INSS 2021
Neutrino oscillations in matter
Group A

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1 Three-Flavor Oscillations in Matter (Earth)

1. **Three-Flavor Oscillations in Matter (Earth)** – As discussed in class, neutrino oscillations can be recast as a discrete, non-relativistic quantum mechanics problem where each mass-eigenstate corresponds to an energy level of the Hamiltonian and time-evolution is replaced with baseline (L) evolution. Using this language, in the so-called flavor basis, where

$$|\nu_e\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\nu_\mu\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\nu_\tau\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

the Hamiltonian, including matter effects, is

$$H = U \frac{1}{2E} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta m_{21}^2 & 0 \\ 0 & 0 & \Delta m_{31}^2 \end{pmatrix} U^\dagger + \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where U is the leptonic mixing matrix, $|\nu_\alpha\rangle = U_{\alpha i}|\nu_i\rangle$, $\alpha = e, \mu, \tau$, $i = 1, 2, 3$. As usual, $A = \pm\sqrt{2}G_F N_e$, where G_F is the Fermi constant and N_e is the electron number density of the medium. The plus (minus) sign applies for the matter potential for (anti)neutrinos. $\Delta m_{ij}^2 \equiv m_i^2 - m_j^2$ are the differences of the neutrino masses-squared.

   (a) Compute $P_{\mu e}$ and $P_{\bar{\mu}\bar{e}}$ – the latter is the oscillation probability for antineutrinos – in the limit where $\Delta m_{21}^2/2E \ll |\Delta m_{31}^2|/2E, |A|$, and A is constant.

   (b) The NOνA and T2K experiments have baselines of L =810 km and L =295 km, respectively. At NOνA, the neutrino energy spectrum peaks at, roughly, 2 GeV; most of the neutrinos have energies between 1 GeV and 3 GeV. At T2K, the neutrino energy spectrum peaks at, roughly, 0.6 GeV; most of the neutrinos have energies between 0.2 GeV and 1 GeV. The matter potential can be expressed as

$$A = 7.6 \times 10^{-5} \left(\frac{Y_e \rho}{\text{g/cm}^3}\right) \left(\frac{\text{eV}^2}{\text{GeV}}\right),$$

   where Y_e is the average number of electrons per nucleon – $Y_e = 0.5$ for the Earth is a good approximation – and ρ is the mass-density of the medium. For the Earth's crust, $\rho \simeq 2.8$ g/cm^3. In the same limit used in (a) ($\Delta m_{21}^2/2E \ll |\Delta m_{31}^2|/2E, |A|$), plot $P_{\mu e}$ and $P_{\bar{\mu}\bar{e}}$, for both mass orderings, as a function of the neutrino energy at NOνA and T2K, concentrating on the relevant energy ranges. For the oscillation parameters, use $\sin^2\theta_{13} = 0.022$, $\sin^2\theta_{23} = 0.55$ and $\Delta m_{13}^2 = \pm 2.5 \times 10^{-3}$ eV^2.

2 Solution

2.1 3 oscillations in matter: general considerations

[1]The 3 Hamiltonian $\tilde{H}$ in flavor basis for arbitrary $N_e(x)$ density profile has a contribution from vacuum oscillations and a potential due to the coherent forward scattering of neutrinos with background electrons; this is similar to light experiencing a refractive index in medium. Thus, the Hamiltonian in the flavor basis is given by:

$$\tilde{H} = \frac{1}{2E} U M^2 U^\dagger + V, \tag{1}$$

where M is the diagonal mass matrix $M^2 = \mathrm{diag}(m_1^2, m_2^2, m_3^2)$ and U is a unitary 3 3 matrix which is effectively parametrized by 3 angles $_{23}, _{13}, _{12}$ and one CP phase (after re-definition of the flavor eigenstates). In particular, by defining the initary matrix U as

$$U = O_{23} \Gamma \ O_{13} \Gamma^\dagger O_{12}, \tag{2}$$

one can check that indeed $U^\dagger U = I$ where $\Gamma = \mathrm{diag}(1, 1, e^{i})$ and $O_{ij}^\dagger = O_{ij}^T$ are rotations matrices. Also, since there's only electrons (no , flavors, the matter potential is simply $V(x) = \mathrm{diag}(\sqrt{2} G_F N_e(x), 0, 0)$.

