HYPER-RADIAL ASYMPTOTICS OF THE WAVE FUNCTION OF THREE-PARTICLES WITH COULOMB INTERACTION IN THE CONTINUUM

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LXXI International Conference NUCLEUS-2021

St Petersburg

September 20-25

2021

Plan of the talk

I. The three-body problem in HS representation
II. Radial asymptotics of the three-particle wave function (short-range case)
III. Radial asymptotics of the three-particle wave function (long-range case)
III. Conclusion

I. The three-body problem in HS representation

The 3-body Hamiltonian in the center of mass frame:

$$H = H_0 + V \equiv -\Delta_X + \sum_{a=1}^{3} V_a(x_a),$$

$X = \{x_a, y_a\} \in \mathbb{R}^6$ is the set of standard mass-weighted Jacobi coordinates, $x_a \in \mathbb{R}^3$ is the two-body relative coordinate

$V_a(x)$ are two body potentials:

$$V_a(x) = \frac{q_b q_c}{|x|} + V_a^s(x)$$

$V_a^s(x)$ is a short-range potential such that $V_a^s(x) \sim O\left(\frac{1}{|x|^{2+\mu}}\right)$, $\mu > 0$
$\{a, b, c\}$ runs over $\{1, 2, 3\}$ cyclically.

Schrödinger equation in HS representation $\varrho = |X|$:

$$\left\{-\frac{d^2}{d\varrho^2} + \frac{L_k(L_k+1)}{\varrho^2} - p^2\right\}\Psi_{[k][n]}(\varrho,p) + \sum_{[k']}\frac{C_{[k][k']}}{\varrho}\Psi_{[k'][n]}(\varrho,p) =$$

$$= -\sum_{[k']} V^s_{[k][k']}(\varrho)\Psi_{[k'][n]}(\varrho,p), \quad L_k = k + 3/2$$

Wave funnction HS componets:

$$\Psi_{[k][n]}(\varrho,p) = \varrho^{5/2}\int d\hat{X}d\hat{P}\Psi(X,P)Y_{[k]}(\hat{X})Y^*_{[n]}(\hat{P})$$

Potentials HS matrix elements:

$$\frac{C_{[k][n]}}{\varrho} = \int d\hat{X}d\hat{P}\sum_a \frac{q_b q_c}{|x_a|}Y_{[k]}(\hat{X})Y^*_{[n]}(\hat{P})$$

Schrödinger equation in HS representation $\varrho = |X|$:

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Wave funnction HS componets:

$$\Psi_{[k][n]}(\varrho,p) = \varrho^{5/2}\int d\hat{X}d\hat{P}\Psi(X,P)Y_{[k]}(\hat{X})Y^*_{[n]}(\hat{P})$$

Potentials HS matrix elements:

$$V^s_{[k][n]}(\varrho) = \int d\hat{X}d\hat{P}\sum_a V^s_a(|x_a|)Y_{[k]}(\hat{X})Y^*_{[n]}(\hat{P})$$

II. Radial asymptotics of the three-particle wave function ($q_a = 0$, $a = 1, 2, 3$ short-range case)

$3 \to 3$ scattering solution of the Schrödinger equation

$$\Psi(P) = \lim_{\epsilon \to 0}(-i\epsilon)(H - P^2 - i\epsilon)^{-1}\Psi_0(P),$$

where $P \in \mathbb{R}^6$ is the incident momentum and $\Psi_0(X, P) = \exp\{i(P, X)\}$ is the incident plain wave, has the following asymptotics of HS components as $\varrho \to \infty$ [Yakovlev TMPh 2021]

$$\Psi_{[k][n]}(\varrho, p) \sim Q^{0-}_{[k][n]}(\varrho, p)(-1)^k - \sum_{[k']} Q^{0+}_{[k][k']}(\varrho, p) S_{[k'][n]}(p)$$

where $p = |P|$, $S_{[k][n]}(p)$ is the S-matrix and $Q^{0\pm}_{[k][n]}(\varrho, p)$ are given by

$$Q^{0\pm}_{[k][n]}(\varrho, p) = \exp\{\pm i(p\varrho - i3\pi/4)\}\delta_{[k][n]}$$

