PHOTONUCLEAR RESEARCHES: STATUS OF EXPERIMENTS

ФОТОЯДЕРНЫЕ ИССЛЕДОВАНИЯ: СТАТУС ЭКСПЕРИМЕНТОВ

Varlamov V.V., Davydov A.I., Orlin V.N.

Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University
Centre for Photonuclear Experiments Data
Physical Faculty, Lomonosov Moscow State University

The talk is devoted to the well-known old problems which at the same time are of modern interest:
significant disagreements between photonuclear reaction cross sections obtained in various experiments.

Those data are widely used in many fields of researches and applications:

- **competition of various Giant Dipole Resonance (GDR) decay channels;**
- **competition of direct and statistical processes in decays of highly-excited nuclear states;**
- **GDR configurational and isospin splitting effects;**
- **astrophysics problems;**
- **monitoring of the beam luminosity in ultra-relativistic heavy-ion colliders, ….**

Experimental data for about 50 nuclei were investigated using the objective physical criteria of data reliability and
the talk provides an overview of status of experiments.

The talk continues the discussions on the modern status of experimental photonuclear data presented for the NUCLEUS-2020 meetings.

The absolute majority of experimental data for partial
$(\gamma, 1n)$, $(\gamma, 2n)$, $(\gamma, 3n)$,
and total

$$(\gamma, sn) = (\gamma, 1n) + (\gamma, 2n) + (\gamma, 3n) + \ldots,$$
$$(\gamma, xn) = (\gamma, 1n) + 2(\gamma, 2n) + 3(\gamma, 3n) + \ldots$$

photonuclear reaction cross sections was obtained at Livermore (USA) and Saclay (France) using beams of quasimonoenergetic annihilation photon beams and the method of photoneutron multiplicity sorting (multiplicity of neutron was determined using the results of measurement of its energy).

Atlas of Photoneutron cross sections obtained with monoenergetic photons
(S.S.Dietrich, B.L.Berman. Atom. Data and Nucl. Data Tables, 38 (1988) 199).

Statistic (numbers of data sets for near all stable nuclei)

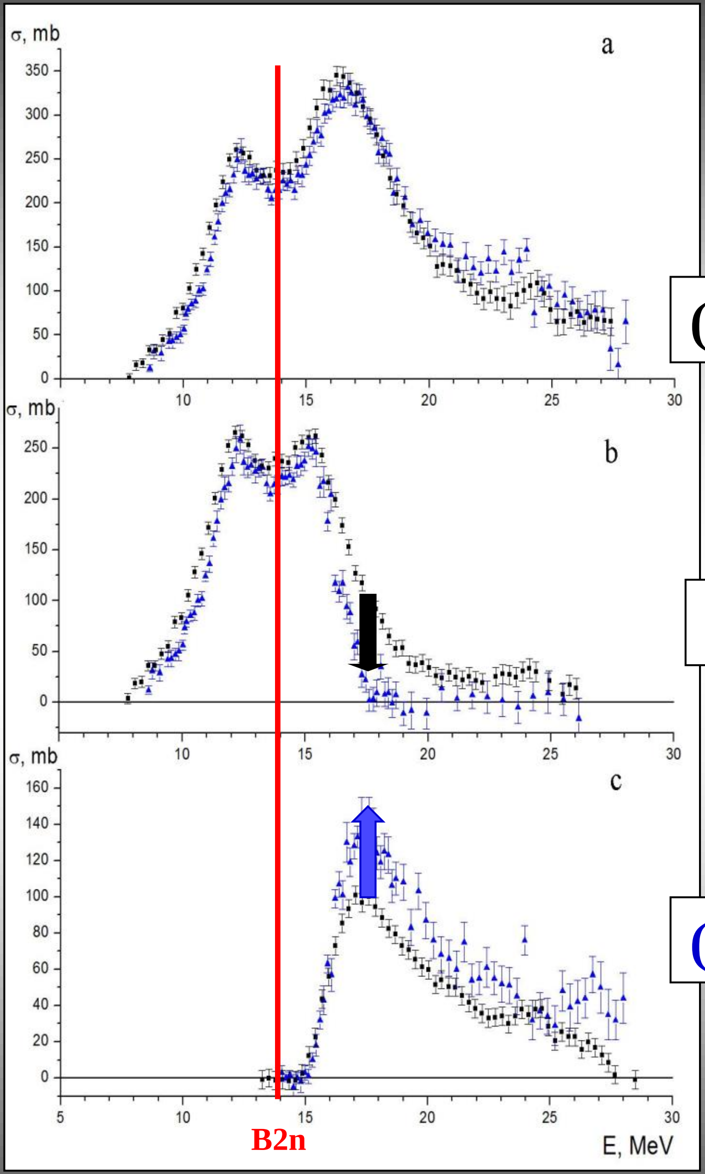
Quasimonoenergetic photons				Bremsstrahlung
Livermore	Both	Saclay	Other	
~240 data sets	~120 data sets (19 nuclei)	~250 data sets	~20 data sets	Several tens

Main problem (users headache): significant disagreements

19 nuclei:

^{51}V, ^{75}As, ^{89}Y, ^{90}Zr, ^{115}In, ^{116,117,118,120,124}Sn, ^{127}I, ^{133}Cs, ^{159}Tb, ^{165}Ho, ^{181}Ta, ^{197}Au, ^{208}Pb, ^{232}Th, ^{238}U

Typical example of disagreements: ^{159}Tb



σ^{int}, MeV mb

(γ, xn)

3194 Saclay

3187 Livermore

almost coincide

(γ, 1n)

1936 Saclay

1413 Livermore

37%

(γ, 2n)

887 Livermore

605 Saclay

47%

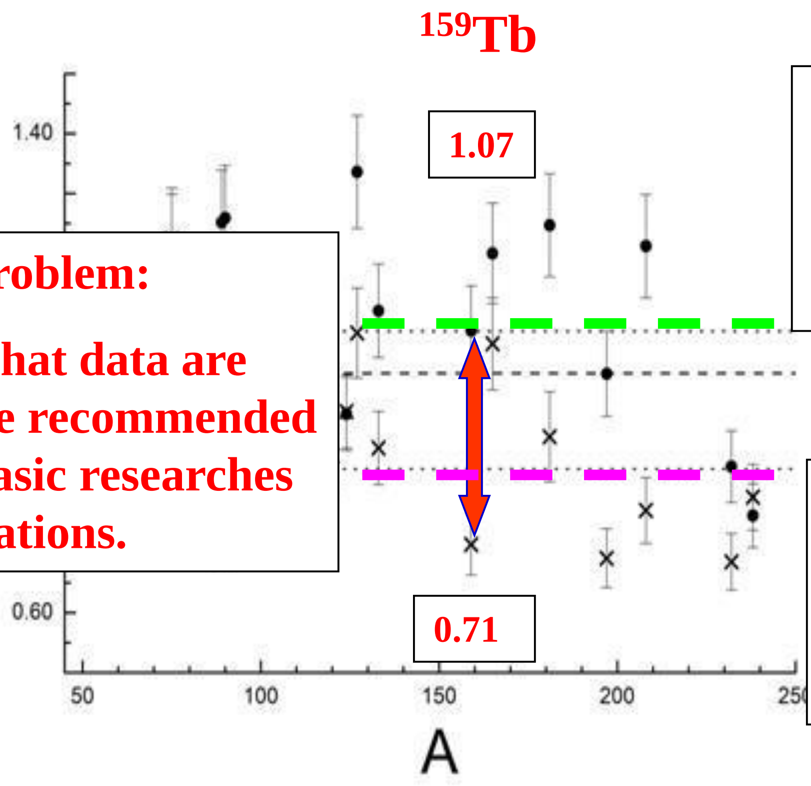
Significant disagreements between partial reaction cross sections

Nucleus	$R_{S/L}^{int}$(1n)	$R_{S/L}^{int}$(2n)
^{51}V	1.07	0.79
^{75}As	1.21	1.22
^{89}Y	1.25	0.87
^{90}Zr	1.26	0.73
^{115}In	0.97	0.76
^{116}Sn	1.10	0.92
^{117}Sn	1.02	
^{118}Sn	1.07	
^{120}Sn	1.00	
^{124}Sn	0.93	
^{127}I	1.34	
^{133}Cs	1.10	
^{159}Tb	1.07	0.71
^{165}Ho	1.20	1.05
^{181}Ta	1.25	0.89
^{197}Au	1.00	0.69
^{208}Pb	1.21	0.77
^{232}Th	0.84	0.69
^{238}U	0.76	0.79

Systematic of disagreements for 19 nuclei mentioned

The main problem:

nobody knows what data are reliable and could be recommended for using in both basic researches and applications.



1.07

0.71

Circles - ratios for (γ, 1n) reactions – are larger than 1.0:

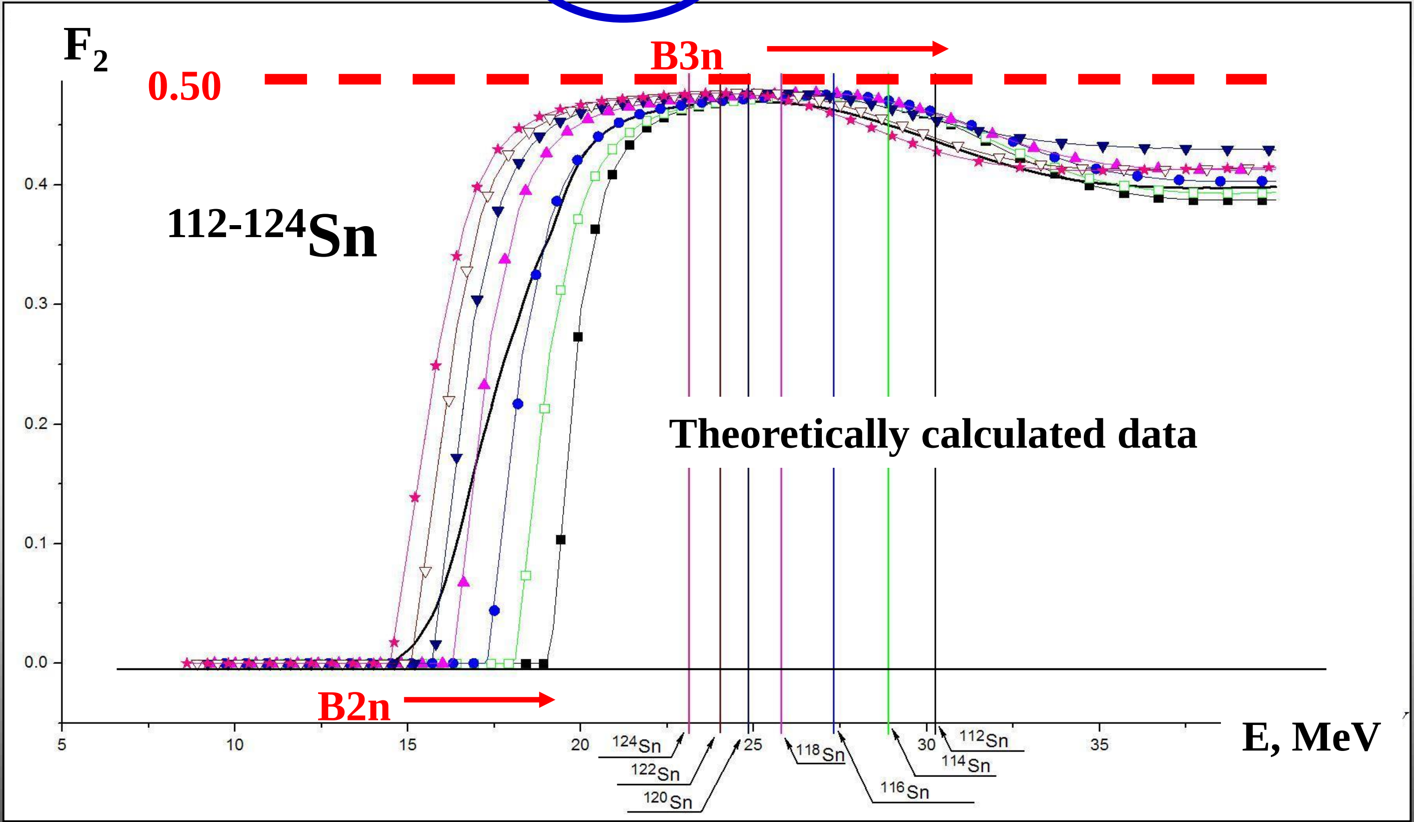
<R> ~ 1.08.