Two important observations are:

1. The matter potential V is invariant under the following transformation (because O_{23} does not touch the $1j, i1$ entries)

$$(O_{23}\Gamma)^\dagger V O_{23}\Gamma = V \tag{3}$$

2. The rotation O_{12} does not touch the $3j, i3$ entries, thus the Γ factors cancel each other when acting together with O_{12} on the M

$$\Gamma^\dagger O_{12} M^2 O_{12}^T \Gamma = O_{12} M^2 O_{12}^T \tag{4}$$

Key point: We can go from the flavor basis to a new "primed flavor basis" where it's easier to compute the evolution operator $\tilde{S}' = e^{-i\tilde{H}' t}$. The primed flavor basis is chosen based on the previous observations 1 and 2. The Hamiltonian in the chosen primed basis is then[2]:

$$\tilde{H}' = (O_{23}\Gamma)^\dagger \tilde{H} O_{23}\Gamma = O_{13} O_{12} \frac{M^2}{2E} (O_{13} O_{12})^T + V. \tag{5}$$

[1] Based on Lecture Notes from Eligio Lisi for the NBI Summer School 2016: "Tutorial on neutrino oscillation probabilities"

[2] The "prime" notation refers to the new basis while the "tilde" notation to the inclusion of matter contributions to the Hamiltonian

Note that $\tilde{H}'$ does not depend at all on nor on $_{23}$. Thus, this transformation will allow us to find the evolution operator more easily on this new basis. We can always transform back to the "real" flavor basis via the inverse prime transformation:

$$\tilde{S} = O_{23}\Gamma\ \tilde{S}'(O_{23}\Gamma\ )^{\dagger}. \tag{6}$$

For arbitrary matter potential $V(x) \neq V(\ \ x)$, one has that $\tilde{S}'_{;} \neq S'_{;}$ even if the Hamiltonian is real and symmetric $\tilde{H}'_{;} = \tilde{H}'_{;}$, because it matters in what other we compute i.e. $\tilde{S}'_{;}$ [original] $= \tilde{S}'_{;}$ [reversed] $\neq \tilde{S}'_{;}$ [original].

This extra complication is gone if we choose a constant matter potential which respects the reflection symmetry and thus the direct and reversed calculations of $\tilde{S}'$ are identical: N_e constant implies $\tilde{S}_{;} = \tilde{S}_{;}(\ \ !\ \ )$ and $\tilde{P}_{;} = \tilde{P}_{;}(\ \ !\ \ )$ from $P_{;} = |\tilde{S}_{;}|^2$

2.2 Computing $P_{e\nu}$ in constant matter density

Let us remind that

$$P_e = |\tilde{S}_e|^2 \text{ with } \tilde{S}_e = \tilde{S}'_e c_{23} + \tilde{S}'_e s_{23} e^{i} ; \tag{7}$$

where the $\tilde{S}_e$ evolution is a linear combination of the $\tilde{S}'$ elements. Expanding this expression one has:

$$\begin{aligned} P_e = |\tilde{S}_e|^2 &= |\tilde{S}'_e c_{23} + \tilde{S}'_e s_{23} e^{i}|^2 \\ &= \ 2\mathrm{Re}[\tilde{S}'_e \tilde{S}'_e] c_{23} s_{23} \cos \\ &\quad\ 2\mathrm{Im}[\tilde{S}'_e \tilde{S}'_e] c_{23} s_{23} \sin \\ &\quad + |\tilde{S}'_e|^2 c_{23}^2 + |\tilde{S}'_e|^2 s_{23}^2. \end{aligned} \tag{8}$$

Spoiler alert: In the limit $\frac{m^2}{2E}\ \ \frac{m^2}{2E}$ we get $\tilde{S}'_e = 0$ which reduces the flavor evolution from 3 to approximately 2 + 1 as we will explain[3]. This "trick" allows us to write the P_e in a simple way analogous to the 2 case.

