Eliminating single-band dominance in dual-band pulsar light curve fitting

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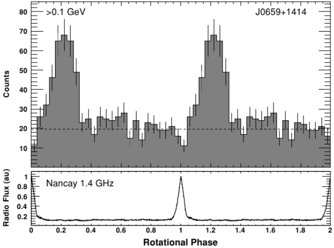
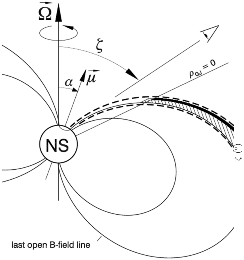
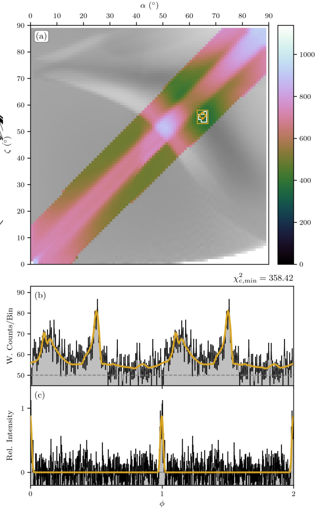
Overview

Background

Statistical LC Fitting



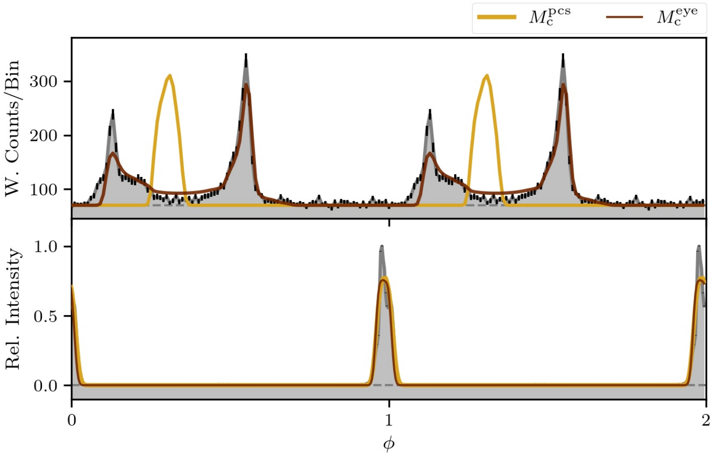




- Minimising $\chi^2_{\mathrm{c}} = \chi^2_{\mathrm{r}} + \chi^2_{\gamma}$ to find $M_{\mathrm{c}}^{\mathrm{pcs}}$ (*yellow* LC pair) yields

$$\vartheta^{\mathrm{pcs}} = \left((65^{+1}_{-1})^{\circ}, (56^{+1}_{-2})^{\circ} \right)$$

(*yellow* diamond); a point estimate of $\vartheta_{\mathrm{true}}$ (*yellow* dot)



LC pair	χ^2_r	χ^2_γ	χ^2_c
M_c^{pcs}	175508	17960	193467
M_c^{eye}	295657	524	296181

Finding a Better Statistic

The Cause of the Failure

Stated simply

Equal decreases in χ^2_r and χ^2_γ do not necessarily correspond to equal increases in the joint GOF. Therefore, the definition of χ^2_c effectively calls for two quantities that express GOF in differing units, to be added together.

To demonstrate this:

- Define the *scaled flux* $\Phi^2 = \chi^2(B)$; the magnitude of the perturbation the pulsar effects in a given band.
- Express χ^2_r and χ^2_γ in units better suited to GOF via the direct (single-band) GOF statistic

$$\Psi^2_\Phi(M) = 1 - \frac{\chi^2(M)}{\Phi^2}$$

The Cause of the Failure

Leveraging $\Psi^2_{\Phi,\mathrm{r}}$ and $\Psi^2_{\Phi,\gamma}$ reveals that χ^2_c is equivalent to this *direct* GOF statistic: *(with $\Phi^2_\mathrm{c} = \Phi^2_\mathrm{r} + \Phi^2_\gamma$)*

$$\left[\Psi^2_\mathrm{c}\right]_\Phi = \frac{\Phi^2_\mathrm{r}}{\Phi^2_\mathrm{c}}\,\Psi^2_{\Phi,\mathrm{r}} + \frac{\Phi^2_\gamma}{\Phi^2_\mathrm{c}}\,\Psi^2_{\Phi,\gamma}$$

