Order parameters of three-flavour chiral symmetry from $\pi\pi$ scattering

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In $\pi\pi$ scattering, we have

$$\pi_1(p_1)\pi_2(p_2) \longrightarrow \pi_3(p_3)\pi_4(p_4) \tag{1}$$

From QFT it follows that the amplitudes of the s,t and u channels should look like

$$A_{\pi\pi}(s,t,u) = B_{\pi\pi}(t,s,u) = C_{\pi\pi}(u,t,s) \tag{2}$$

Unitarity

Assuming unitarity of S-matrix

$$S^{+}S = 1, \tag{3}$$

we can introduce the transition matrix

$$iT = S - 1 \tag{4}$$

$$-i(T - T^{+}) = T^{+}T \tag{5}$$

and finally the amplitude A_{fi}

$$\langle f|T|i\rangle = (2\pi)^4 N_{P_f} N_{P_i} \delta^{(4)}(P_f - P_i) i A_{fi}, \tag{6}$$

where i, f are initial and final states, P_i, P_f sum of momenta in these states. N_P are the standard normalisation factors

$$N_p = \frac{1}{(2\pi)^{\frac{3}{2}}(2p_0)^{\frac{1}{2}}}. \tag{7}$$

Inserting intermediate states and assuming time-reversal invariance, we arrive at the so called "Cutkosky rule"

$$2\mathrm{Im}A_{fi} = \sum_n (2\pi)^4 N_{P_n} \delta^{(4)}(P_n - P_i) A^*_{nf} A_{ni}. \tag{8}$$

Dispersion relations $\rightarrow$ useful tool allowing us to relate higher order amplitudes to the amplitudes of the lower order

Consider a complex function $F(s)$ with s a complex argument, using these assumptions:

- $F(s)$ has a branch cut for real $s > M^2$.
- $F(s)$ is real for $s < M^2$.
- $F(s)$ is analytic for any complex s (except along the branch cut).

For some point s_0 inside the "Pac-Man" curve along the branch cut, we get

$$F(s_0) = P_n(s_0) + \frac{s_0^n}{\pi} \int_{M^2}^{\infty} \frac{\mathrm{Im} F(s)}{(s - s_0 - i\epsilon)s^n} ds, \tag{9}$$

with $P_n(s_0)$ a polynomial of (n-1)-th order.

In second order of QFT, the dispersion relations + analycity + crossing symmetry + unitarity give us (for a fixed $u = u_0$)

$$\begin{aligned} A_{\pi\pi}(s,t,u_0) =& P^{(2)}(s,t,u_0) + \frac{s^3}{\pi}\int_{m_s^2}^{\infty} \frac{\mathrm{Im} A_{\pi\pi}(x,y,u_0)}{x-s}\frac{dx}{x^3} + \\ &+ \frac{t^3}{\pi}\int_{m_t^2}^{\infty} \frac{\mathrm{Im} B_{\pi\pi}(x,y,u_0)}{x-t}\frac{dx}{x^3}, \end{aligned} \tag{10}$$

with the polynom of at most second order in Mandelstam variables, defining $y = \sum_{i=1}^{4} m_i^2 - u_0 - x$.

Roy equations - essentially dispersion relations with partial wave decomposition
Roy's representation for the partial wave amplitudes t_l^I of elastic $\pi\pi$ scattering reads

$$t_l^I(s) = k_l^I(s) + \sum_{I'=0}^{2} \sum_{l'=0}^{\infty} \int_{4M_\pi^2}^{\infty} ds^{'} K_{ll'}^{II'}(s, s^{'}) \mathrm{Im} t_{l'}^{I'}(s^{'}), \tag{11}$$

with I as isospin and l as angular momentum.
The $k_l^I(s)$ is the partial wave projection of the subtraction term (present only in s and p-waves).
Validity of these equations has been established on the interval $4M_\pi^2 < s < 60M_\pi^2$.