Crosses - ratios for (γ, 2n) reactions – are smaller than 1.0:

<R> ~ 0.83.

The objective physical criteria of data reliability were proposed.

$$F_2 = \frac{\sigma(\gamma, 2n)}{\sigma(\gamma, 1n) + 2\sigma(\gamma, 2n) + 3\sigma(\gamma, 3n) + \ldots} < 0.50\ (!)$$



Theoretically calculated data

The natural and physically reliable energy dependence of F_2 should be following:

– Below the (γ, 2n) reaction threshold B2n only the (γ, 1n) reaction is possible: $F_2 = 0$;

– Above B2n both (γ, 1n) and (γ, 2n) reactions are possible, F_2 increases due to competition between decreasing $\sigma(\gamma, 1n)$ and increasing $\sigma(\gamma, 2n)$, going to the theoretical limit of 0.50, but never reach it because of a high–energy part in $\sigma(\gamma, 1n)$;

– Above the B3n threshold the (γ, 3n) reaction is also possible, F_2 decreases due to a $3\sigma(\gamma, 3n)$.

The natural physical additions:

- $F_1 < 1.00$,
- $F_2 < 0.50$,
- $F_3 < 0.33$,
- $F_4 < 0.25$,
- $F_5 < 0.20\ldots$

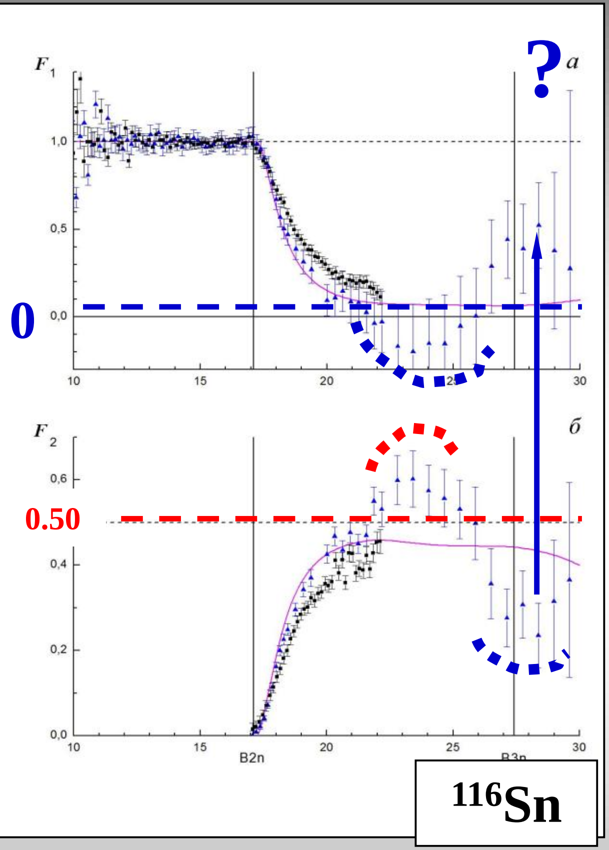
Using the physical data reliability criteria the experimental cross sections of partial reactions were investigated for about 50 nuclei obtained using quasimonoenergetic annihilation photon beams and the method of neutron multiplicity sorting (first of all for majority of 19 nuclei from systematics mentioned above):

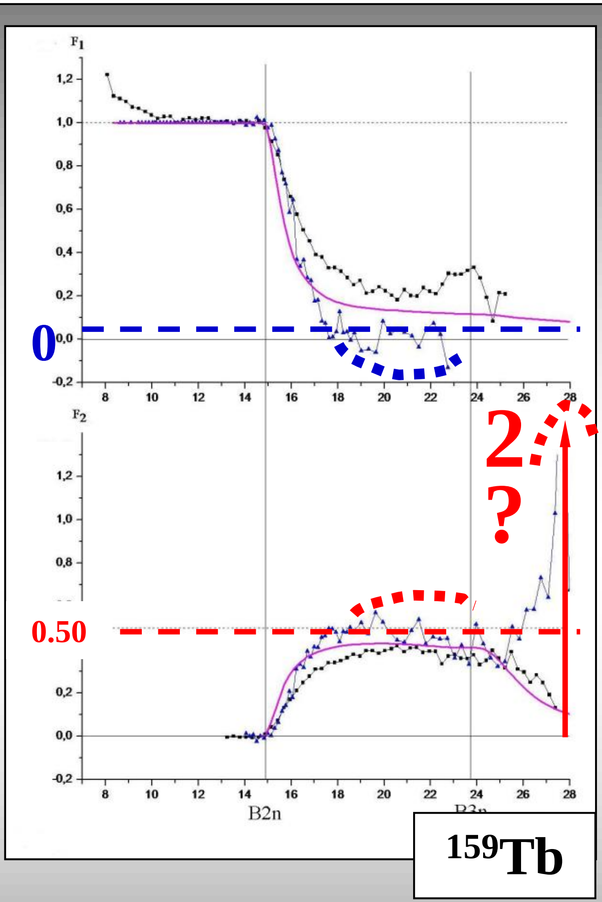
$^{51}\mathrm{V}_S$, $^{59}\mathrm{Co}_L$, $^{63,65}\mathrm{Cu}_L$, $^{75}\mathrm{As}_S$, $^{76,78,80,82}\mathrm{Se}_S$, $^{89}\mathrm{Y}_S$, $^{90,91,92,94}\mathrm{Zr}_S$, $^{103}\mathrm{Rh}_S$, $^{115}\mathrm{In}_L$, $^{116,117,118,120,124}\mathrm{Sn}_L$, $^{127}\mathrm{I}_S$, $^{129}\mathrm{Xe}_S$, $^{133}\mathrm{Cs}_L$, $^{138}\mathrm{Ba}_L$, $^{139}\mathrm{La}_S$, $^{140,142}\mathrm{Ce}_S$, $^{141}\mathrm{Pr}_L$, $^{145,148}\mathrm{Nd}_S$, $^{153}\mathrm{Eu}_S$, $^{160}\mathrm{Gd}_L$, $^{159}\mathrm{Tb}_S$, $^{165}\mathrm{Ho}_S$, $^{181}\mathrm{Ta}_S$, $^{186}\mathrm{W}_L$, $^{186,188,190,192}\mathrm{Os}_L$, $^{197}\mathrm{Au}_S$, $^{206,207,208}\mathrm{Pb}_L$, $^{208}\mathrm{Pb}_S$, $^{209}\mathrm{Bi}_L$,

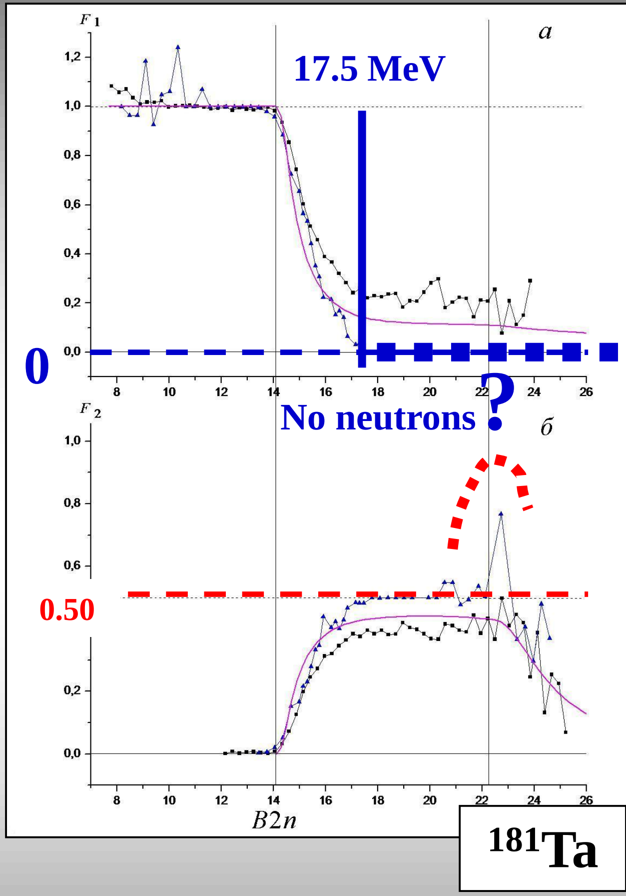
and obtained using alternative methods:

**bremsstrahlung and subtraction method (^{112,114,119}Sn),
bremsstrahlung and activation method (^{197}Au, ^{181}Ta, ^{209}Bi),
quasimonoenergetic photons from laser Compton scattering (^{159}Tb, ^{197}Au, ^{209}Bi).**

A lot of unreliable data







The comparison of ratios F_i^{exp} obtained for Livermore (triangles) Saclay (squares) cross sections with calculated F_i^{theor} (lines).