Spherical waves $Q^{0\pm}_{[k][n]}$ as solutions to

$$\left\{-\frac{d^2}{d\varrho^2} - p^2\right\} Q^{0\pm}_{[k][n]}(\varrho, p) = 0$$

are the asymptotic solutions to the full Schrödinger equation with the discrepancy $O(\varrho^{-2})$:

$$\left\{-\frac{d^2}{d\varrho^2} + \frac{L_k(L_k+1)}{\varrho^2} - p^2\right\} Q^{0\pm}_{[k][n]}(\varrho, p) +$$
$$+ \sum_{[k']} V^s_{[k'][n]}(\varrho) Q^{0\pm}_{[k][n]}(\varrho, p) = O(\varrho^{-2})$$

Better asymptotic solutions are given in terms of Riccati-Hankel functions $h^{\pm}_{L_k}(p\varrho)$

$$Q^{\pm}_{[k][n]}(\varrho,p) = h^{\pm}_{L_k}(p\varrho)\delta_{[k][n]}\exp\{\mp i(3\pi/4 - L_k\pi/2)\},$$

which as the solutions to

$$\left\{-\frac{d^2}{d\varrho^2} + \frac{L_k(L_k+1)}{\varrho^2} - p^2\right\} h^{\pm}_{L_k}(p\varrho) = 0$$

obey the full Schrödinger equation with better discrepancy:

$$\left\{-\frac{d^2}{d\varrho^2} + \frac{L_k(L_k+1)}{\varrho^2} - p^2\right\} Q^{\pm}_{[k][n]}(\varrho,p) +$$
$$+\sum_{[k']} V^{s}_{[k'][n]}(\varrho) Q^{\pm}_{[k][n]}(\varrho,p) = O\left(\frac{1}{\varrho^{2+\mu}}\right)$$

Thanks to the asymptotics of the Riccati-Hankel function

$$h_L^{\pm}(z) = \exp\{\pm i(z - L\pi/2)\}\left(1 \pm i\frac{L(L+1)}{2z} + O(z^{-2})\right)$$

the leading term of $Q^{\pm}_{[k][n]}(\varrho, p)$ coincides with $Q^{0\pm}_{[k][n]}(\varrho, p)$ and the wave function asymptotics can now be taken in the form

$$\Psi_{[k][n]}(\varrho, p) \sim (-1)^k Q^{-}_{[k][n]}(\varrho, p) - \sum_{[k']} Q^{+}_{[k][k']}(\varrho, p) S_{[k'][n]}(p)$$

Resumé:

- Potential therms of the order $O(\varrho^{-2})$ do not make contribution in the leading terms of the asymptotic solutions of the Schrödinger equation
- Solve the Schrödinger equations for asymptotic solutions which in the leading order are incoming and outgoing spherical waves
- Take the linear combination of this asymptotic solutions as the boundary conditions for scattering problem

III. Radial asymptotics of the three-particle wave function (long-range case)

Schrödinger equation with Coulomb interaction in HS representation:

$$\left\{-\frac{d^2}{d\varrho^2} - p^2\right\}\Psi_{[k][n]}(\varrho,p) + \sum_{[k']}\frac{C_{[k][k']}}{\varrho}\Psi_{[k'][n]}(\varrho,p) =$$

$$= -\frac{L_k(L_k+1)}{\varrho^2}\Psi_{[k][n]}(\varrho,p) - \sum_{[k']}V^s_{[k][k']}(\varrho)\Psi_{[k'][n]}(\varrho,p), \quad L_k = k + 3/2$$

Two stage solution

- 1) Solving the asymptotic equation without the $O(\varrho^{-2})$ terms
- 2) Correction of the solution by taking into account the $O(\varrho^{-2})$ terms

Solution of equations of stage 1)

Matrix Coulomb problem:

$$\left\{-\frac{d^2}{d\varrho^2} - p^2\right\} \Phi^0_{[k][n]}(\varrho, p) + \sum_{[k']} \frac{C_{[k][k']}}{\varrho} \Phi^0_{[k'][n]}(\varrho, p) = 0 \qquad (1)$$