Now we may finally express the amplitude of $\pi\pi$ scattering up to $O(p^6)$ as

$$\begin{aligned} A_{\pi\pi}(s|t,u) = & \frac{\alpha_{\pi\pi}}{3F_\pi^2} M_\pi^2 + \frac{\beta_{\pi\pi}}{3F_\pi^2}(3s - 4M_\pi^2) + \\ & + \frac{\lambda_1}{F_\pi^4}\left(s - 2M_\pi^2\right)^2 + \frac{\lambda_2}{F_\pi^4}\left[(t - 2M_\pi)^2 + (u - 2M_\pi)^2\right] + \\ & + \frac{\lambda_3}{F_\pi^6}\left(s - 2M_\pi^2\right)^2 + \frac{\lambda_4}{F_\pi^6}\left[(t - 2M_\pi)^3 + (u - 2M_\pi)^3\right] + \\ & + \overline{J}(s|t,u) + O\left[\left(\frac{p}{\Lambda_H}\right)^8\right]], \end{aligned} \tag{12}$$

where $\alpha_{\pi\pi}$, $\beta_{\pi\pi}$, $\lambda_1 \dots \lambda_4 \equiv$ *subthreshold parameters*
$\overline{J}(s|t,u)$ collects the unitary cuts arising from elastic $\pi\pi$ intermediate states.
I will focus on the LO subthreshold parameters $\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$.

Parametrization obviously isn't fixed $\rightarrow$ one may use other parameters instead of $\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$ (representation from S. Descotes-Genon, N. H. Fuchs, L. Girlanda and J. Stern), i.e. scattering lengths a_0^0 and a_0^2 ($\equiv$ "phenomenological representation" from S. M. Roy)

$$\begin{aligned} A_{\pi\pi}(s,t,u) =& 16\pi a_0^2 + \frac{4\pi}{3M_\pi^2}\left(2a_0^0 - 5a_0^2\right)s + P(s,t,u) + 32\pi[\frac{1}{3}\overline{W}^0(s) + \\ & \frac{3}{2}(s-u)\overline{W}^1(t) + \frac{3}{2}(s-t)\overline{W}^1(u) + \frac{1}{2}\overline{W}^2(u) + \frac{1}{2}\overline{W}^2(u) - \\ & \frac{1}{3}\overline{W}^2(s)] + O(p^8), \end{aligned} \tag{13}$$

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Standard Lagrangian density of QCD can be written as

$$\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \overline{q} \left(i\gamma_\mu D^\mu - M \right) q \tag{14}$$

with invariants called *color* and *baryon number*.
Now looking at masses of quarks → construct an additional approximate symmetry SU(N_f), where N_f is the number of quarks we consider to be of the same mass.
Next we may assume the small masses → splits the symmetries to 2 - left and right helicities.

Introduced by S. Weinberg
Idea of expanding Lagrangian in degrees of freedom below a certain boundary.
Such expansion of QCD $\equiv$ *chiral perturbation theory*.
This approach has its difficulties - NLO gives us 10 coupling constants and NNLO terrifying 90.
QCD effective Lagrangian by Gasser and Leutwyler

$$\mathcal{L}_{eff} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \ldots \tag{15}$$

$$\mathcal{L}^{(2)} = \frac{F_0^2}{4} Tr[D_\mu U D^\mu U^+ + (U^+\chi + \chi^+ U)] \tag{16}$$

$$\mathcal{L}^{(4)} = \mathcal{L}^{(4)}(L_1, .., L_{10}) + \mathcal{L}^{(4)}_{WZ} \tag{17}$$

$$U(x) = e^{\frac{i}{F_0}\phi^a(x)\lambda^a} \tag{18}$$

$$\chi = 2B_0 M. \tag{19}$$

Effective Lagrangian

$$
\begin{aligned}
\mathcal{L}^{(4)}(L_1,..,L_{10}) \quad = \quad & L_1\, Tr[D_\mu U^+ D^\mu U]^2 + L_2\, Tr[D_\mu U^+ D_\nu U]\, Tr[D^\mu U^+ D^\nu U] + \\
& + L_3\, Tr[D_\mu U^+ D^\mu U D_\nu U^+ D^\nu U] + \\
& + L_4\, Tr[D_\mu U^+ D^\mu U]\, Tr[\chi^+ U + \chi U^+] + \\
& + L_5\, Tr[D_\mu U^+ D^\mu U(\chi^+ U + U^+ \chi)] + L_6\, Tr[\chi^+ U + \chi U^+]^2 + \\
& + L_7\, Tr[\chi^+ U - \chi U^+]^2 + L_8\, Tr[\chi^+ U \chi^+ U + \chi U^+ \chi U^+] - \\
& - i L_9\, Tr[F_R^{\mu\nu} D_\mu U D_\nu U^+ + F_L^{\mu\nu} D_\mu U^+ D_\nu U] + \\
& + L_{10}\, Tr[U^+ F_R^{\mu\nu} U F^L_{\mu\nu}] \qquad (20)
\end{aligned}
$$

$$
\mathcal{L}^{(6)} = \mathcal{L}^{(6)}(C_1,..,C_{90}) + \mathcal{L}^{(6)}_{WZ}(C_1^W,..,C_{23}^W) \qquad (21)
$$