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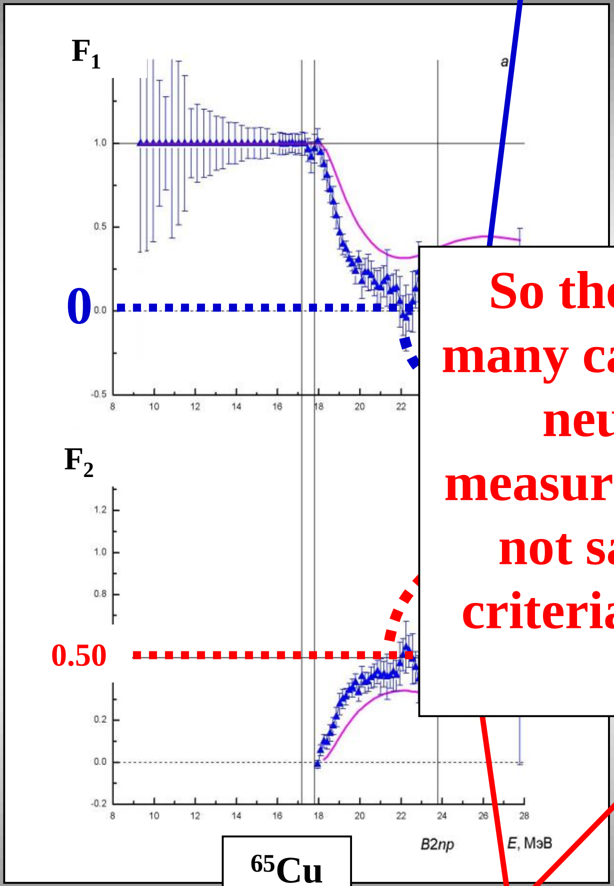
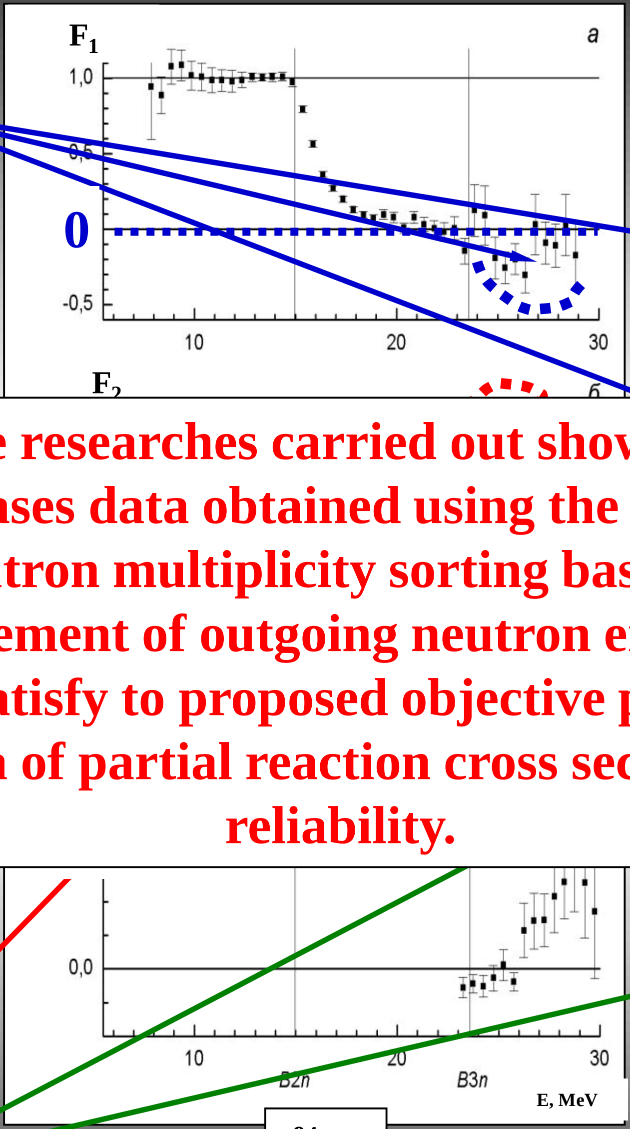
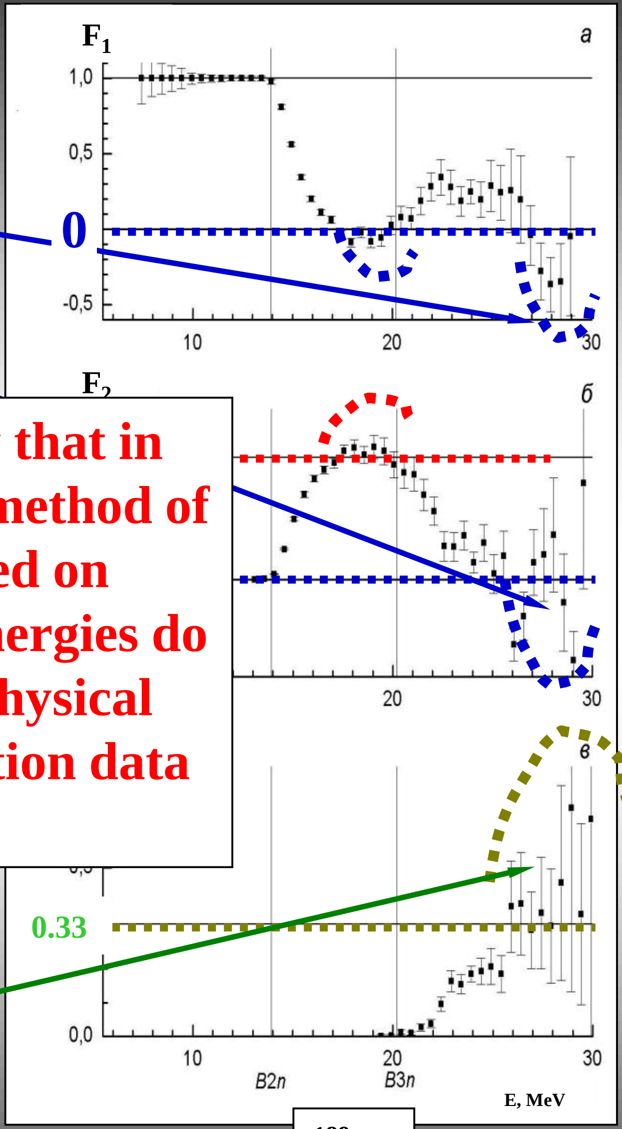
Physically forbidden negative values



^{65}Cu



^{94}Zr



^{189}Os

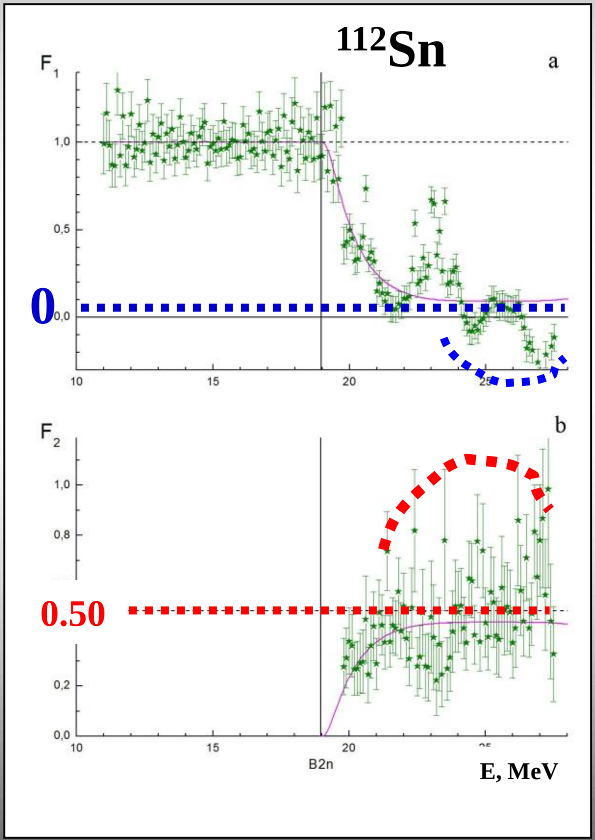
So the researches carried out show that in many cases data obtained using the method of neutron multiplicity sorting based on measurement of outgoing neutron energies do not satisfy to proposed objective physical criteria of partial reaction cross section data reliability.

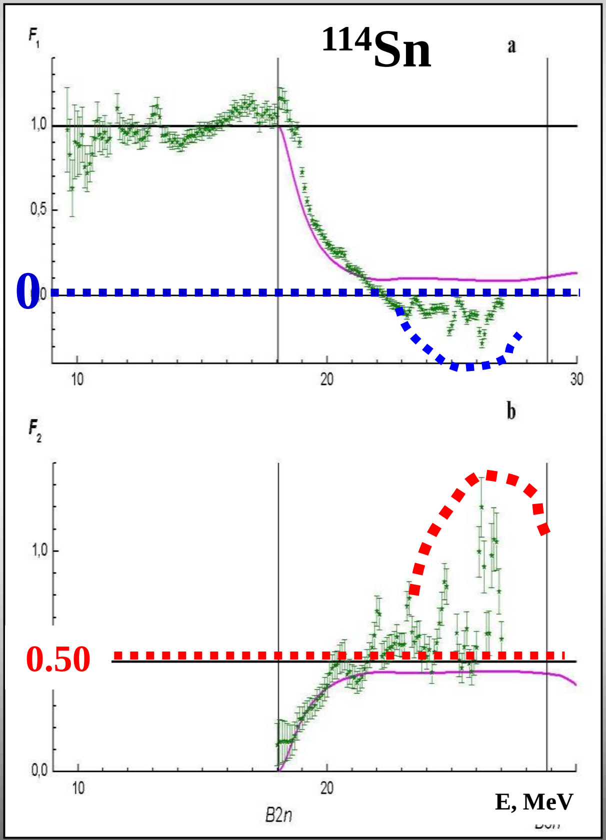
Physically unreliable
$F_i > 0.50$ (0.33)

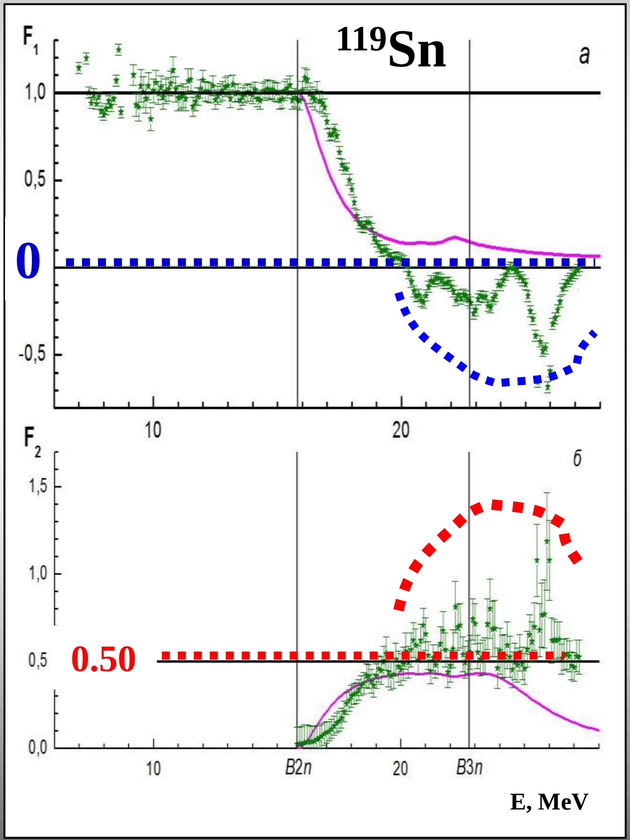
Cu - Zr - Os

Similar unreliable data obtained in experiments using beams of bremsstrahlung (statistical theory corrections of neutron yield cross section (γ, xn))

(Sov. J. Nucl. Phys. 20, 123 (1975), Bull.Acad.Sci.USSR, Phys.Ser. 39, 98 (1975)).







