OPTIMAL BOUNDS ON THE QUANTUM SPEED OF SUBSPACE EVOLUTION*

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*Based on a joint work with Sergio Albeverio.

Introduction

In Q.M., a ***quantum speed limit*** is a ***lower bound on the time*** required for a quantum system to evolve between two (given) distinguishable states.

For shortness, we will assume that the measurement units are chosen in such a way that

$$\hbar = 1.$$

Let us consider an isolated Q.M. system with a Hamiltonian H, which is supposed to be a time-independent self-adjoint operator in the Hilbert space $\mathfrak{H}$. So that the unit (i.e. normalized) vectors of $\mathfrak{H}$ represent possible (pure) states of the system. Evolution of a state vector $\psi(t) \in \mathfrak{H}$, $t \in \mathbb{R}$, is governed by the Schrödinger equation (of course, under the assumption that $\psi \in C^1(\mathbb{R}, \mathfrak{H})$)

$$\mathrm{i}\frac{d}{dt}\psi = H\psi, \tag{1}$$

$$\psi(t)\Big|_{t=t_0} = \psi_0 \ \ \big(\in \mathrm{Dom}(H)\big), \tag{2}$$

where ψ_0 represents an initial state of the system.

(Under the condition $\psi_0 \in \mathrm{Dom}(H)$, the inclusion $\psi(t) \in \mathrm{Dom}(H)$ holds automatically for any t under consideration :-).

Let $t_0 = 0$. Then the solution to (1), (2) is given by

$$\psi(t) = U(t)\psi_0, \qquad \text{where } U(t) = \mathrm{e}^{-\mathrm{i}Ht}, \quad t \in \mathbb{R}; \tag{3}$$

the operators $U(t)$, $t \in \mathbb{R}$, form a strongly continuous unitary group.

Studies of quantum speed limits originate from the very basic question:

How fast can the isolated system with the Hamiltonian H evolve to a state orthogonal to its initial state ψ_0?

The importance of this question is obvious in many respects. Probably, the very latest motivation comes from quantum information theory and quantum computing.

On the concept of state in Q.M. A system state $\mathscr{S}$ is a class of equivalence of vectors on a unit sphere in the Hilbert space $\mathfrak{H}$ of the system. It is assumed that the (normalized) vectors $\phi \in \mathfrak{H}$ and $\psi \in \mathfrak{H}$ belong to (and represent) the same state if there is $\alpha \in [0, 2\pi)$ such that $\psi = \mathrm{e}^{\mathrm{i}\alpha}\phi$.

In an obvious way, one may identify the system state $\mathscr{S}$ with a the one-dimensional subspace $\mathfrak{P}_{\mathscr{S}}$ which is a linear span of an arbitrary vector $\psi \in \mathscr{S}$, $\mathfrak{P}_{\mathscr{S}} := \{\phi = \lambda\psi \,|\, \lambda \in \mathbb{C}\}$.

Known answers to the above basic question — lower bounds for the orthogonalization time $T_\perp$:

Mandelstam–Tamm inequality (1945)

$$T_\perp \geq \frac{\pi}{2\,\Delta E}, \qquad (4)$$

Margolis–Levitin inequality (1998)

$$T_\perp \geq \frac{\pi}{2\,\delta E}, \qquad (5)$$

where

$$\Delta E = \sqrt{\|H\psi_0\|^2 - \langle H\psi_0, \psi_0\rangle^2} \quad \text{and} \quad \delta E = \langle H\psi_0, \psi_0\rangle - \min\bigl(\mathrm{spec}(H)\bigr) \qquad (6)$$

are the energy spread (dispersion) for the state ψ_0 and the average energy for this state measured relative to the lower bound of H.

Both inequalities recall the uncertainty relation for energy and time but are very different in the essence since these inequalities are related not to the standard deviation in the measurement of t but to the well-founded time for a given state to evolve into an orthogonal state.

Fleming bound (1973)

$$T_\theta \geq \frac{\theta}{\Delta E}, \tag{7}$$

where T_θ is the time moment at which the acute angle

$$\angle\big(\psi_0, \psi(t)\big) := \arccos |\langle \psi_0, \psi(t) \rangle|$$

between the vectors ψ_0 and $\psi(t)$ reaches the value of $\theta \in (0, \pi/2]$.

The Mandelstam-Tamm bound is a particular case of the Fleming bound for $\theta = \dfrac{\pi}{2}$.

All the three bounds (4), (5), and (7) have been proven to be sharp.