Let's remind that in the primed basis (and with normal hierarchy):

$$\begin{aligned} \tilde{H}' &= O_{13} O_{12} \frac{M^2}{2E} (O_{13} O_{12})^T + V \quad \text{(independent of } _{23} \text{ and }) &(9)\\ M^2 &= \mathrm{diag}(\ \ m^2/2; +\ m^2/2; \Delta m^2) &(10)\\ V &= \mathrm{diag}(\sqrt{2} G_F N_e; 0; 0). &(11) \end{aligned}$$

In this primed basis, the flavor evolution decouples as 3 = 2 + 1 in the following two cases:

[3] where $\delta m^2 \equiv \ \ m_{12}^2,\ \ m^2 \equiv \ \ m_{13}^2$

1. $s_{13} \to 0 \Rightarrow O_{13} = 1$.

2. $m^2 \to 0 \Rightarrow O_{12} M^2 O_{12}^T = M^2$.

We focus on the second case. Under this limit we have that:

$$\lim_{m^2 \to 0} \tilde{H}' = \frac{1}{2E}\left(O_{13} \begin{bmatrix}0&0&0\\0&0&0\\0&0&\Delta m^2\end{bmatrix} O_{13}^T + \begin{bmatrix}A&0&0\\0&0&0\\0&0&0\end{bmatrix}\right)$$
$$= \left(\frac{\Delta m^2}{4E} + \frac{A}{4E}\right) I + \frac{1}{4E}\begin{bmatrix} A \quad \cos 2_{13} & 0 & \sin 2_{13}\Delta^2_m \\ 0 & \Delta m^2 \quad A & 0 \\ \sin 2_{13}\Delta m^2 & 0 & \cos 2_{13}\Delta m^2 \quad A. \end{bmatrix}$$

The structure of the Hamiltonian in the prime basis in the limit ($m^2 \to 0$) results in the $(e, {}')$ flavors evolving independently from the $'$ i.e. they do not mix and no $_e \to {}_{'}$ occur, thus

$$\lim_{m^2 \to 0} \tilde{S}'_{\ e} = 0. \tag{12}$$

On the other hand $\tilde{S}'_{\ e} \neq 0$ in this limit. From the 2 flavor case in matter we have develop all the tools that we need to compute $\tilde{S}'_{\ e}$.

Therefore, we have learned that in the limit $m^2 \to 0$ we have

$$P_e = \lim_{m^2 \to 0} |\tilde{S}\ _e|^2 \tag{13}$$
$$= \lim_{m^2 \to 0} |\tilde{S}'_{\ e} c_{23} + \tilde{S}'_{\ e} s_{23} e^{i}\ |^2 \tag{14}$$
$$= |\tilde{S}'_{\ e}|^2 s_{23}^2. \tag{15}$$

where from the 2 case in the presence of constant matter we know that

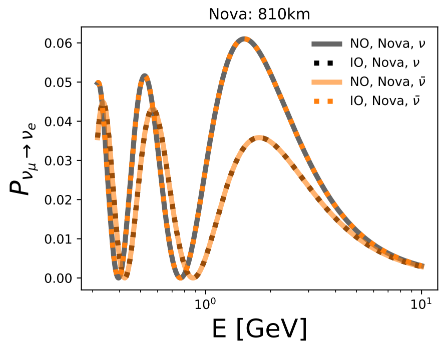
$$|S'_{\ e}|^2 = \sin^2 2\tilde{\ }_{13} \sin\left(\frac{\Delta\tilde{m}^2}{4E}\right), \tag{16}$$