The experimental-theoretical method of evaluation was proposed:

$$\sigma^{eval}(\gamma, in) = F_i^{theor}(\gamma, in) \bullet \sigma^{exp}(\gamma, xn).$$

This approach means that partial reactions (γ, 1n), (γ, 2n) and (γ, 3n) competitions are independent on the problems of experimental

The research carried out for about 50 nuclei

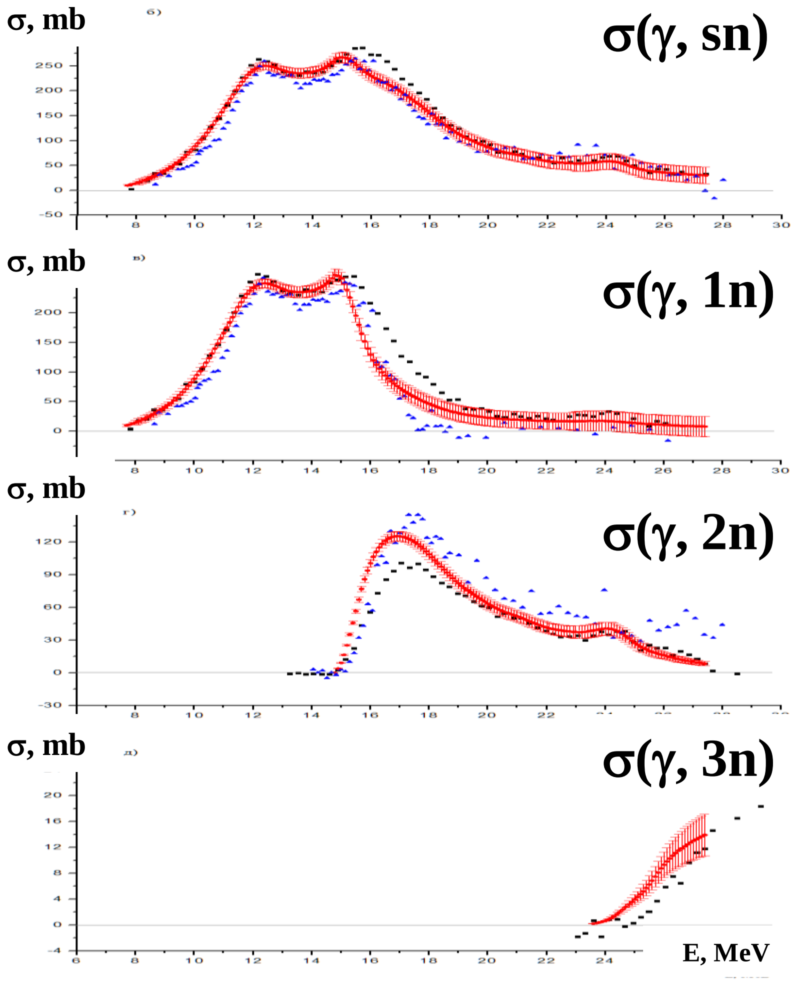
(^{51}V, ^{59}Co, ^{63,65}Cu, ^{75}As, ^{76,78,80,82}Se, ^{89}Y, ^{90,91,92,94}Zr, ^{103}Rh, ^{115}In, ^{116,117,118,120,124}Sn, ^{127}I, ^{129}Xe, ^{133}Cs, ^{138}Ba, ^{139}La, ^{140,142}Ce, ^{141}Pr, ^{145,148}Nd, ^{153}Eu, ^{160}Gd, ^{159}Tb, ^{165}Ho, ^{181}Ta, ^{186}W, ^{186,188,190,192}Os, ^{197}Au, ^{206,207,208}Pb, ^{209}Bi)

and the correspondent sum of evaluated cross sections

$$\sigma^{eval}(\gamma, xn) = \sigma^{eval}(\gamma, 1n) + 2\sigma^{eval}(\gamma, 2n) + 3\sigma^{eval}(\gamma, 3n) + \ldots$$

is equal to the experimental σ^{exp}(γ, xn) and also relatively independent on multiplicity problems.

Typical example of evaluation results: ^{159}Tb



The evaluated cross sections differ noticeably from the experimental once

	σ^{int}, MeV•mb				
	Livermore		Evaluation		Saclay
(γ, xn)	3187	≈	3200	≈	3194
(γ, sn)	2300	< 4%	2383	< 7%	2557
(γ, 1n)	1413	< 16%	1642	< 18%	1936
(γ, 2n)	887	24% >	714	20% >	605
(γ, 3n)	46	77% >	26	63% >	16

Using the analysis in detail the differences between evaluated and experimental cross sections it was found that there are several sources of significant systematic uncertainties

Typical examples of differences

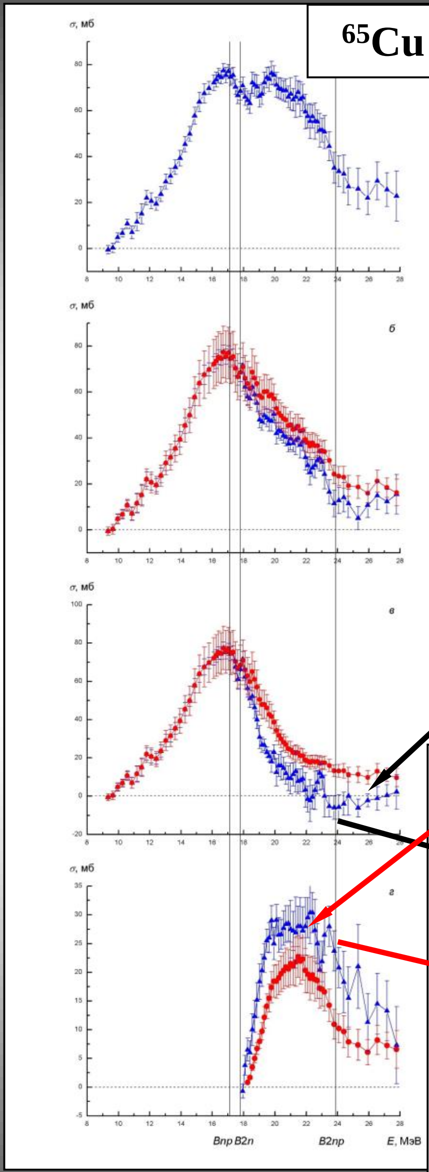
The number of neutrons unreliably extracted from one reaction is close to the number of neutrons unreliably added to another one

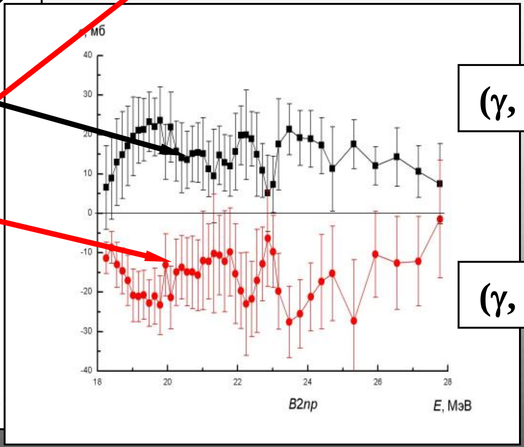
$[\sigma^{exp}(\gamma, 1n) - \sigma^{eval}(\gamma, 1n)]$

$[\sigma^{eval}(\gamma, 2n) - \sigma^{exp}(\gamma, 2n)]$

Physically forbidden negative values

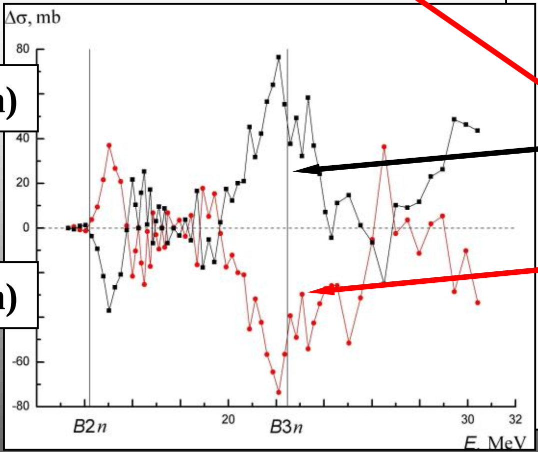
Physically unreliable values ($F_2 > 0.50$)

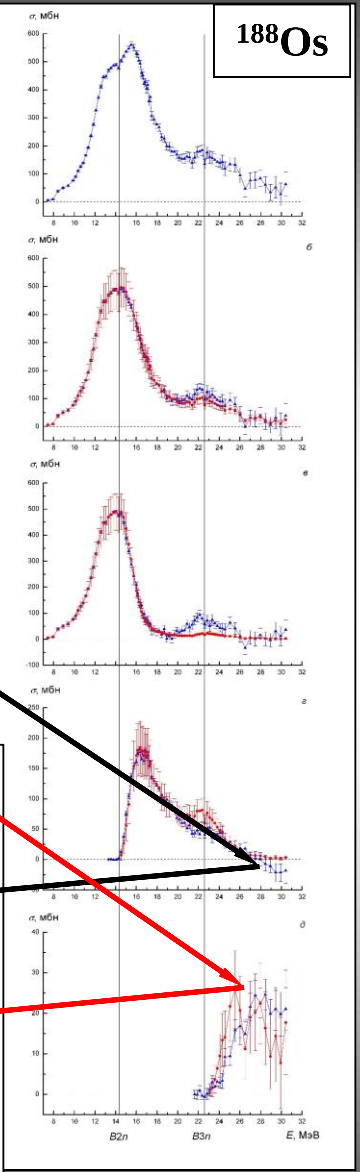




(γ, 1n)

(γ, 2n)



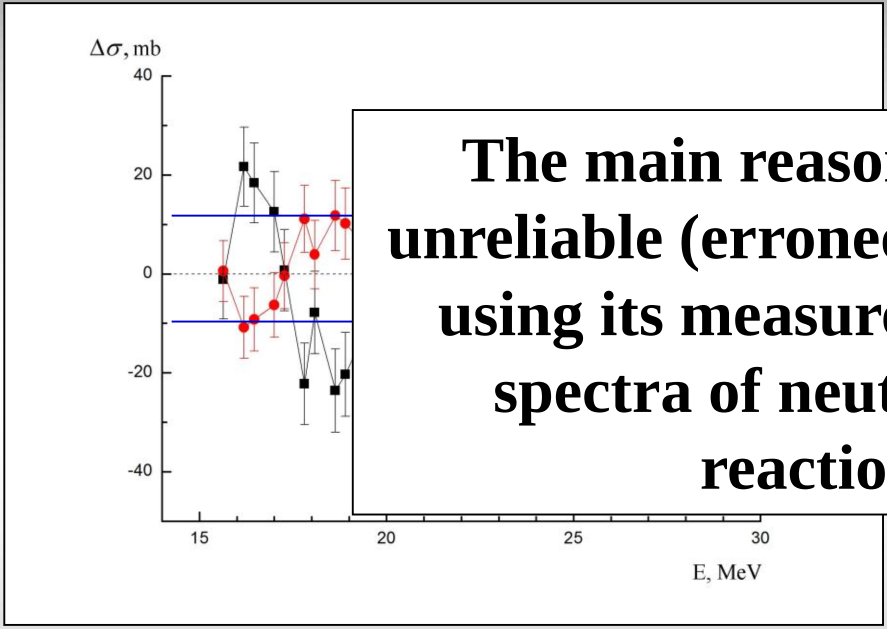


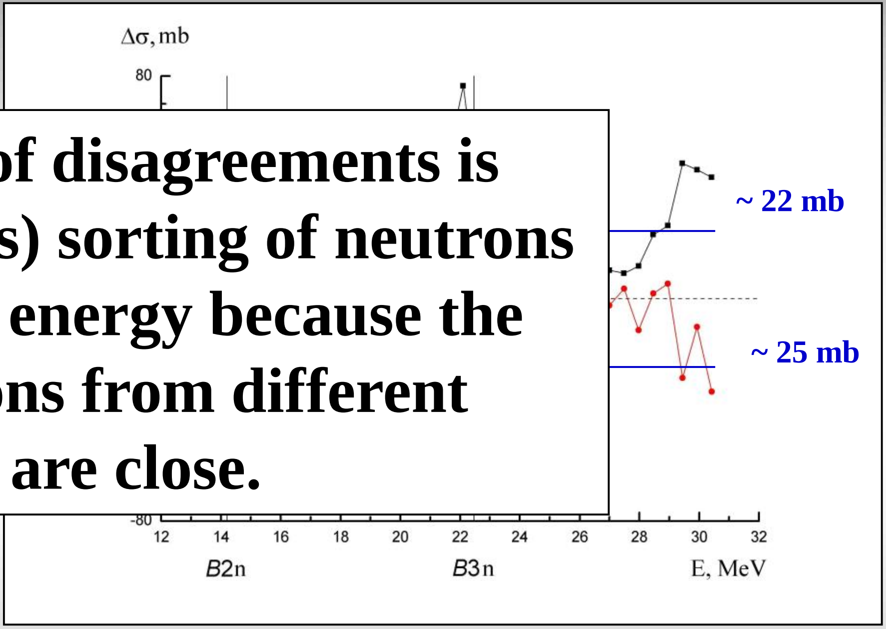
^{98}Mo: Mos. Univ. Phys. Bull., 73, 68 (2018)