Notice that, through the years, the Mandelstam-Tamm bound has been rediscovered several times by various researchers. Also, there are generalizations of this bound to the evolution of mixed states. Furthermore, there are more detail evolution speed estimates for particular classes of quantum-mechanical evolutionary problems. For details, look, e.g., at the review articles [1,2,3]. For the very latest developments look at [4,5].

[1] S. Deffner and S. Campbell, *Quantum speed limits: From Heisenberg's uncertainty principle to optimal quantum control*, J. Phys. A: Math. Gen **50** (2017), 453001 (49 pages).

[2] M. R. Frey, *Quantum speed limits — primer, perspectives, and potential future directions*, Quant. Information Processing **15** (2016), 3919–3950.

[3] C. M. Bender and D. C. Brody, *Optimal time evolution for Hermitian and non-Hermitian Hamiltonians*, Lect. Notes Phys. **789** (2009), 341–361.

[4] N. Il'in and O. Lychkovskiy, *Quantum speed limit for thermal states*, Phys. Rev. A **103** (2021), 062204.

[5] S. Albeverio and A.K. Motovilov, *Quantum speed limits for time evolution of a system subspace*, arXiv:2011.02778 (2020) [8 pages].

Our results: Bounds for the speed of the subspace evolution

We are concerned not with a single state but with a whole (possibly infinite-dimensional) subspace that is the subject to the Schrödinger evolution. That is, we consider a subspace $\mathfrak{P}_0 \subset \mathfrak{H}$ such that every vector taken in $\mathfrak{P}_0 \cap \mathrm{Dom}(H)$ evolves according to the Schrödinger equation (1), that is, we consider the family of Cauchy problems

$$\mathrm{i}\frac{d}{dt}\psi = H\psi, \tag{8}$$

$$\psi(t)\Big|_{t=t_0} = \psi_0, \quad \psi_0 \in \mathfrak{P}_0 \cap \mathrm{Dom}(H). \tag{9}$$

It is also assumed that (a quite natural but strong assumption!!)

$$\mathfrak{P}_0 \cap \mathrm{Dom}(H) \oplus \mathfrak{P}_0^{\perp} \cap \mathrm{Dom}(H) = \mathrm{Dom}(H). \tag{10}$$

The assumption (10) implies the 2×2 block matrix representation with respect to the decomposition $\mathfrak{H} = \mathfrak{P}_0 \oplus \mathfrak{P}_0^{\perp}$:

$$H = H_{\mathrm{diag}} + H_{\mathrm{off}}, \quad H_{\mathrm{diag}} = \begin{pmatrix} H_{\mathfrak{P}_0} & 0 \\ 0 & H_{\mathfrak{P}_0^{\perp}} \end{pmatrix}, \quad H_{\mathrm{off}} = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}$$

with $H_{\mathfrak{P}_0} = PH|_{\mathfrak{P}_0}$, $\mathrm{Dom}(H_{\mathfrak{P}_0}) = \mathfrak{P}_0 \cap \mathrm{Dom}(H)$, $H_{\mathfrak{P}_0^{\perp}} = PH|_{\mathfrak{P}_0}$, $\mathrm{Dom}(H_{\mathfrak{P}_0^{\perp}}) = \mathfrak{P}_0^{\perp} \cap \mathrm{Dom}(H)$, and $B = PH|_{\mathfrak{P}_0^{\perp}}$.

Sometimes, the Hamiltonian H will be assumed to be **a bounded operator**. But, in general it is **unbounded**.

Recalling: A linear operator H on a space with norm is called bounded if there is a number $c > 0$ such that

$$\|Hx\| \leq c\|x\| \quad \text{for any} \quad x \in \mathrm{Dom}(H).$$

Otherwise, H is called unbounded.

If H is bounded Hermitian operator (on a Hilbert space) then

$$\|H\| = \max\{|E_{\min}(H)|, |E_{\max}(H)|\},$$

where

$$E_{\min}(H) = \min(\mathrm{spec}(H)) \quad \text{and} \quad E_{\max}(H) = \max(\mathrm{spec}(H))$$

are respectively the upper and lower bounds of the spectrum of H.

Unfortunately for us, few-particle Hamiltonians are usually unbounded.