- The single-band GOF contributions are implicitly weighted
- Defining the *scaled flux ratio* $\lambda_{\mathrm{r}\gamma} = \Phi_\mathrm{r}/\Phi_\gamma$, the relative weight of the radio GOF is $\omega^{\mathrm{pcs}}_\mathrm{r} = \lambda^2_{\mathrm{r}\gamma}$
- Single-band dominance: e.g., $\Phi^2_\mathrm{r} \gg \Phi^2_\gamma \Rightarrow \left[\Psi^2_\mathrm{c}\right]_\Phi (M_\mathrm{c}) \simeq \Psi^2_{\Phi,\mathrm{r}}(M_\mathrm{r})$; i.e., that χ^2_c effectively disregards the γ-ray–band GOF.

The Scaled-flux Standardised $\Psi^2_{\Phi,\mathrm{c}}$ Statistic

A more appropriate extension of the single-band χ^2 statistic (for parameter estimation) would be one that grants Ψ^2_{r} and Ψ^2_{γ} equal weight. Adding in the requirement that it should be scaled-flux normalised yields the *scaled-flux standardised* statistic

$$\begin{aligned}\Psi^2_{\Phi,\mathrm{c}}(M_{\mathrm{c}}) &= \frac{1}{2}\,\Psi^2_{\mathrm{r}}(M_{\mathrm{r}}) + \frac{1}{2}\,\Psi^2_{\gamma}(M_{\gamma}) \\ &= 1 - \frac{1}{2}\,\left[\frac{1}{\Phi^2_{\mathrm{r}}}\,\chi^2_{\mathrm{r}}(M_{\mathrm{r}}) + \frac{1}{\Phi^2_{\gamma}}\,\chi^2_{\gamma}(M_{\gamma})\right].\end{aligned}$$

- $0 < \Psi^2_{\Phi,\mathrm{c}} < 1$ for a favourable joint fit
- $\Psi^2_{\Phi,\mathrm{c}} = 0$ for a background-equivalent (or "null") joint fit
- $\Psi^2_{\Phi,\mathrm{c}} < 0$ for a bad joint fit

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This intuitive argument is just the tip of the iceberg! Seyffert et al. 2021, wherein we derive $\Psi^2_{\Phi,\mathrm{c}}$ with full mathematical rigour, is coming soon!

Synthetic and Real-world Tests

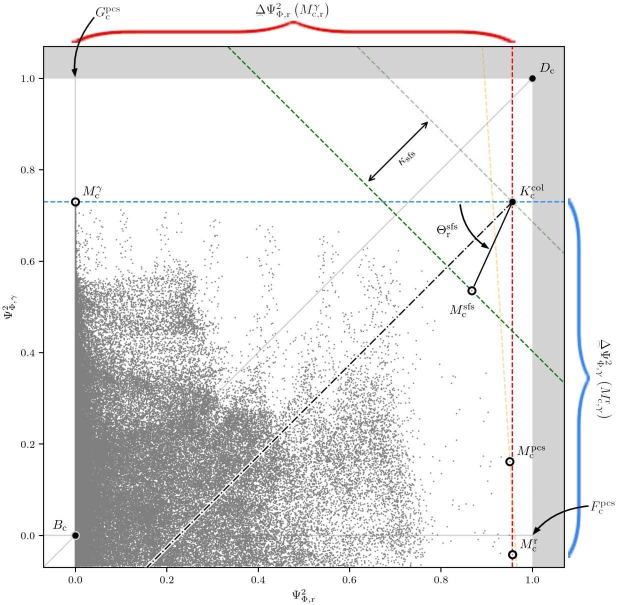
Characterising Dual-band Statistics

Causative factors

- Uncertainty disparity — $\lambda_{\mathrm{r}\gamma}$ (per fit)
- Single-band best-fit non-colocation — η_{s} (per fit)

Effect size

- Single-band priority — f_{r} (per fit)
- ~~Non-colocation penalty — κ~~ (per fit)
- Precision-determined dominance — F (population)