Came from Stern et al.
Resummed approach aims to bypass this problem - it resumms the higher order terms and doesn't omit them.
The resummed approach yields on calculating this remainder directly, but assumes we may describe it as the observable itself multiplied by an unknown variable.
One may place restrictions on this remainder, when calculating the original observable.
So an observable with resummed approach may look i.e. like this

$$A = A_{LO} + A_{NLO} + A\delta A \quad (22)$$

χPT LO parameters X,Y and Z

The principal order parameters of QCD in spontaneous symmetry breaking of chiral symmetry are the quark condensate and pseudoscalar decay constants

$$\Sigma(N_f) = - < 0 \mid \overline{q}q|0 >|_{m_q \to 0} \tag{23}$$

$$F(N_f) = F_P^a \mid_{m_q \to 0} \tag{24}$$

$$ip_\mu F_P^a =< 0 \mid A_\mu^a \mid P >, \tag{25}$$

where N_f = number of light quarks with mass m_q, A_μ^a are axial vector currents and F_P^a are the decay constants of light pseudoscalar mesons P.
We can reparametrize these and relate them to physical quantities connected with pion two point Green functions

$$X(N_f) = \frac{2\hat{m}\Sigma(N_f)}{F_\pi^2 M_\pi^2} \tag{26}$$

$$Z(N_f) = \frac{F(N_f)^2}{F_\pi^2}, \tag{27}$$

with $\hat{m} = \frac{m_u + m_d}{2}$.

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We may calculate the subthreshold parameters as (working in SU(3) and identifying X(3)$\equiv$ X, Z(3)$\equiv$ Z)

$$\begin{aligned}\alpha_{\pi\pi} =& 1+\frac{3r}{r+2}\epsilon(r)-\frac{2Yr}{r+2}\eta(r)+\frac{2(1-X)}{r+2}+\frac{4(1-Y)}{r+2}-\frac{1}{2}Y^2\left(\frac{M_\pi}{4\pi F_\pi}\right)^2\cdot\\ &\cdot\left(\frac{r}{(r-1)(r+2)}\left((r+2)\log\left(\frac{M_\eta^2}{M_K^2}\right)-(r-2)\log\left(\frac{M_K^2}{M_\pi^2}\right)\right)+\frac{7}{3}\right)-\\ &-\frac{6}{r+2}\left(\frac{r+1}{r-1}\delta_{M_\pi}-\left(\epsilon(r)+\frac{2}{r-1}\right)\delta_{M_K}\right)-\\ &-Y\frac{2r}{r+2}\left(\frac{r+1}{r-1}\delta_{F_K}-\left(\eta(r)+\frac{2}{r-1}\right)\delta_{F_K}\right)+2Y\delta_{F_\pi}+\delta_{\alpha_{\pi\pi}}\end{aligned} \tag{28}$$

$$\begin{aligned}\beta_{\pi\pi} =& 1+\frac{r\eta(r)}{r+2}+\frac{2(1-Z)}{r+2}+\frac{1}{2}Y\left(\frac{M_\pi}{4\pi F_\pi}\right)^2\cdot\\ &\cdot\left(\frac{r}{(r-1)(r+2)}\left((2r+1)\log\left(\frac{M_\eta^2}{M_K^2}\right)+(4r+1)\log\left(\frac{M_K^2}{M_\pi^2}\right)\right)-5\right)-\\ &-\frac{2}{r+2}\left(\frac{r+1}{r-1}\delta_{F_\pi}-\left(\eta(r)+\frac{2}{r-1}\right)\delta_{F_K}\right)+\delta_{\beta_{\pi\pi}}\end{aligned} \tag{29}$$

where

$$Y = \frac{X}{Z} \tag{30}$$

$$r = \frac{m_s}{\hat{m}} \tag{31}$$

$$\epsilon(r) = \frac{2}{r^2 - 1}\left(2\frac{F_K^2 M_K^2}{F_\pi^2 M_\pi^2} - r - 1\right) \tag{32}$$

$$\eta(r) = \frac{1}{r - 1}\left(\frac{F_K^2}{F_\pi^2} - 1\right). \tag{33}$$

$\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$ from $\pi\pi$ scattering