Given $t \geq 0$, by $\mathfrak{P}(t)$ we will denote the closure of the span of the values $\psi(t)$ of the vector functions that solve (8), (9) for various $\psi_0 \in \mathfrak{P}_0 \cap \mathrm{Dom}(H)$. So that we deal with a path: $\mathfrak{P}(t)$, $t \geq 0$, in the set of all subspaces of the Hilbert space $\mathfrak{H}$. Or (and this is the same) with the path of the orthogonal projections onto $\mathfrak{P}(t)$,

$$P(t), \quad t \geq 0, \qquad \mathrm{Ran}\big(P(t)\big) = \mathfrak{P}(t) \quad \text{(and } P(0) = P_0\text{)}. \tag{11}$$

Surely, the family $P(t)$, $t \in \mathbb{R}$, is explicitly given by

$$P(t) = U(t)P_0U(t)^* = \mathrm{e}^{-\mathrm{i}Ht}P_0\mathrm{e}^{\mathrm{i}Ht}. \tag{12}$$

It is well known (and this is easily verified) that the path $P(t)$, $t \geq 0$, is a *strong solution* to the Cauchy problem

$$\mathrm{i}\frac{d}{dt}P = [P, H], \tag{13}$$

$$P(t)\Big|_{t=t_0} = P_0, \tag{14}$$

where $[P,H] := PH - HP$ is the commutator of $P = P(t)$ and H.

[The solution $P(t)$ should be strong in the sense that both (13) and (14) are assumed to hold being applied to any $\psi \in \mathrm{Dom}(H)$.]

It is well known that the set of all orthogonal projections in the Hilbert space $\mathfrak{H}$ (and hence the set of all subspaces of $\mathfrak{H}$) is a metric space with distance given by the standard operator norm,

$$\rho(Q_1,Q_2) := \|Q_1 - Q_2\|, \qquad \rho(\mathfrak{Q}_1,\mathfrak{Q}_2) := \rho(Q_1,Q_2),$$

where Q_1, Q_2 are arbitrary orthogonal projections and $\mathfrak{Q}_1$, $\mathfrak{Q}_2$, their ranges.

It is, however, much less known that there is ANOTHER natural metric on the set of all the orthogonal projections in/ all the subspaces of the Hilbert space $\mathfrak{H}$. The corresponding distance is defined by

$$\vartheta(\mathfrak{Q}_1,\mathfrak{Q}_2) := \vartheta(Q_1,Q_2) := \arcsin(\|Q_1 - Q_2\|). \tag{15}$$

That (15) is a metric has been first proven in 1993 by Lawrence Brown. An alternative (and, we think, somewhat simpler) proof may be found in our joint paper with Sergio Albeverio (2013).

The quantity $\vartheta(\mathfrak{Q}_1,\mathfrak{Q}_2)$ is called the ***maximal angle*** between the subspaces $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$.

Remark The concept of maximal angle between subspaces is traced back to Krein, Krasnoselsky, and Milman (1948). Assuming that $(\mathfrak{Q}_1, \mathfrak{Q}_2)$ is an ordered pair of subspaces with $\mathfrak{Q}_1 \neq \{0\}$, they applied the notion of the (relative) maximal angle between $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ to the number $\varphi(\mathfrak{Q}_1, \mathfrak{Q}_2) \in [0, \pi/2]$ such that

$$\sin\varphi(\mathfrak{Q}_1, \mathfrak{Q}_2) = \sup_{x \in \mathfrak{Q}_1,\, \|x\|=1} \operatorname{dist}(x, \mathfrak{Q}_2).$$

If both $\mathfrak{Q}_1 \neq \{0\}$ and $\mathfrak{Q}_2 \neq \{0\}$ then

$$\vartheta(\mathfrak{Q}_1, \mathfrak{Q}_2) = \max\big\{\varphi(\mathfrak{Q}_1, \mathfrak{Q}_2), \varphi(\mathfrak{Q}_2, \mathfrak{Q}_1)\big\}.$$

Unlike $\varphi(\mathfrak{Q}_1, \mathfrak{Q}_2)$, the maximal angle $\vartheta(\mathfrak{Q}_1, \mathfrak{Q}_2)$ is always symmetric with respect to the interchange of the arguments $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$. Furthermore,

$$\varphi(\mathfrak{Q}_2, \mathfrak{Q}_1) = \varphi(\mathfrak{Q}_1, \mathfrak{Q}_2) = \vartheta(\mathfrak{Q}_1, \mathfrak{Q}_2) \quad \text{whenever } \|Q_1 - Q_2\| < 1.$$

To give a QM interpretation of the maximal angle $\vartheta(\mathfrak{Q}_1, \mathfrak{Q}_2)$ we follow the concept of a *subspace-state* of a quantum system. Namely, given a subspace $\mathfrak{Q} \subset \mathfrak{H}$, one says that the system is in the $\mathfrak{Q}$-state if it is in a pure state described by a (non-specified) vector $x \in \mathfrak{Q}$, $\|x\| = 1$. Then the quantity $\cos^2\phi(\mathfrak{Q}_1, \mathfrak{Q}_2)$ is understood as a minimum probability for a quantum system which is in a $\mathfrak{Q}_1$-state to be found also in a $\mathfrak{Q}_2$-state. The $\cos^2\theta(\mathfrak{Q}_1, \mathfrak{Q}_2)$ is the minimum of the quantities $\cos^2\phi(\mathfrak{Q}_1, \mathfrak{Q}_2)$ and $\cos^2\phi(\mathfrak{Q}_2, \mathfrak{Q}_1)$.