which leads us to our final expression for the conversion probability

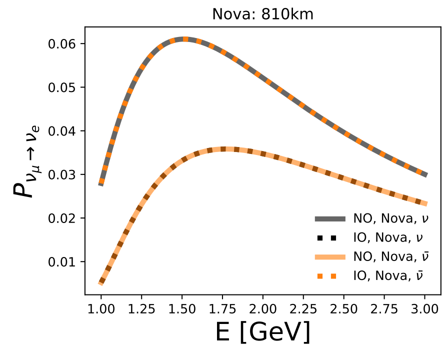
$$P_e = \sin^2 {}_{23} \sin^2 2\tilde{\ }_{13} \sin\left(\frac{\Delta\tilde{m}^2}{4E}\right), \tag{17}$$

where as before the effecting mixing parameters in matter are defined as

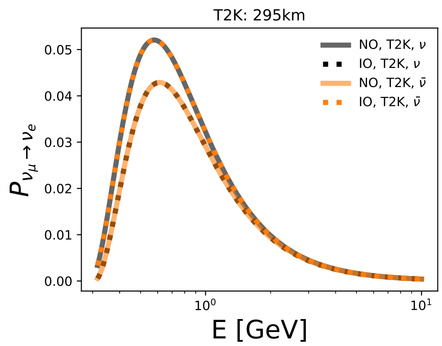
$$\Delta\tilde{m}^2 = \Delta m^2 \sqrt{\left(\cos(2\ _{13}) \quad \frac{2AE}{\Delta m^2}\right)^2 + \sin^2(2\ _{13})} \tag{18}$$
$$\sin^2(2\tilde{\ }_{13}) = \frac{\sin(2\ _{13})}{\sqrt{\left(\cos(2\ _{13}) \quad \frac{2AE}{m^2}\right)^2 + \sin^2(2\ _{13})}}. \tag{19}$$



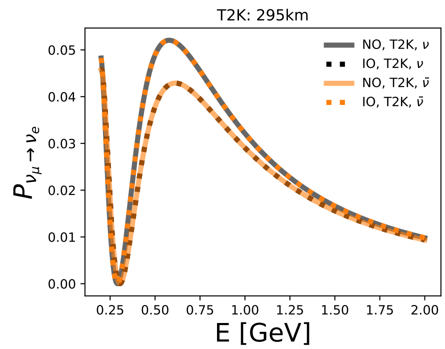
(a) Noνa, logarithmic energy scale



(b) Noνa, linear energy scale



(a) T2K, logarithmic energy scale



(b) T2K, linear energy scale

Mass ordering $\Delta m^2 \leftrightarrow -\Delta m^2$ and matter potential sign $A \leftrightarrow -A$ Normal and inverted change the sign of the vacuum oscillation frequency i.e. the sign of the mass squared difference $\Delta m^2 \to -\Delta m^2$. At the level of the Hamiltonian, it's clear that such a change of sign can be absolved by the matter potential by doing $A \to -A$. Notice the dependence on relative sign between NMO and particle/antiparticle:

$$P^{NO}_{\alpha\to\beta} = P^{IO}_{\alpha\to\beta} \neq P^{IO}_{\alpha\to\beta} = P^{NO}_{\alpha\to\beta}. \quad (20)$$

The latter transformation corresponds to swapping $\nu \leftrightarrow \bar{\nu}$ in the evolution of flavor, as we will see in our results.

Unit conversion Knowing the values of Planck constant ($\hbar = 6.58 \times 10^{-22}$ MeV s) and speed of light ($c = 3 \times 10^{8}$ m s^{-1}), we can switch to SI unit as shown below:

$$\frac{\Delta m^2_{ij} L}{4E} = \hbar^{-1} c^3 \frac{\Delta m^2_{ij} L}{4E} \approx 1.27 \frac{\Delta m^2_{ij} c^4}{\mathrm{eV}^2} \frac{\mathrm{MeV}}{E} \frac{L}{\mathrm{km}}. \quad (21)$$