Representation matching can be found in "$\pi\pi$ scattering" by Colangelo, Gasser and Leutwyler

Using solutions to the Roy equations

$\downarrow$

Phenomenological representation (p_i's)

$\downarrow$

Transform them into chiral representation (c_i's)

$\downarrow$

Transform them into representation introduced by Bijnens ($\bar{b}_i$'s)

$\downarrow$

Get $\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$

$$A_{\pi\pi}(s,t,u) = 16\pi a_0^2 + \frac{4\pi}{3M_\pi^2}\left(2a_0^0 - 5a_0^2\right)s + P(s,t,u) + 32\pi[\frac{1}{3}\overline{W}^0(s) + \frac{3}{2}(s-u)\overline{W}^1(t) + \frac{3}{2}(s-t)\overline{W}^1(u) + \frac{1}{2}\overline{W}^2(u) + \frac{1}{2}\overline{W}^2(u) - \frac{1}{3}\overline{W}^2(s)] + O(p^8), \tag{34}$$

where $P(s,t,u)$ is a crossing symmetry polynomial

$$P(s,t,u) = p_1 + p_2 s + p_3 s^2 + p_4(t-u)^2 + p_5 s^3 + p_6 s(t-u)^2$$

$$p_1 = -128\pi M_\pi^4\left(\bar{I}_0^1 + \bar{I}_0^2 + 2M_\pi^2\bar{I}_1^1 + 2M_\pi^2\bar{I}_1^2 + 8M_\pi^4\bar{I}_2^2\right)$$

$$p_2 = \frac{-64\pi M_\pi^2}{3}\left(2\bar{I}_0^0 - 6\bar{I}_0^1 - \bar{I}_0^2 - 15M_{/pi}^2\bar{I}_1^1 - 3M_\pi^2\bar{I}_1^2 - 36M_\pi^4\bar{I}_2^2 + 6M_\pi^2 H\right)$$

$$p_3 = \frac{8\pi}{3}\left(4\bar{I}_0^0 - 9\bar{I}_0^1 - \bar{I}_0^2 - 16M_\pi^2\bar{I}_1^0 - 42M_\pi^2\bar{I}_1^1 + 22M_\pi^2\bar{I}_1^2 - 72M_\pi^4\bar{I}_2^2 + 24M_\pi^2 H\right)$$

$$p_4 = 8\pi\left(4\bar{I}_0^1 + \bar{I}_0^2 + 2M_\pi^2\bar{I}_1^1 + 2M_\pi^2\bar{I}_1^2 - 24M_\pi^4\bar{I}_2^2\right)$$

$$p_5 = \frac{4\pi}{3}\left(8\bar{I}_1^0 + 9\bar{I}_1^1 - 11\bar{I}_1^2 - 32M_\pi^2\bar{I}_2^0 + 44M_\pi^2\bar{I}_2^2 - 6H\right)$$

$$p_6 = 4\pi\left(\bar{I}_1^1 - 3\bar{I}_1^2 + 12M_\pi^2\bar{I}_2^2 + 2H\right),$$

with $\overline{I}_n^I$ and H defined as

$$\overline{I}_n^I = \sum_{l=0}^{\infty} \frac{2l+1}{\pi} \int_{4M_\pi^2}^{\infty} \frac{\mathrm{Im}\, t_l^I(s)}{s^{n+2}(s-4M_\pi^2)} \quad (35)$$

$$H = \sum_{l=2}^{\infty} \frac{(2l+1)l(l+1)}{\pi} \int_{4M_\pi^2}^{\infty} \frac{2\mathrm{Im}\, t_l^0(s) + 4\mathrm{Im}\, t_l^2(s)}{9s^3(s-4M_\pi^2)}. \quad (36)$$