Let

$$0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t, \quad n \geq 1, \tag{16}$$

be an arbitrary partition of the interval $[0,t]$, $t > 0$. If $Q(t)$ it a projection path, then by the triangle inequality for the operator angle we have

$$\begin{aligned} \vartheta\left(\mathfrak{Q}_0, \mathfrak{Q}_t\right) &\leq \sum_{k=1}^{n} \vartheta\left(\mathfrak{Q}_{t_{k-1}}, \mathfrak{Q}_{t_k}\right) \\ &= \sum_{k=1}^{n} \arcsin \|Q(t_{k-1}) - Q(t_{k-1})\|, \end{aligned} \tag{17}$$

where $\mathfrak{Q}_s = \mathrm{Ran}\big(Q(s)\big)$, $s \geq 0$.

In particular, if the path $Q(t)$, $t \geq 0$, is piecewise smooth w.r.t. the operator norm topology, then (17) implies (see, e.g., Makarov-Seelmann, 2015) that

$$\vartheta\left(\mathfrak{Q}_0, \mathfrak{Q}_t\right) \leq \int_0^t \|\dot{Q}(t)\| dt, \tag{18}$$

where $\dot{Q}(t) = \frac{dQ(t)}{dt}$, $t \geq 0$.

The following statement is the main tool that we use below in establishing quantum speed limits for evolution of subspaces. It represents itself the first of such limits. Its general proof is based directly on (17).

Theorem 1. *Assume that the projection path $P(t)$ is given by $P(t) = \mathrm{e}^{-\mathrm{i}Ht}P_0\mathrm{e}^{\mathrm{i}Ht}$, $t \geq 0$, where P_0 is an orthogonal projection in $\mathfrak{H}$ such that the closure of the commutator $[P_0, H]$ is a* ***bounded operator****. Then the following inequality holds*

$$\vartheta\left(\mathfrak{P}_0, \mathfrak{P}_t\right) \leq V_{H,P_0}\, t, \tag{19}$$

where $\mathfrak{P}_0 = \mathrm{Ran}(P_0)$, $\mathfrak{P}_t = \mathrm{Ran}\big(P(t)\big)$, $t \geq 0$, and

$$V_{H,P_0} := \|P_0HP_0^{\perp}\| = \|P_0^{\perp}HP_0\| \quad \big(= \|[P_0, H]\|\big). \tag{20}$$

Proof. More precisely, this is a hint for the proof in the case of bounded H. Observe that the commutator of $P(t)$ and H does not depend on t, thus,

$$\big\|[P(t), H]\big\| = \big\|[P_0, H]\big\|, \quad t \geq 0. \tag{21}$$

Furthermore, the commutator $[P_0, H]$ is block off-diagonal with respect to the orthogonal decomposition $\mathfrak{H} = \mathfrak{P}_0 \oplus \mathfrak{P}_0^{\perp}$, more precisely,

$$[P_0, H] = P_0HP_0^{\perp} - P_0^{\perp}HP_0 = \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix}. \tag{22}$$

From (22) it immediately follows that $\big\|[P_0, H]\big\| = \|P_0HP_0^{\perp}\| = \|P_0^{\perp}HP_0\|$. Thus, in order to conclude with (19) it only remains to combine (13) with (21) and (22) and then to apply to $P(t)$ the inequality (18). □

It is worth to notice that by (19), (20) only the off-diagonal entries $P_0HP_0^{\perp}$ and $P_0^{\perp}HP_0$ of H contribute into the variation of the subspace $\mathfrak{P}_0$. If the Hamiltonian

H is block diagonal with respect to the decomposition $\mathfrak{H} = \mathfrak{P}_0 \oplus \mathfrak{P}_0^{\perp}$ and, thus, the subspace $\mathfrak{P}_0$ is reducing for H it does not vary in time at all. This concerns, in particular the case where $\mathfrak{P}_0$ is a spectral subspace of H.