G_c^{pcs}
$\Delta\Psi^2_{\Phi,r}(M^\gamma_{c,r})$
D_c
κ_{sfs}
M_c^γ
K_c^{col}
Θ_r^{sfs}
M_c^{sfs}
M_c^{pcs}
F_c^{pcs}
B_c
M_c^r
$\Delta\Psi^2_{\Phi,\gamma}(M^r_{c,\gamma})$
$\Psi^2_{\Phi,\gamma}$
$\Psi^2_{\Phi,r}$
1.0
0.8
0.6
0.4
0.2
0.0
0.0 0.2 0.4 0.6 0.8 1.0

Characterising Dual-band Fits—Causative Factors

1. Uncertainty disparity:

$$\log(\lambda_{\mathrm{r}\gamma}) = \log(\Phi_{\mathrm{r}}) - \log(\Phi_{\gamma})$$

   - Positive (negative) if the radio (γ-ray) LC is more sensitive
   - Symmetric about $\log(\lambda_{\mathrm{r}\gamma}) = 0$ *(because of the logarithm)*

2. Single-band best-fit non-colocation:

$$\eta_{\mathrm{s}} = \underset{\frown}{\Delta}\Psi^2_{\Phi,\mathrm{r}}\left(M^{\gamma}_{\mathrm{c,r}}\right) \cdot \underset{\frown}{\Delta}\Psi^2_{\Phi,\gamma}\left(M^{\mathrm{r}}_{\mathrm{c},\gamma}\right) \geq 0$$

   - $\eta_{\mathrm{s}} = 0$ for colocated single-band fits
   - Surface area of the rectangle defined by M^{γ}_{c}, $M^{\mathrm{r}}_{\mathrm{c}}$ and $K^{\mathrm{col}}_{\mathrm{c}}$
   - *For some context:* If the single-band fits were to lie where the arrows point, then η_{s} would be 1

Characterising Dual-band Fits—Effect Size

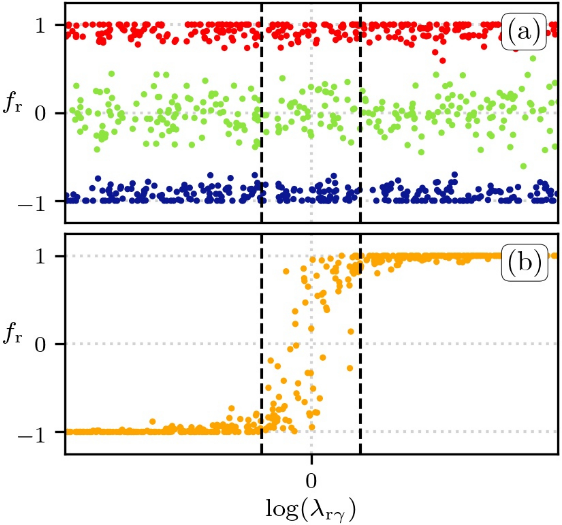
1. Radio priority:

$$f_{\mathrm{r}} = \begin{cases} \dfrac{\Theta_{\mathrm{r}} - \pi/4}{\pi/4}, & \eta_{\mathrm{s}} > 0 \\ 0, & \text{otherwise,} \end{cases}$$

   - $f_{\mathrm{r}} = 0$ if no waveband is prioritised
   - $0 < f_{\mathrm{r}} \leq 1$ if the radio GOF is prioritised
   - $-1 \leq f_{\mathrm{r}} < 0$ if the γ-ray GOF is prioritised

2. ~~Non-colocation penalty~~

Characterising Dual-band Statistics



Characterising Dual-band Statistics—Effect Size

1. Precision-determined dominance: *(for a pulsar population of size N)*

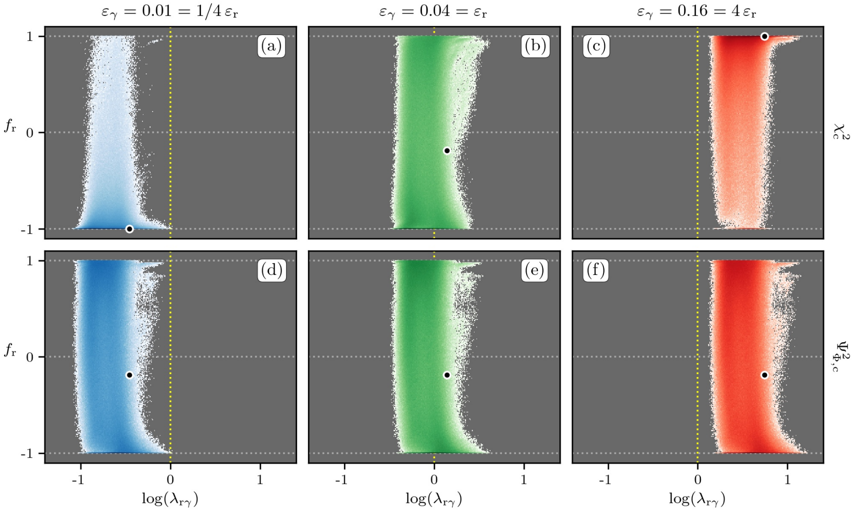
$$F = \frac{\sum_{k=1}^{N} \log(\lambda_{\mathrm{r}\gamma,k}) \cdot \widetilde{\eta}_{\mathrm{s},k} \cdot f_{\mathrm{r},k}}{\sum_{k=1}^{N} \left| \log(\lambda_{\mathrm{r}\gamma,k}) \right| \cdot \widetilde{\eta}_{\mathrm{s},k}},$$