The amplitude can be expressed as

$$t_l^I(s) = \frac{1}{\sigma(s)} e^{i\delta_l^I(s)} \sin(\delta_l^I(s)), \quad (37)$$

where $\delta(s)$ is real. We can then use Schenk parametrization

$$\tan(\delta_l^I) = \sqrt{1 - \frac{4M_\pi^2}{s}}\, q^{2l} \left(A_l^I + B_l^I q^2 + C_l^I q^4 + D_l^I q^4\right) \frac{4M_\pi^2 - s_l^I}{s - s_l^I}. \quad (38)$$

$A_l^I, B_l^I, C_l^I, D_l^I, s_l^I$ are called Schenk parameters.

$\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$ from $\pi\pi$ scattering

$$
\begin{aligned}
c_1 &= 16\pi a_0^2 + p_1 + O(p^8) \\
c_2 &= \frac{4\pi}{3M_\pi^2}(2a_0^0 - 5a_0^2) + p_2 + O(p^6) \\
c_3 &= p_3 + O(p^4) \\
c_4 &= p_4 + O(p^4) \\
c_5 &= p_5 + O(p^2) \\
c_6 &= p_6 + O(p^2)
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
c_1 =& -\frac{M_\pi^2}{F_\pi^2}[1+\xi\left(-\overline{b}_1-\frac{68}{315}\right) \\
&+\xi^2\left(-\frac{8\overline{b}_1}{105}-\frac{32\overline{b}_2}{63}-\frac{464\overline{b}_3}{315}-\frac{3824\overline{b}_4}{315}+\frac{601\pi^2}{945}-\frac{17947}{2835}\right)] \\
c_2 =& \frac{1}{F_\pi^2}[1+\xi\left(\overline{b}_2-\frac{323}{1260}\right)+ \\
&\xi^2\left(-\frac{11\overline{b}_1}{70}-\frac{211\overline{b}_2}{315}-\frac{628\overline{b}_3}{315}-\frac{5164\overline{b}_4}{315}-\frac{3977}{630}+\frac{5237\pi^2}{7560}\right)] \\
c_3 =& \frac{1}{NF_\pi^4}\left(\overline{b}_3+\frac{1}{42}+\xi\left(\frac{18\overline{b}_1}{35}+\frac{59\overline{b}_2}{105}+\frac{731\overline{b}_3}{315}+\frac{3601\overline{b}_4}{315}-\frac{5387\pi^2}{15120}-\frac{19121}{7560}\right)\right] \\
c_4 =& \frac{1}{NF_\pi^4}\left(\overline{b}_4-\frac{31}{2520}+\xi\left(-\frac{43\overline{b}_1}{420}-\frac{8\overline{b}_2}{63}+\frac{23\overline{b}_3}{63}+\frac{997\overline{b}_4}{315}+\frac{467\pi^2}{7560}-\frac{63829}{45360}\right)\right] \\
c_5 =& \frac{1}{N^2F_\pi^6}\left(\frac{137}{1680\xi}+\frac{\overline{b}_1}{16}+\frac{379\overline{b}_2}{1680}-\frac{25\overline{b}_3}{28}-\frac{731\overline{b}_4}{180}+\overline{b}_5+\frac{269\pi^2}{15120}+\frac{61673}{18144}\right) \\
c_6 =& \frac{1}{N^2F_\pi^6}\left(-\frac{31}{1680\xi}+\frac{\overline{b}_1}{112}-\frac{47\overline{b}_2}{1680}-\frac{65\overline{b}_3}{252}-\frac{547\overline{b}_4}{420}+\overline{b}_6+\frac{\pi^2}{15120}+\frac{44287}{90720}\right),
\end{aligned}
\tag{40}
$$

To finally receive

$$\alpha_{\pi\pi} = 1 + \xi(3\bar{b}_1 + 4\bar{b}_2 + 4\bar{b}_3 - 4\bar{b}_4) - \frac{11}{36}\pi^2\xi^2 - \frac{152}{9}\xi^2 \quad (41)$$

$$\beta_{\pi\pi} = 1 + \xi(\bar{b}_2 + 4\bar{b}_3 - 4\bar{b}_4) + 4\xi^2(3\bar{b}_5 - \bar{b}_6) - \frac{13}{72}\pi^2\xi^2 + \frac{152}{9}\xi^2, \quad (42)$$

where $\xi = \left(\frac{M_\pi}{4\pi F_\pi}\right)^2$ and $N = 16\pi^2$.

