases in the joint* GOF. *Therefore, the definition of* $\chi^2_c$ *effectively calls for two quantities that express GOF in differing units, to be added together.*

To demonstrate this:

- Define the *scaled flux* $\Phi^2 = \chi^2(B)$; the magnitude of the perturbation the pulsar effects in a given band.

- Express $\chi^2_r$ and $\chi^2_\gamma$ in units better suited to GOF via the direct (single-band) GOF statistic

$$\Psi^2_\Phi(M) = 1 - \frac{\chi^2(M)}{\Phi^2}$$
The Cause of the Failure

Leveraging $\Psi_{\Phi,r}^2$ and $\Psi_{\Phi,\gamma}^2$ reveals that $\chi_c^2$ is equivalent to this *direct* GOF statistic:

\[
\begin{align*}
\left[ \Psi_c^2 \right]_{\Phi} &= \frac{\Phi_r^2}{\Phi_c^2} \Psi_{\Phi,r}^2 + \frac{\Phi_{\gamma}^2}{\Phi_c^2} \Psi_{\Phi,\gamma}^2 
\end{align*}
\]

\[\text{(with } \Phi_c^2 = \Phi_r^2 + \Phi_{\gamma}^2 \text{)}\]

- The single-band GOF contributions are implicitly weighted
- Defining the *scaled flux ratio* $\lambda_{r\gamma} = \Phi_r / \Phi_{\gamma}$, the relative weight of the radio GOF is $\omega_{r}^{\text{pcs}} = \lambda_{r\gamma}^2$
- Single-band dominance: e.g., $\Phi_r^2 \gg \Phi_{\gamma}^2 \Rightarrow \left[ \Psi_c^2 \right]_{\Phi} (M_c) \simeq \Psi_{\Phi,r}^2 (M_r)$; i.e., that $\chi_c^2$ effectively disregards the $\gamma$-ray–band GOF.
A more appropriate extension of the single-band $\chi^2$ statistic (for parameter estimation) would be one that grants $\Psi_r^2$ and $\Psi_\gamma^2$ equal weight. Adding in the requirement that it should be scaled-flux normalised yields the scaled-flux standardised statistic

$$\Psi_{\Phi,c}^2(M_c) = \frac{1}{2} \Psi_r^2(M_r) + \frac{1}{2} \Psi_\gamma^2(M_\gamma)$$

$$= 1 - \frac{1}{2} \left[ \frac{1}{\Phi_r^2} \chi_r^2(M_r) + \frac{1}{\Phi_\gamma^2} \chi_\gamma^2(M_\gamma) \right].$$

- $0 < \Psi_{\Phi,c}^2 < 1$ for a favourable joint fit
- $\Psi_{\Phi,c}^2 = 0$ for a background-equivalent (or “null”) joint fit
- $\Psi_{\Phi,c}^2 < 0$ for a bad joint fit
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This intuitive argument is just the tip of the iceberg! Seyffert et al. 2021, wherein we derive $\Psi_{\Phi,c}^2$ with full mathematical rigour, is coming soon!
Synthetic and Real-world Tests
## Characterising Dual-band Statistics

### Causative factors

- Uncertainty disparity: $\lambda_{\gamma \gamma}$ (per fit)
- Single-band best-fit non-colocation: $\eta_{s}$ (per fit)

### Effect size

- Single-band priority: $f_{r}$ (per fit)
- Non-colocation penalty: $\kappa$ (per fit)
- Precision-determined dominance: $F$ (population)
Synthetic and Real-world Tests

Characterising Dual-band Statistics

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### Characterising Dual-band Fits—Causative Factors

1. **Uncertainty disparity:**

   \[
   \log(\lambda_{r\gamma}) = \log(\Phi_{r}) - \log(\Phi_{\gamma})
   \]

   - Positive (negative) if the radio (\(\gamma\)-ray) LC is more sensitive
   - Symmetric about \(\log(\lambda_{r\gamma}) = 0\) (because of the logarithm)

