PHYSICS 442 – WINTER 2021
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Homework 4 - **due Thursday, March 4**

1. **Three-Flavor Oscillations in Matter (Earth)** – As discussed in class, neutrino oscillations can be recast as a discrete, non-relativistic quantum mechanics problem where each mass-eigenstate corresponds to an energy level of the Hamiltonian and time-evolution is replaced with baseline (L) evolution. Using this language, in the so-called flavor basis, where

$$|\nu_e\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\nu_\mu\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\nu_\tau\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

the Hamiltonian, including matter effects, is

$$H = U\frac{1}{2E}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta m^2_{21} & 0 \\ 0 & 0 & \Delta m^2_{31} \end{pmatrix} U^\dagger + \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where U is the leptonic mixing matrix, $|\nu_\alpha\rangle = U_{\alpha i}|\nu_i\rangle$, $\alpha = e, \mu, \tau$, $i = 1, 2, 3$. As usual, $A = \pm\sqrt{2}G_F N_e$, where G_F is the Fermi constant and N_e is the electron number density of the medium. The plus (minus) sign applies for the matter potential for (anti)neutrinos. $\Delta m^2_{ij} \equiv m^2_i - m^2_j$ are the differences of the neutrino masses-squared.

   (a) Compute $P_{\mu e}$ and $P_{\bar{\mu}\bar{e}}$ – the latter is the oscillation probability for antineutrinos – in the limit where $\Delta m^2_{21}/2E \ll |\Delta m^2_{31}|/2E, |A|$, and A is constant.

   (b) The NOνA and T2K experiments have baselines of L =810 km and L =295 km, respectively. At NOνA, the neutrino energy spectrum peaks at, roughly, 2 GeV; most of the neutrinos have energies between 1 GeV and 3 GeV. At T2K, the neutrino energy spectrum peaks at, roughly, 0.6 GeV; most of the neutrinos have energies between 0.2 GeV and 1 GeV. The matter potential can be expressed as

$$A = 7.6\times 10^{-5}\left(\frac{Y_e\rho}{\text{g/cm}^3}\right)\left(\frac{\text{eV}^2}{\text{GeV}}\right),$$

   where Y_e is the average number of electrons per nucleon – $Y_e = 0.5$ for the Earth is a good approximation – and ρ is the mass-density of the medium. For the Earth's crust, $\rho \simeq 2.8$ g/cm^3. In the same limit used in (a) ($\Delta m^2_{21}/2E \ll |\Delta m^2_{31}|/2E, |A|$), plot $P_{\mu e}$ and $P_{\bar{\mu}\bar{e}}$, for both mass orderings, as a function of the neutrino energy at NOνA and T2K, concentrating on the relevant energy ranges. For the oscillation parameters, use $\sin^2\theta_{13} = 0.022$, $\sin^2\theta_{23} = 0.55$ and $\Delta m^2_{13} = \pm 2.5\times 10^{-3}$ eV^2.

2. **Three-Flavor Oscillations in Matter (Sun)** – For neutrinos produced in the Sun, it turns out, $|\Delta m^2_{31}/2E| \gg \Delta m^2_{21}/2E, A$. Show that, in this limit,

$$P_{ee} \simeq \cos^4\theta_{13} \times P_{ee}^{(2f)} + \sin^4\theta_{13},$$

when $|\Delta m^2_{31}L/2E \gg 1$, i.e., when the short-wavelength oscillations "average out." Here, $P_{ee}^{(2f)}$ is the two-flavor oscillation expression one would obtain with $\theta \to \theta_{12}$, $\Delta m^2 \to \Delta m^2_{21}$ and $A \to A\cos^2\theta_{13}$. This result does not depend on A being a constant (only that it is small relative to the 1–3 term for all L) and hence applies to solar and terrestrial neutrinos. In practice, $\sin^2\theta_{13} \sim 0.02$ so solar data are mostly sensitive to Δm^2_{21} and $\sin^2\theta_{12}$.

3. Show that, assuming there are three neutrinos, $|P_{\alpha\beta} - P_{\beta\alpha}|$ is the same for all $\alpha, \beta = e, \mu, \tau$ when $\alpha \neq \beta$ (they are also trivially the same when $\alpha = \beta$, of course). This is independent of whether the neutrinos are propagating in matter or in vacuum; the only restriction is that the baseline evolution of the neutrino states is unitary (which it is if you include matter effects). Nonetheless, if you happen to be stuck, you can choose to show it for oscillations in vacuum.