^{188}Os: Phys. Atom. Nucl., 78, 746 (2015)

H. Beil, et. al., Nucl. Phys. 227 427 (1974)

B.L.Berman, et.al., Phys. Rev. C 19, 1205 (1979)



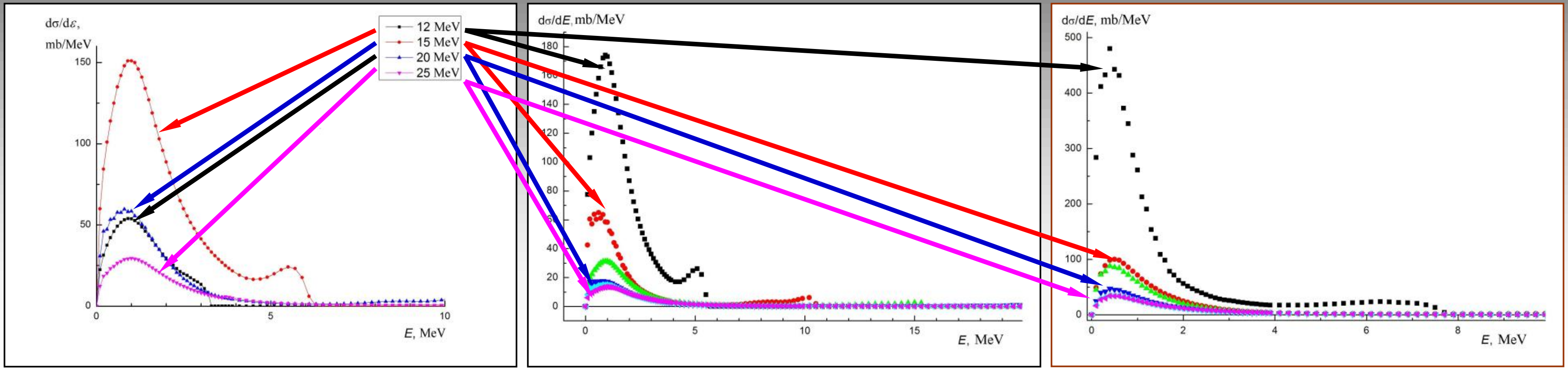


The main reason of disagreements is unreliable (erroneous) sorting of neutrons using its measured energy because the spectra of neutrons from different reactions are close.

Saclay:
The main reason of those differences is erroneous many neutrons moving from “2n” channel to “1n”.

Livermore:
The main reason of those differences is erroneous many neutrons moving from “1n” channel to “2n”.

^{139}La **^{141}Pr** **^{186}W**



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The profiles and average energies of the spectra are quite similar (main peaks energies ~0.7–1.0 MeV).

The reason is that the final nuclei are left not only in the ground states but in excited once: the neutron’s energy relation to its multiplicity is really unclear.

This greatly complicates the procedure for determining neutron multiplicity from this energy and makes the neutron multiplicity sorting procedure ambiguous.

The first and main source of systematic uncertainties.

The main reason of disagreements between Livermore and Saclay data and between both of them and evaluated once is the unreliable (erroneous) separation of detected neutrons between the reactions with different multiplicities because of incorrect classification of some of the detected neutrons, where, e.g., a neutron originating from the 1n reaction is assigned to the 2n channel, and vice versa.

This kind of errors arises from using the kinetic energy to classify neutrons from different reaction channels in which energy spectra overlap.

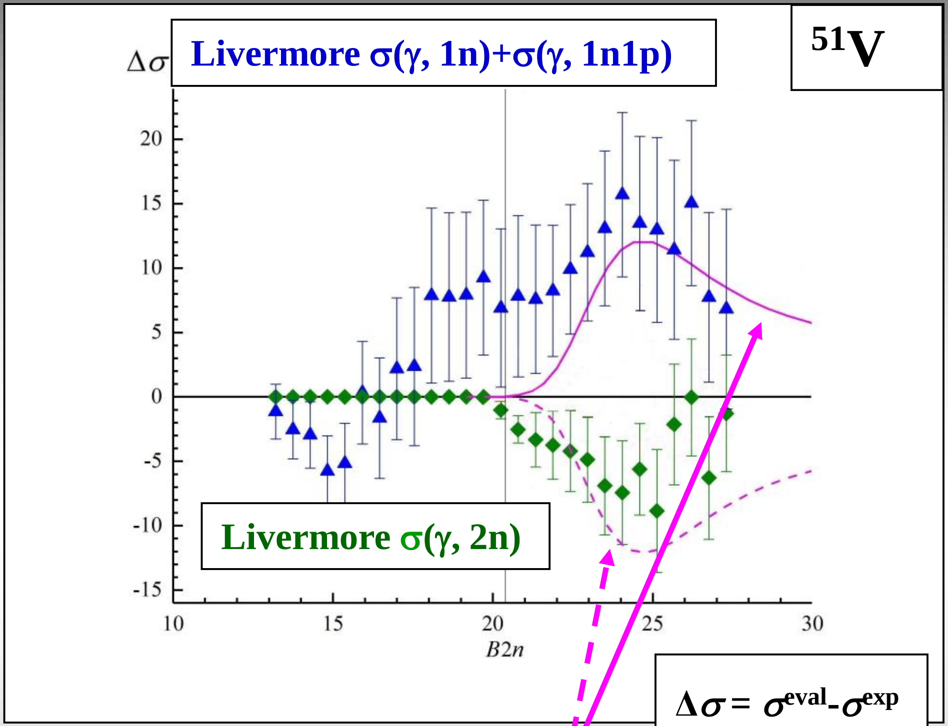
The second source of systematic uncertainties.

Because of direct detection of neutrons σ(γ, 1n) cross sections obtained in experiments under discussion for reaction in fact are really the sums σ(γ, 1n)+σ(γ, 1n1p).

For relatively light nuclei the (γ, 1n1p) reaction is the important additional source of the systematic uncertainties of the neutron multiplicity determination procedure because its features are close to those of reaction (γ, 2n).

The source of ambiguity in this case is the similar sharing of nuclear excitation energy between products of both two-nucleon reactions: the multiplicity of outgoing neutrons in the reaction (γ, 1n1p) is equal to 1 but in the reaction (γ, 2n) – equal to 2 .

Typical examples of this kind systematic disagreements



Livermore σ(γ, 1n)+σ(γ, 1n1p)

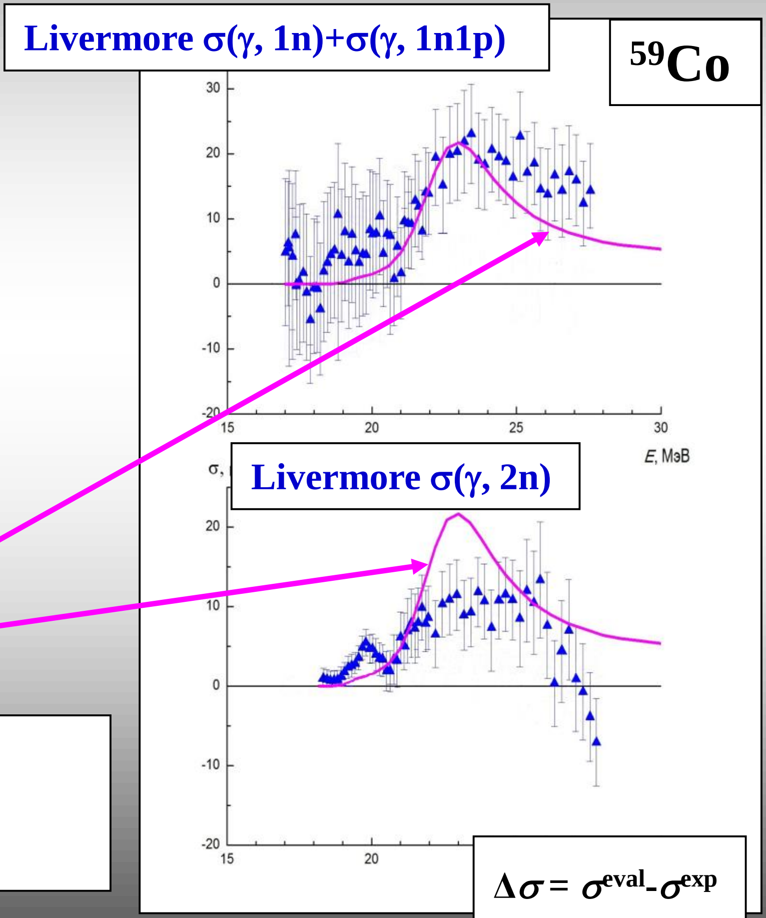
^{51}V

Livermore σ(γ, 2n)

$\Delta\sigma = \sigma^{eval} - \sigma^{exp}$

Phys. Atom.Nucl., 84, 389 (2021)

Eur. Phys. J. A, 53 (2017) 180



Livermore σ(γ, 1n)+σ(γ, 1n1p)

^{59}Co

Livermore σ(γ, 2n)

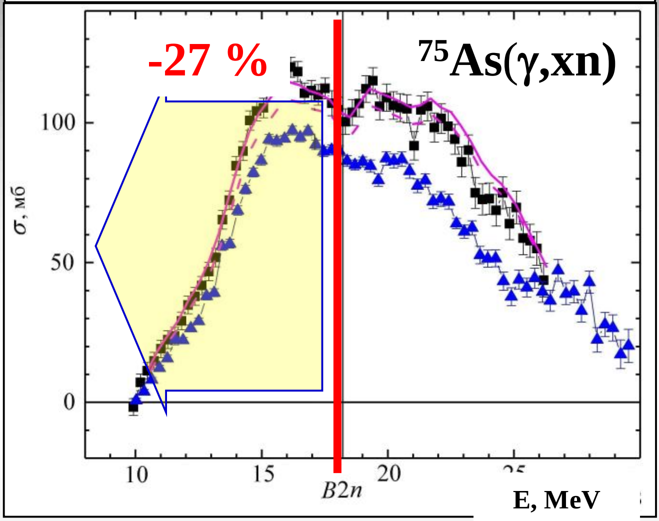
$\Delta\sigma = \sigma^{eval} - \sigma^{exp}$

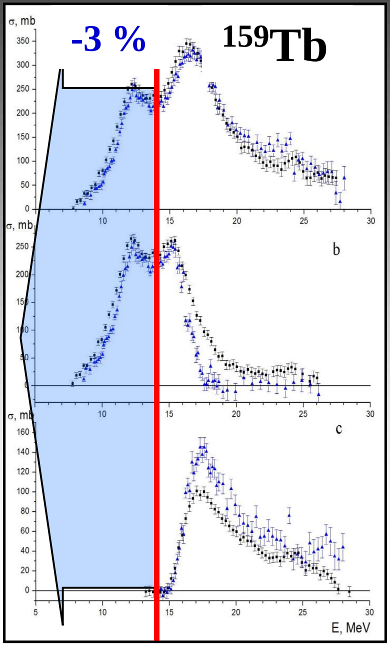
σ^{theor}(γ, 1n1p) from CMPNR

Analogous disagreements:
^{63,65}Cu (Bull. Rus. Acad. Sci. Phys., 80, 317 (2016)),
^{75}As (Phys. Rev. C 99, 024608 (2019)).