Corollary 2. *Under the hypothesis of Theorem 1, assume that T_θ is a time moment for which the maximal angle between the initial subspace $\mathfrak{P}_0$ and a subspace in the path $\mathfrak{P}_t$, $t \geq 0$, reaches the value of θ, $0 < \theta \leq \frac{\pi}{2}$, that is,*

$$\vartheta\left(\mathfrak{P}_0, \mathfrak{P}(T_\theta)\right) = \theta. \tag{23}$$

Then

$$T_\theta \geq \frac{\theta}{V_{H,P_0}}. \tag{24}$$

Example 3. Let the Hamiltonian H correspond to a two-level quantum system with non-degenerate bound states e_1 and e_2, that is, $\|e_1\| = \|e_2\| = 1$, $\langle e_1, e_2 \rangle = 0$, and

$$H = E_1 \langle \cdot, e_1 \rangle e_1 + E_2 \langle \cdot, e_2 \rangle e_2$$

where the binding energies E_1 and E_2 are different, $E_1 \neq E_2$. Assume that P_0, $P_0 = \langle \cdot, e \rangle e$, is projection on the one-dimensional subspace spanned by the vector $e = \frac{1}{\sqrt{2}}(e_1 + e_2)$. In this case (24) turns into equality, which means that the lower bound (24) is sharp.

Notice that Example 3 is employed in many papers on quantum speed limits (see the surveys mentioned above). In particular, this example proves tightness of both the Mandelstam-Tamm and Margolus-Levitin inequalities.

Theorem 4. *Assume the hypothesis of Theorem 1. Let θ and T_θ be the same as in Corollary 2. Then the following inequality holds:*

$$T_\theta \geq \frac{\theta}{\Delta E_{\mathfrak{P}_0}}, \tag{25}$$

where

$$\Delta E_{\mathfrak{P}_0} := \sup_{\psi \in \mathfrak{P}_0 \cap \mathrm{Dom}(H),\, \|\psi\|=1} \left(\|H\psi\|^2 - \langle H\psi, \psi \rangle^2 \right)^{1/2} \tag{26}$$

Skipping the proof, we only notice that (25) is proven by Theorem 1 by taking into account that $V_{H,P_0} \leq \Delta E_{\mathfrak{P}_0}$.

Remark 5. The same Example 3 shows that the bound (25) is sharp. (In the case of a one-dimensional subspace this bound simply turns into the Fleming bound for the speed of a state evolution).

The next statement is rather well known. We present it only for convenience.

Lemma 6. *Assume that that H is a bounded self-adjoint operator on $\mathfrak{H}$ and let $\mathfrak{P}_0$ be an arbitrary subspace of $\mathfrak{H}$. Then the maximal energy dispersion*

$$\Delta E_{\mathfrak{P}_0} := \sup_{\psi \in \mathfrak{P}_0 \cap \mathrm{Dom}(H),\, \|\psi\|=1} \left(\|H\psi\|^2 - \langle H\psi, \psi \rangle^2 \right)^{1/2}$$

satisfies the following (optimal) bound

$$\Delta E_{\mathfrak{P}_0} \leq \frac{E_{\max}(H) - E_{\min}(H)}{2}, \tag{27}$$

where

$$E_{\min}(H) = \min\big(\mathrm{spec}(H)\big) \quad \textit{and} \quad E_{\max}(H) = \max\big(\mathrm{spec}(H)\big)$$

are respectively the upper and lower bounds of the spectrum of the Hermitian Hamiltonian H.

By using Lemma 6 one derives the following corollary.

Corollary 7. *Assume that* Ω *is a non-negative number and let* $\mathscr{B}_\Omega(\mathfrak{H})$ *be the set of all bounded self-adjoint operators* H *on the Hilbert space* $\mathfrak{H}$ $($*with* $\dim\mathfrak{H} \geq 2)$ *such that*

$$E_{\max}(H) - E_{\min}(H) = \Omega.$$

Then

$$\inf_{H\in\mathscr{B}_\Omega(\mathfrak{H})} T_\theta(H) = \frac{2\theta}{\Omega}, \tag{28}$$

where $T_\theta(H)$ *is a time moment for which the maximal angle between the initial subspace* $\mathfrak{P}_0$ *and a subspace in the path* $\mathfrak{P}_t$, $t \geq 0$, *given by* (12) *reaches the value of* $\theta \leq \dfrac{\pi}{2}$.

The bound (28) represents a generalization to subspaces of the optimal passage time estimate established for the quantum brachistochrone problem (see, e.g. [6]. The latter estimate is nothing but the equality in the Fleming bound (7) with ΔE replaced by $\frac{1}{2}\Omega$ where the quantity Ω is introduced in Corollary 7.

[6]C. M. Bender and D. C. Brody, *Optimal time evolution for Hermitian and non-Hermitian Hamiltonians*, Lect. Notes Phys. **789** (2009), 341–361.

THANK YOU FOR YOUR ATTENTION!