Two independent solutions are given in terms of the Coulomb-Hankel wave functions $H_0^{\pm}(\eta, z)$ after diagonalization of the Matrix Coulomb potential

$$\Phi^{0\pm}_{[k][n]}(\varrho, p) = \sum_{[k']} u^{([k'])}_{[k]} H_0^{\pm}(d_{[k']}/(2p), p\varrho) u^{([k'])*}_{[n]}$$

where $d_{[n]}$ and $u^{([n])}_{[k]}$ are eigenvalues and corresponding eigenvectors of the charge-matrix

$$\sum_{[k']} C_{[k][k']} u^{([n])}_{[k']} = u^{([n])}_{[k]} d_{[n]}$$

Asymptotic solution of equations of stage 2)

Schrödinger equation with Coulomb plus $O(\varrho^{-2})$ interaction:

$$\left\{-\frac{d^2}{d\varrho^2} - p^2\right\} \Phi_{[k][n]}(\varrho, p) + \sum_{[k']} \frac{C_{[k][k']}}{\varrho} \Phi_{[k'][n]}(\varrho, p) =$$

$$= -\frac{L_k(L_k+1)}{\varrho^2} \Phi_{[k][n]}(\varrho, p)$$

In matrix notations

$$\left\{-\frac{d^2}{d\varrho^2} - p^2\right\} \mathbf{\Phi} + \frac{\mathbf{C}}{\varrho} \mathbf{\Phi} = -\frac{\mathbf{L}}{\varrho} \mathbf{\Phi}$$

Asymptotic solution

$$\mathbf{\Phi}^{\pm}(\varrho, p) = \mathbf{T}^{\pm}(\varrho, p) \mathbf{\Phi}^{0\pm}(\varrho, p)$$

Asymptotic solution of equations of stage 2)

Matrix $\mathbf{T}^{\pm}$:

$$\mathbf{T}^{\pm}(\varrho, p) = \mathbf{T}^{0\pm}(p) + \frac{\mathbf{T}^{1\pm}(p)}{\varrho} + \ldots$$

Matrix elements of $\mathbf{T}^{\#\pm}(p)$ matrices are given by

$$T^{0\pm}_{[k][n]} = \delta_{[k][n]} \exp\{\mp i3\pi/4\}$$

$$T^{1\pm}_{[k][n]} = \sum_{[k'][n']} u^{([k'])}_{[k]} \tilde{T}^{1\pm}_{[k'][n']} u^{([n'])*}_{[n]} \exp\{\mp i3\pi/4\}$$

$$\tilde{T}^{1\pm}_{[k][n]} = \frac{\tilde{L}_{[k][n]}}{d_{[k]} - d_{[n]} \pm 2ip}$$

$$\tilde{L}_{[k][n]} = \sum_{[k'][n']} u^{([k])*}_{[k']} L_{k'}(L_{k'} + 1) u^{([n])}_{[k']}$$

Asymptotic boundary conditions for 3 → 3 scattering problem with Coulomb Interaction:

$$\Psi_{[k][n]}(\varrho, p) \sim (-1)^k \Phi^-_{[k][n]}(\varrho, p) - \sum_{[k']} \Phi^+_{[k][k']}(\varrho, p) S_{[k'][n]}(p)$$

Asymptotic boundary conditions for $3 \to 3$ scattering problem without Coulomb Interaction:

$$\Psi_{[k][n]}(\varrho, p) \sim (-1)^k Q^{-}_{[k][n]}(\varrho, p) - \sum_{[k']} Q^{+}_{[k][k']}(\varrho, p) S_{[k'][n]}(p)$$

III. Conclusion

- The asymptotic boundary conditions for $3 \to 3$ scattering problem derived by weak asymptotics of the wave function have been corrected to include the contribution of the $O(\varrho^{-2})$ term for short-range case
- The asymptotic boundary conditions for $3 \to 3$ scattering for a system of three particles with Coulomb interactions have been constructed for the first time

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