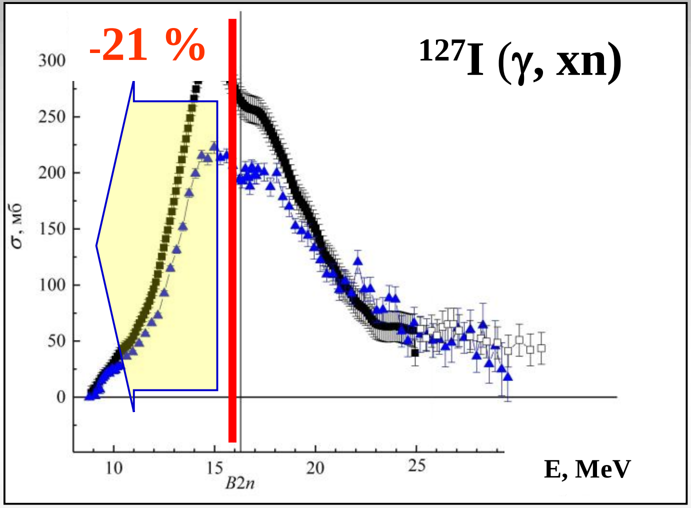
Nucleus-2020 results

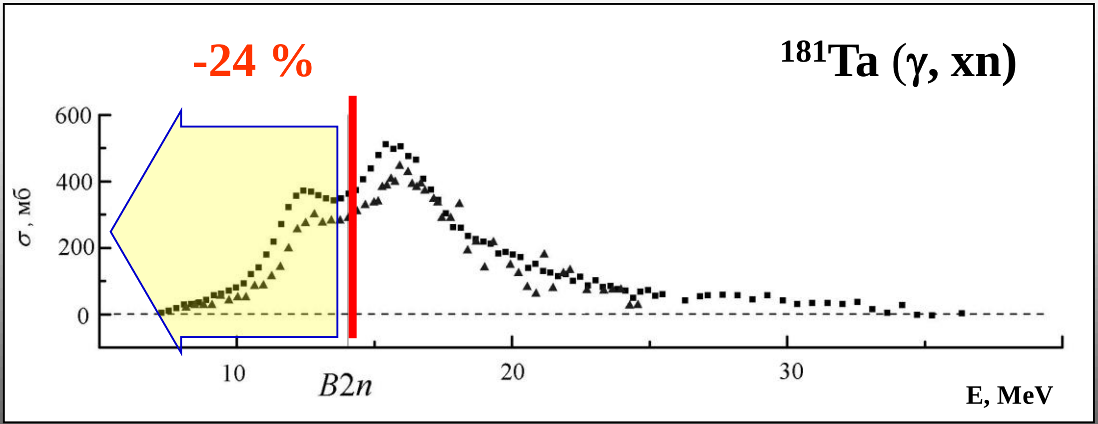
The third source of systematic disagreements

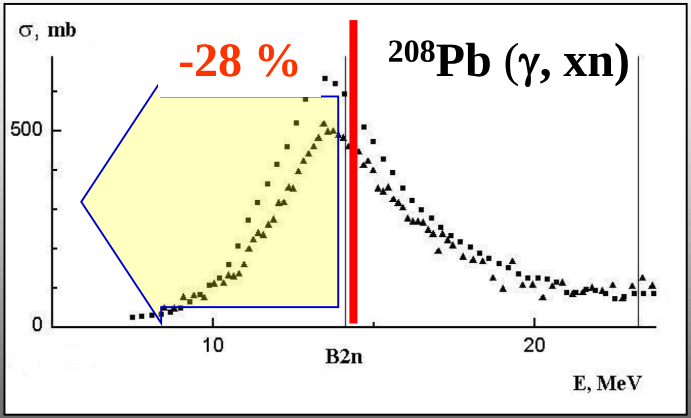




Disagreements in energy ranges E < B2n (no multiplicity problems)





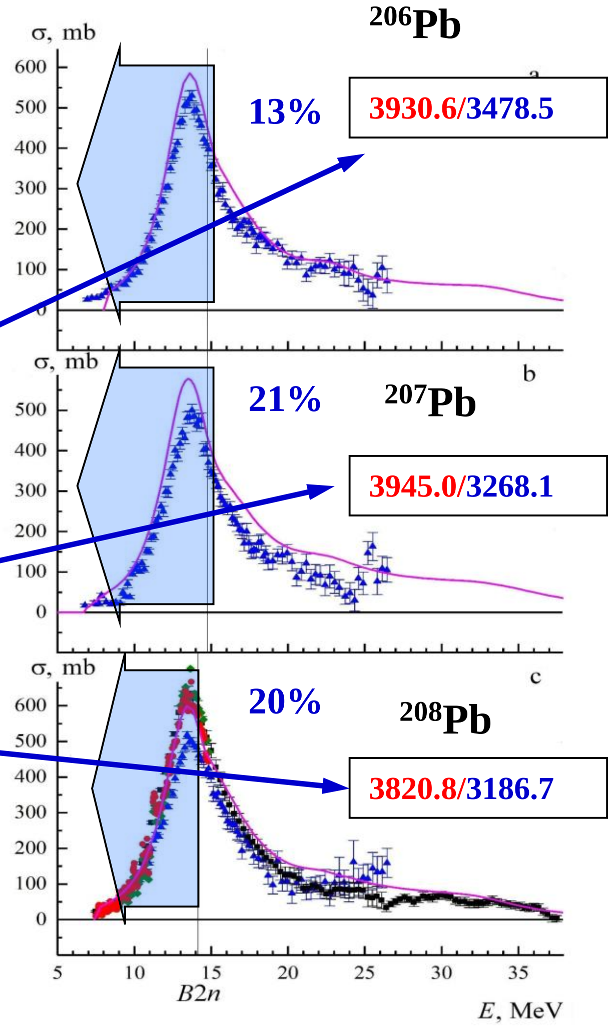


Nucleus-2021 results, Section 2, 22.09.

In the cases of all 3 isotopes ^{206,207,208}Pb Livermore experiment neutron yield cross sections σ(γ, xn) are significantly underestimated in comparison with those calculated (and Saclay data in the case of ^{208}Pb).

	$\sigma^{int}_{eval}/\sigma^{int}_{L}$		
Nucleus **Reaction**	^{206}Pb	^{207}Pb	^{208}Pb
(γ, xn)	1.13	1.21	1.20
(γ, sn)	1.15	1.24	1.30
(γ, 1n)	1.19	1.30	1.40
(γ, 2n)	1.02	1.02	0.85

$$\sigma(\gamma, xn) = \sigma(\gamma, 1n) + 2\sigma(\gamma, 2n) + 3\sigma(\gamma, 3n)$$



Livermore cross sections unreliable competitions

	$\sigma^{int}_{eval}/\sigma^{int}_{L}$					
	^{75}As	^{127}I	^{181}Ta	^{208}Pb	^{207}Pb*)	^{206}Pb*)
(γ, xn)	1.27	1.20	1.24	1.28	1.21	1.13
(γ, sn)	1.30	1.25	1.30	1.37	1.24	1.15
(γ, 1n)	1.34	1.33	1.46	1.42	1.30	1.19
(γ, 2n)	1.14	0.98	1.05	0.83	1.02	1.02

*) Special evaluations of Livermore data (no Saclay data), talk in NUCLEUS-2021, Section 2, 22.09.

The larger the fraction of the simple σ(γ, 1n) reaction in the cross-section for the complex reactions the higher the degree to which the latter is underestimated in comparison with evaluated one.

(γ, xn) = (γ, 1n) + [2(γ, 2n) + 3(γ, 3n) + …]
some contribution of (γ, 1n) reaction

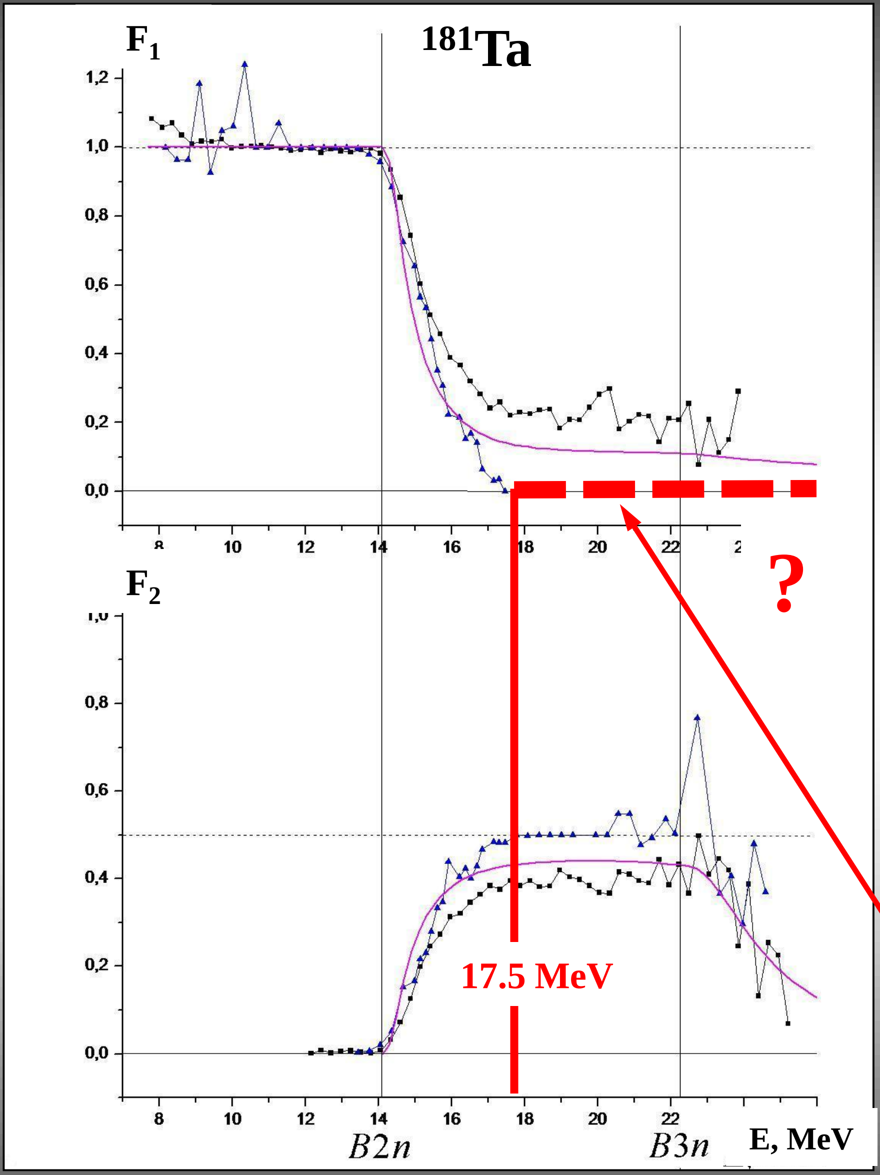
(γ, sn) = (γ, 1n) + [(γ, 2n) + (γ, 3n) + …]
larger contribution of (γ, 1n) reaction