2. **Single-band best-fit non-colocation:**

   \[
   \eta_{s} = \Delta \Psi_{\Phi,r}^{2}(M_{c,r}^{\gamma}) \cdot \Delta \Psi_{\Phi,\gamma}^{2}(M_{c,\gamma}^{r}) \geq 0
   \]

   - \(\eta_{s} = 0\) for colocated single-band fits
   - Surface area of the rectangle defined by \(M_{c,\gamma}^{}, M_{c,r}^{r}\) and \(K_{c}^{col}\)
   - *For some context:* If the single-band fits were to lie where the arrows point, then \(\eta_{s}\) would be 1
Characterising Dual-band Fits—Effect Size

1. Radio priority:

\[
f_r = \begin{cases} \frac{\Theta_r - \pi/4}{\pi/4}, & \eta_s > 0 \\ 0, & \text{otherwise,} \end{cases}
\]

- \( f_r = 0 \) if no waveband is prioritised
- \( 0 < f_r \leq 1 \) if the radio GOF is prioritised
- \( -1 \leq f_r < 0 \) if the \( \gamma \)-ray GOF is prioritised

2. Non-colocation penalty
Characterising Dual-band Statistics

(a)

(b)

$fr$

$fr$

$log(\lambda_{r\gamma})$
Characterising Dual-band Statistics—Effect Size

1 Precision-determined dominance:  (for a pulsar population of size $N$)

$$F = \frac{\sum_{k=1}^{N} \log(\lambda_{\gamma},k) \cdot \tilde{\eta}_{s,k} \cdot f_r,k}{\sum_{k=1}^{N} \left| \log(\lambda_{\gamma},k) \right| \cdot \tilde{\eta}_{s,k}},$$

where

$$\tilde{\eta}_{s,k} = \begin{cases} 
\eta_s, & \eta_s \leq 1 \\
1, & \text{otherwise}, 
\end{cases}$$

- Characterises how close the $f_r$ profile is to an “S” shape
- $F \simeq 0$ for undominated statistics (green) and purely radio– or $\gamma$-ray–dominated statistics (red or blue)
- $F \simeq 1$ if the more precise band dominates
- $F \simeq -1$ if the less precise band dominates
Results—Synthetic Test

\[ \varepsilon_\gamma = 0.01 = \frac{1}{4} \varepsilon_r \]

\[ \varepsilon_\gamma = 0.04 = \varepsilon_r \]

\[ \varepsilon_\gamma = 0.16 = 4 \varepsilon_r \]
Results—Synthetic Test

The graph illustrates the results of synthetic tests, showing the relationship between $\varepsilon_\gamma / \varepsilon_r$ and $F$, with $\chi^2_c$ and $\Psi_{\Phi,c}^2$ as indicated by the markers and lines on the graph.
Joint fits for 23 Fermi LAT pulsars

- For the $\chi^2_c$ fits: $F = 0.92$ and $0.96$ (OG+Cone and TPC+Cone)
- For the $\Psi^2_{\Phi,c}$ fits: $F = 0.37$ and $-0.11$ (OG+Cone and TPC+Cone)
Conclusion
Conclusion

- The absence of an “S” shape in the $f_r$ profiles for the $\Psi^2_{\Phi,c}$ statistic (in both the synthetic and real-world profiles) demonstrates that it effectively eliminates precision-determined single-band dominance.

- Moreover, it achieves this in a robust fashion, without the need for sporadic intervention.

- The general nature of the form of the $\Psi^2_{\Phi,c}$ statistic’s definition (i.e., the fact that it’s effectively just an average) means that it can easily be extended to more than two wavebands. (*With the caveat that all the LCs must be the result of perturbations.*)

- Interestingly, taking the average $\Psi^2_{\Phi,c}$ value across a population in essence yields a $\Psi^2_{\Phi,c}$-like performance score for the model used.
Thank you!