where

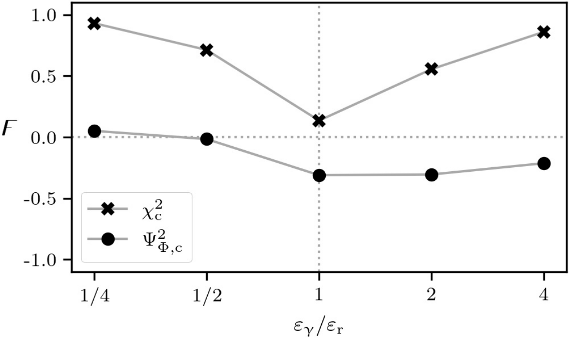
$$\widetilde{\eta}_{\mathrm{s},k} = \begin{cases} \eta_{\mathrm{s}}, & \eta_{\mathrm{s}} \leq 1 \\ 1, & \text{otherwise}, \end{cases}$$

- Characterises how close the f_{r} profile is to an "S" shape
- $F \simeq 0$ for undominated statistics (*green*) and purely radio– or γ-ray–dominated statistics (*red* or *blue*)
- $F \simeq 1$ if the more precise band dominates
- $F \simeq -1$ if the less precise band dominates

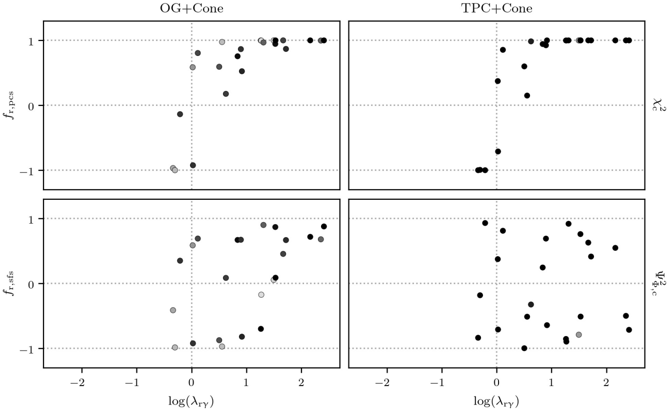
Results—Synthetic Test



Results—Synthetic Test



Results—Real-world Test



- Joint fits for 23 Fermi LAT pulsars
- For the χ^2_c fits: $\mathcal{F} = 0.92$ and 0.96 (OG+Cone and TPC+Cone)
- For the $\Psi^2_{\Phi,c}$ fits: $\mathcal{F} = 0.37$ and -0.11 (OG+Cone and TPC+Cone)

Conclusion

Conclusion

- The absence of an "S" shape in the f_{r} profiles for the $\Psi^2_{\Phi,\mathrm{c}}$ statistic (in both the synthetic and real-world profiles) demonstrates that it effectively eliminates precision-determined single-band dominance.
- Moreover, it achieves this in a robust fashion, without the need for sporadic intervention.
- The general nature of the form of the $\Psi^2_{\Phi,\mathrm{c}}$ statistic's definition (i.e., the fact that it's effectively just an average) means that it can easily be extended to more than two wavebands. (*With the caveat that all the LCs must be the result of perturbations.*)
- Interestingly, taking the average $\Psi^2_{\Phi,\mathrm{c}}$ value across a population in essence yields a $\Psi^2_{\Phi,\mathrm{c}}$-like performance score for the model used.

Thank you!