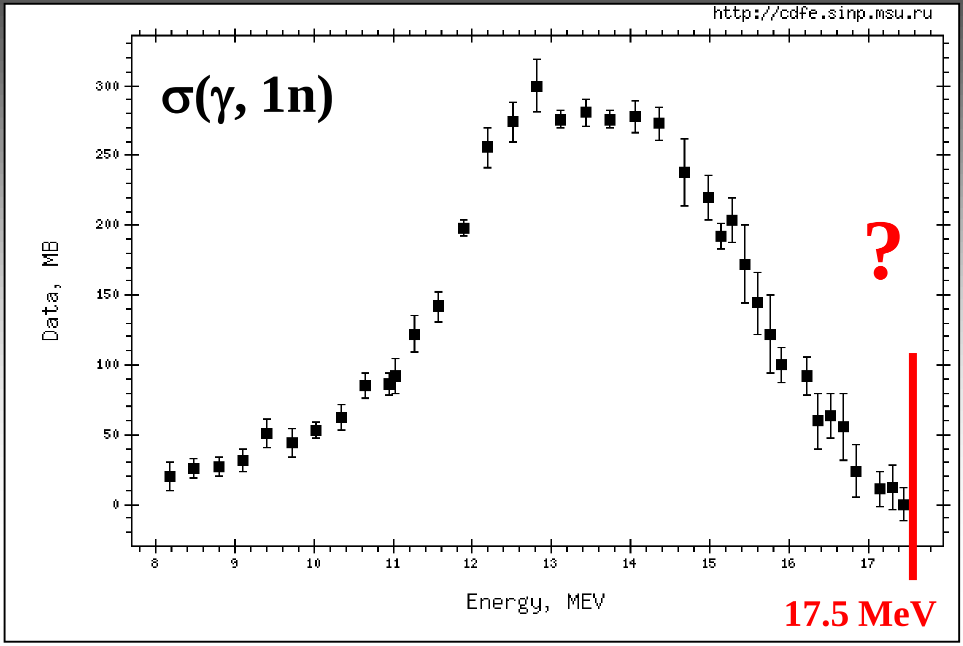
(γ, 1n) = (γ, 1n) + [0]
maximal 100%-contribution of (γ, 1n) reaction

(γ, 2n) = [0]
zero contribution of (γ, 1n) reaction

The ratios $\sigma^{int}_{eval}/\sigma^{int}_{L}$ for (γ, 2n) reaction for are very small, but for (γ, 1n) reaction are very large. It means that namely the very large underestimation of the cross-section for reaction (γ, 1n) is responsible for a substantial underestimations of the cross-section for the reaction (γ, xn). One is forced to conclude that in the relevant experiments many neutrons from (γ, 1n) reaction were lost.

This could be resulted from some technical problems.





The extremly underestimation of σ (γ, 1n) is because no neutrons from this reaction reaction were detected: they were lost.

This is the reason of very specific competitions of total and partial reaction cross sections.

^{181}Ta F_{12}

So the results of researches carried out for about 50 nuclei

(^{51}V, ^{59}Co, ^{63,65}Cu, ^{75}As, ^{76,78,80,82}Se, ^{89}Y, ^{90,91,92,94}Zr, ^{103}Rh, ^{115}In, ^{116,117,118,120,124}Sn, ^{127}I, ^{129}Xe, ^{133}Cs, ^{138}Ba, ^{139}La, ^{140,142}Ce, ^{141}Pr, ^{145,148}Nd, ^{153}Eu, ^{160}Gd, ^{159}Tb, ^{165}Ho, ^{181}Ta, ^{186}W, ^{186,188,190,192}Os, ^{197}Au, ^{206,207,208}Pb, ^{209}Bi)

using the experimental-theoretical method of evaluation based on objective physical criteria of data reliability

forced one to conclude that generally the experimental photoneutron reaction cross sections obtained using the method of neutron multiplicity sorting contain significant uncertainties of various nature and therefore can not be interpreted as reliable once.

Therefore the alternative methods for partial reaction cross section determination are needed.

Activation experiment for ^{181}Ta
(separation partial reactions using not outgoing neutrons but final nuclei features) MSU Institute of Nuclear Physics, race-track microtron)

Decays of ^{181}Ta(γ, 1n) and ^{181}Ta(γ, 2n) reactions final nucleus differ significantly:

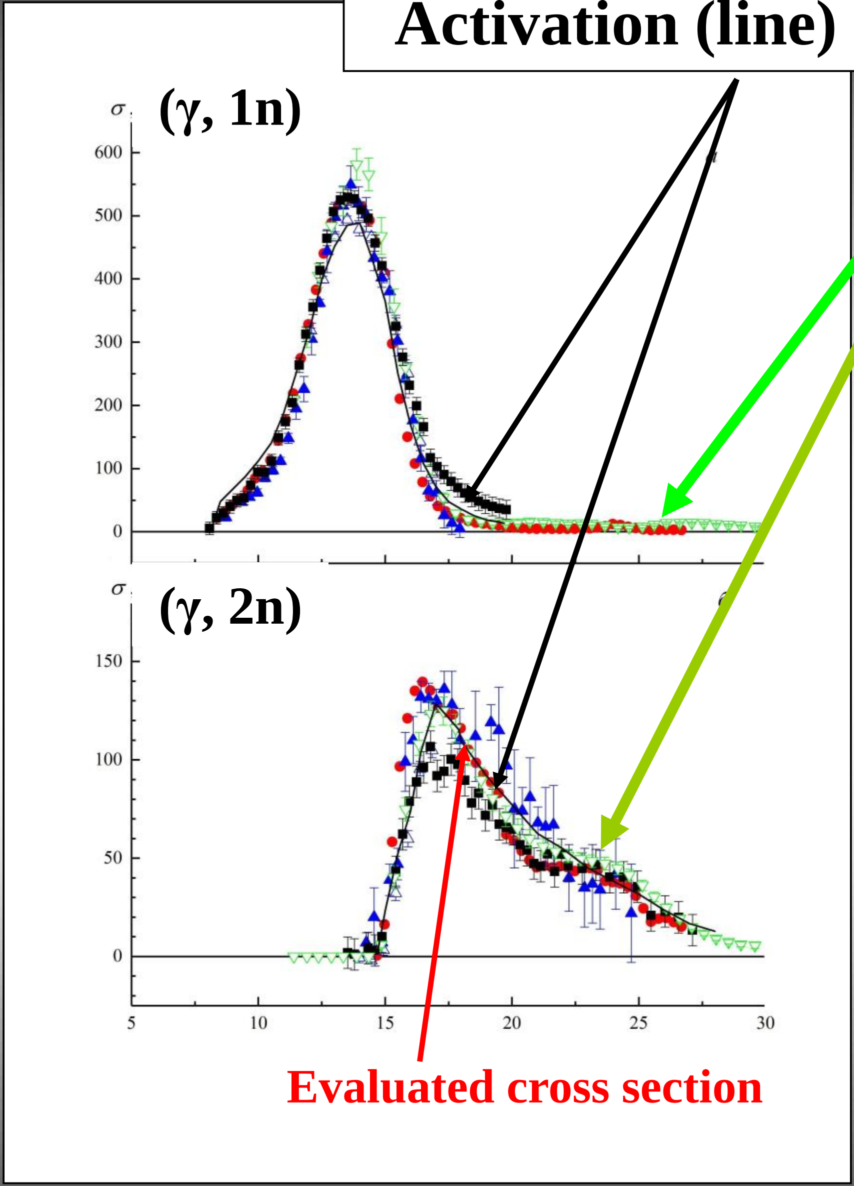
^{181}Ta(γ, 1n)^{180}Ta, $T_{1/2}$ = 8.154 hour, E = 93.326 кэВ
E = 103.557 кэВ
^{181}Ta(γ, 2n)^{179}Ta, $T_{1/2}$ = 1.820 year, E = 63.0 кэВ

The comparison of ratios of reaction yields Y and integrated cross sections σ^{int} obtained for experimental and evaluated data for ^{181}Ta at E^{int} = 65 MeV.

Ratios	Experiments			Evaluation
	Saclay	Livermore	Activity	$F_{1,2,3}$
of cross sections σ(γ,2n)/σ(γ,n)	0.36 (797/2190)	0.67 (887/1316)		0.49 (958/1956)
of yields Y(γ,2n)/Y(γ,n)	0.24	0.42	0.34	0.33
of cross sections σ(γ,3n)/σ(γ,n)	0.063 (137/2190)			0.055 (107/1956)

Analogous agreement was obtained for ^{209}Bi investigated at E^{int} = 55 MeV.