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I'm going to use Bayesian approach, which is described by this equation

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}. \tag{43}$$

For physical observables, we can use

$$P(data|true) = \prod_k \frac{1}{\sigma_k\sqrt{2\pi}} \exp\left[-\frac{(O_k^{exp} - O_k^{true})^2}{2\sigma_k^2}\right], \tag{44}$$

I calculate $\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$ from $\pi\pi$ scattering, take formula from χPT fo these subthreshold parameters as functions of X, Z $\rightarrow$ information about X, Y, Z

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$\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$ from $\pi\pi$ scattering

Using more recent results for scattering length from the NA48/2 collaboration

$$a_0^0 = 0.2196 \pm 0.00024 \tag{45}$$

$$\begin{aligned} a_0^2 = & -0.0444 \pm 0.0008 + 0.236(a_0^0 - 0.22) - \\ & - 0.61(a_0^0 - 0.22)^2 - 9.9(a_0^0 - 0.22)^3, \end{aligned} \tag{46}$$

I obtained the subthreshold parameters $\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$. The results were

$$\begin{aligned} \alpha_{\pi\pi} &= 1.08 \pm 0.08 \\ \beta_{\pi\pi} &= 1.11 \pm 0.01 \end{aligned} \tag{47}$$

	$\alpha_{\pi\pi}$	$\beta_{\pi\pi}$
Stern et al.	1.381 ± 0.242	1.081 ± 0.023
Colangelo et al.	1.08 ± 0.07	1.12 ± 0.01
My results	1.08 ± 0.08	1.11 ± 0.01

Table: Comparison of subthreshold parameters

Application on χPT parameters X,Y and Z

Then I used Monte Carlo with 10^6 entries to simulate χPT parameters X,Y and Z, calculated the subthreshold parameters and applied Bayes's theorem, resulting in following probability distributions

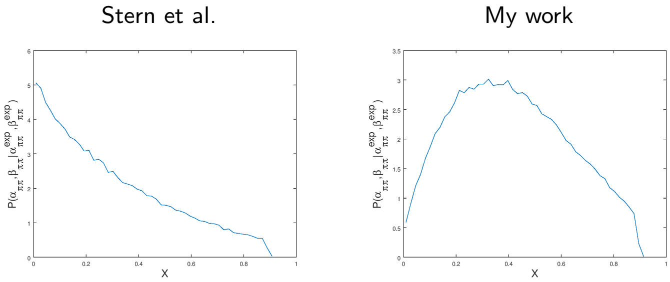

Table: Comparison of probability distributions between my work and Stern et al.

Comparison of subthreshold parameters

Stern et al. | My work

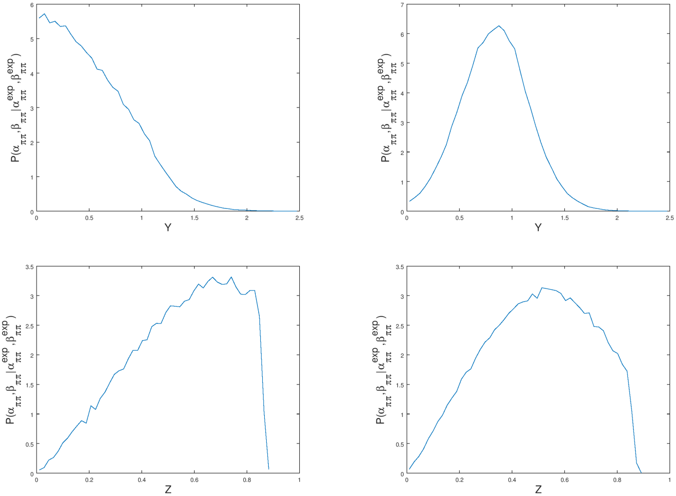

Table: Comparison of probability distributions between my work and Stern et al.

Summary

- I have calculated the subthreshold parameters $\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$ from Roy equations
- I have used the resummed approach with Bayesian approach to produce probability distributions for χPT LO parameters X, Y, Z
- Significant shift in probability distributions $\rightarrow$ more consistent with theoretical expectations