Activation (line) and LCS experiments for ^{197}Au

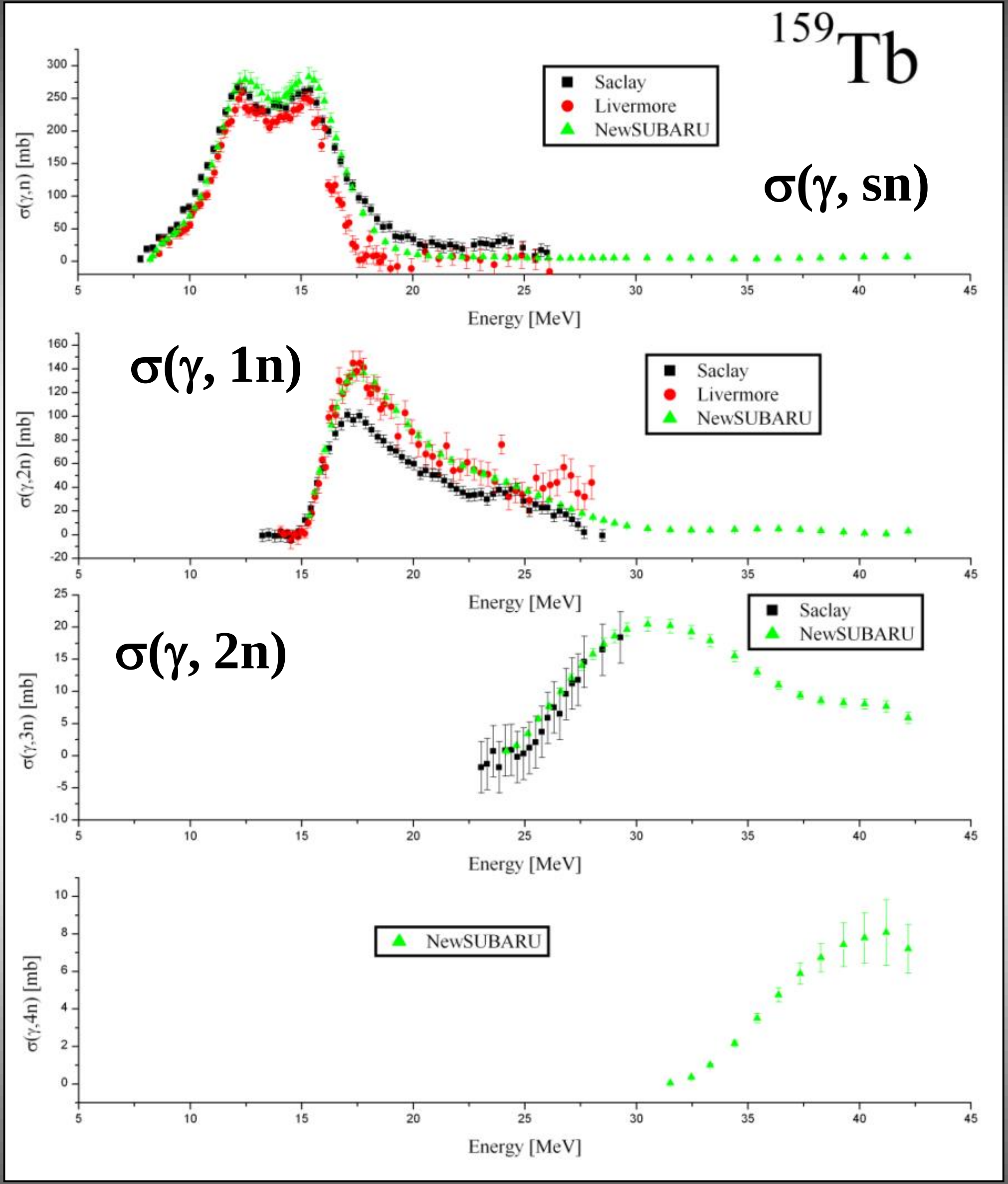


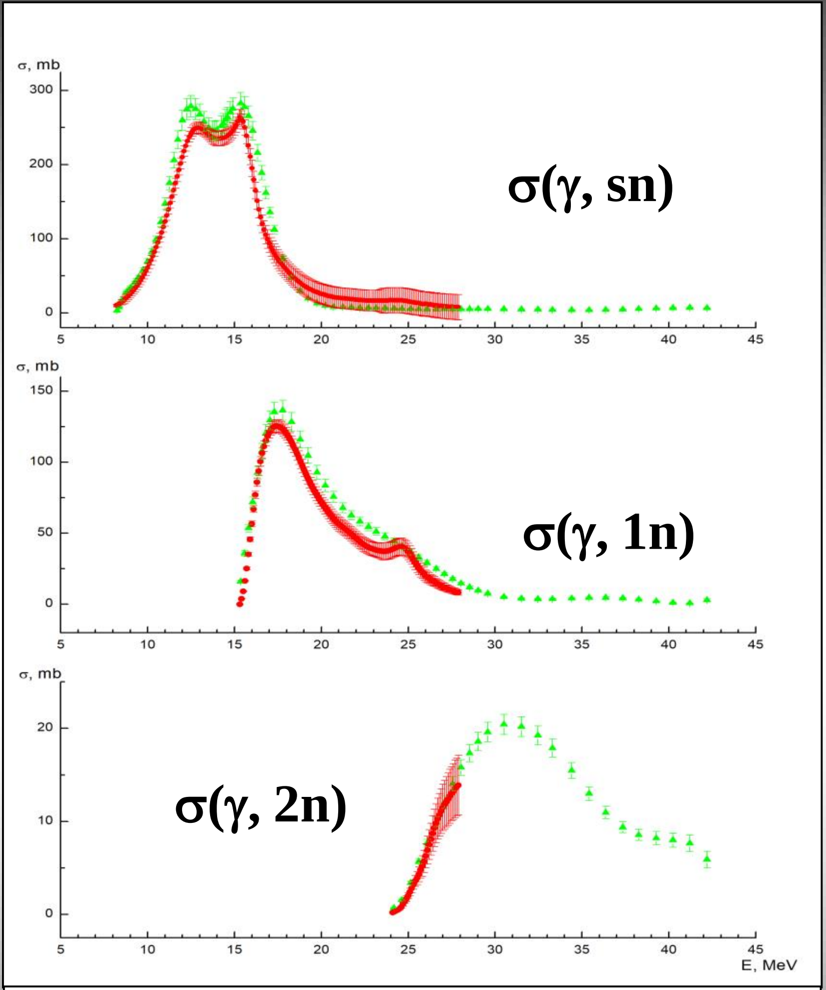
Evaluated cross section

LCS – laser Compton scattering gamma rays

and
direct measurement of neutron multiplicity using flat-efficiency detector (FED), NewSUBARU (Japan)

	E^{int} = 25.0 МэВ		
	σ^{int} (MeV mb)		$\sigma^{int}_{exp}/\sigma^{int}_{eval}$
Livermore	**784.53**	**10-16%**	**1.10**
Saclay	**627.08**		**0.84**
Activation	**720.52**	**2-3%**	**0.97**
LCS	**721.61**		**0.98**
Evaluation	**739.40**		**1.00**





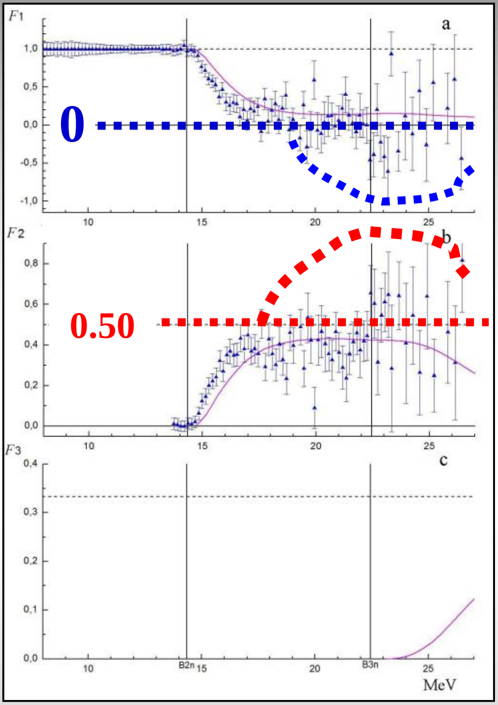
Data evaluated using experimental-theoretical method contradict to the data obtained at Livermore and Saclay but agree with new LCS-data (Japan, NewSUBARU)

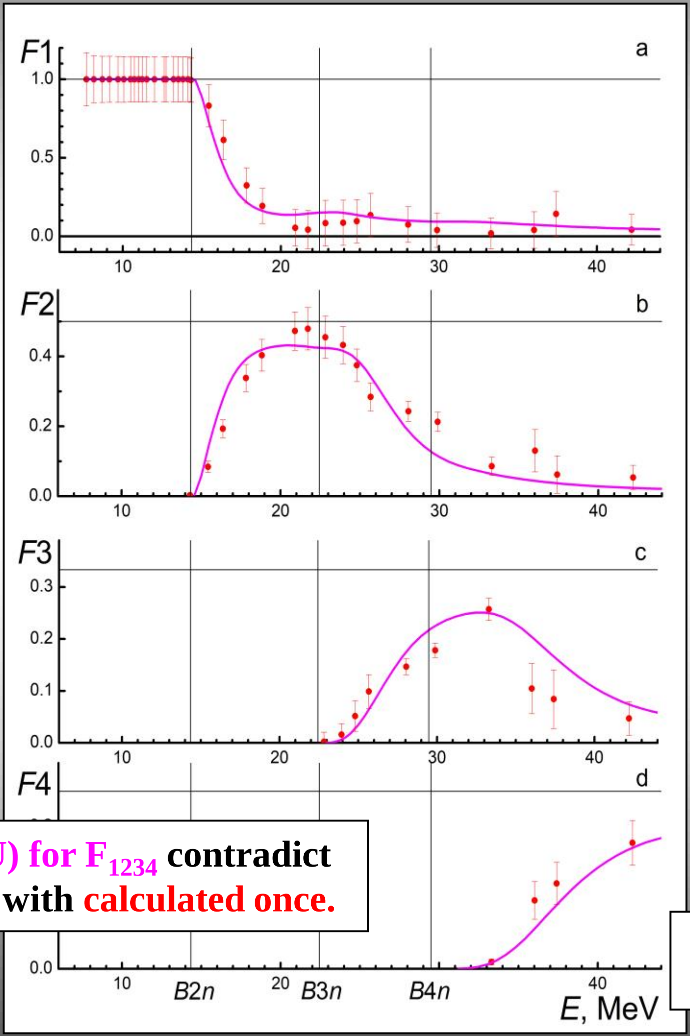
^{159}Tb -LCS

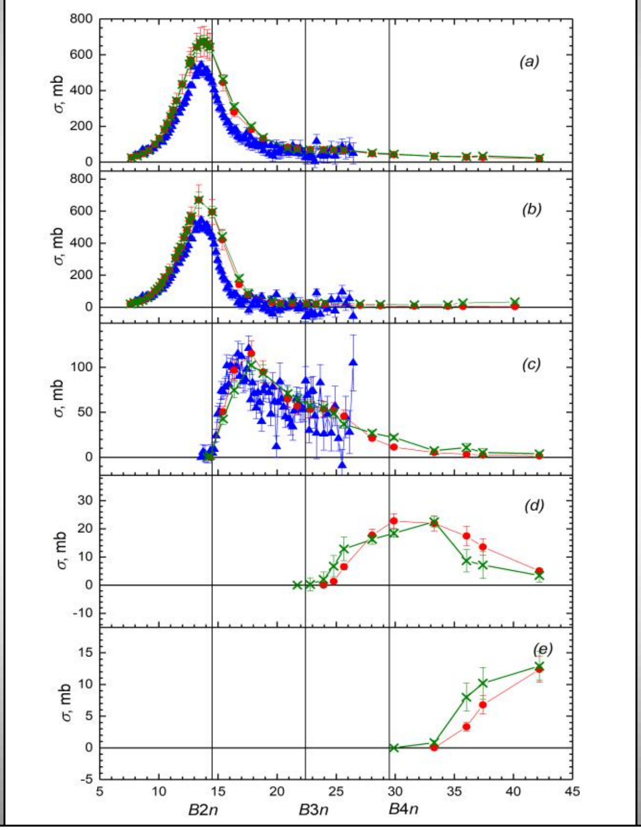
^{209}Bi, F_{1234}

R. R. Harvey, et al.,
Phys. Rev. 136, B126 (1964)

LCS, NewSUBARU - H.Utsunomiya, I.Gheorghe et. al., Phys. Rev., C 96, 044604 (2017)







LCS-data (Japan, NewSUBARU) for F_{1234} contradict to the Livermore data but agree with calculated once.

LCS cross section data contradict to the Livermore data but agree with evaluated once.

SUMMARY AND CONCLUSIONS: STATUS OF EXPERIMENTS

1. The results of many various photonuclear experiments were investigated using the objective physical criteria of data reliability.
2. It was found that the experimental partial photoneutron reaction cross sections obtained for about 50 nuclei using neutron multiplicity sorting method in general do not satisfy those criteria because of significant systematic uncertainties of several reasons:
   - erroneous classification of many detected neutron multiplicity (once originating from the 1n reaction are assigned to the 2n channel, and vice versa);
   - erroneous sorting of many neutrons not only between (γ, 1n) and (γ, 2n) reactions, but between (γ, 2n) and (γ, 1n1p) reactions also in the cases of relatively light nuclei;
   - loss of many neutrons from reaction (γ, 1n) in several experiments.
3. For all nuclei under discussion new cross section satisfying the data reliability criteria were evaluated using experimental-theoretical method.
4. Significant differences between investigated experimental data and once evaluated using experimental-theoretical method mean that those experimental photoneutron reaction cross sections are not reliable.

CONCLUSIONS: CONTINUAION

At the same time it was shown that the experimental data obtained for several nuclei using alternative methods

- activation method;
- the method of direct determination of neutron multiplicities (laser Compton scattering (LCS) gamma rays, the flat-efficiency detector (FED))

agree with evaluated data, are free of systematic uncertainties under discussion are reliable and therefore could be recommended for using in researches and applications .

Thanks a lot for attention!

Большое спасибо за внимание!

Model

B.S.Ishkhanov, V.N.Orlin. Physics of Particles and Nuclei, 38, 232 (2007),
Physics of Atomic Nuclei, 71, 493 (2008):

- **semiclassical exiton preequilibrium model of photonuclear reaction based on the Fermi gas densities;**
- **effects of nucleus deformation;**
- **effects of Giant Dipole Resonance isospin splitting.**

The Combined Photonucleon Reaction Model (CPNRM)

Semiclassical exiton preequilibrium model of photonuclear reaction based on the Fermi gas densities and taking into account the effects of nucleus deformation and of GDR isospin splitting.

Bohr description of σ(*γ*, *lpkn*):

$$\sigma(\gamma, lpkn; E_\gamma) = \sum_i \sigma^{(i)}_{\text{ГДР}}(E_\gamma) W^{(i)}_{\text{ГДР}}(l, k, E_\gamma) + \\ + \sigma_{\text{КД}}(E_\gamma) W_{\text{КД}}(l, k, E_\gamma),$$

σ^i - one of 4 components (2 isospins - T_0 and $T_0 + 1$ and 2 directions of vibration),

σ_{GDR} - Lorenz lines with

$$\Gamma^{\downarrow}_{\text{рез}} \approx G I(a_0/R_0)[E_{\text{рез}} - \Delta(Z, N)\delta_{TT_>}]^2,$$

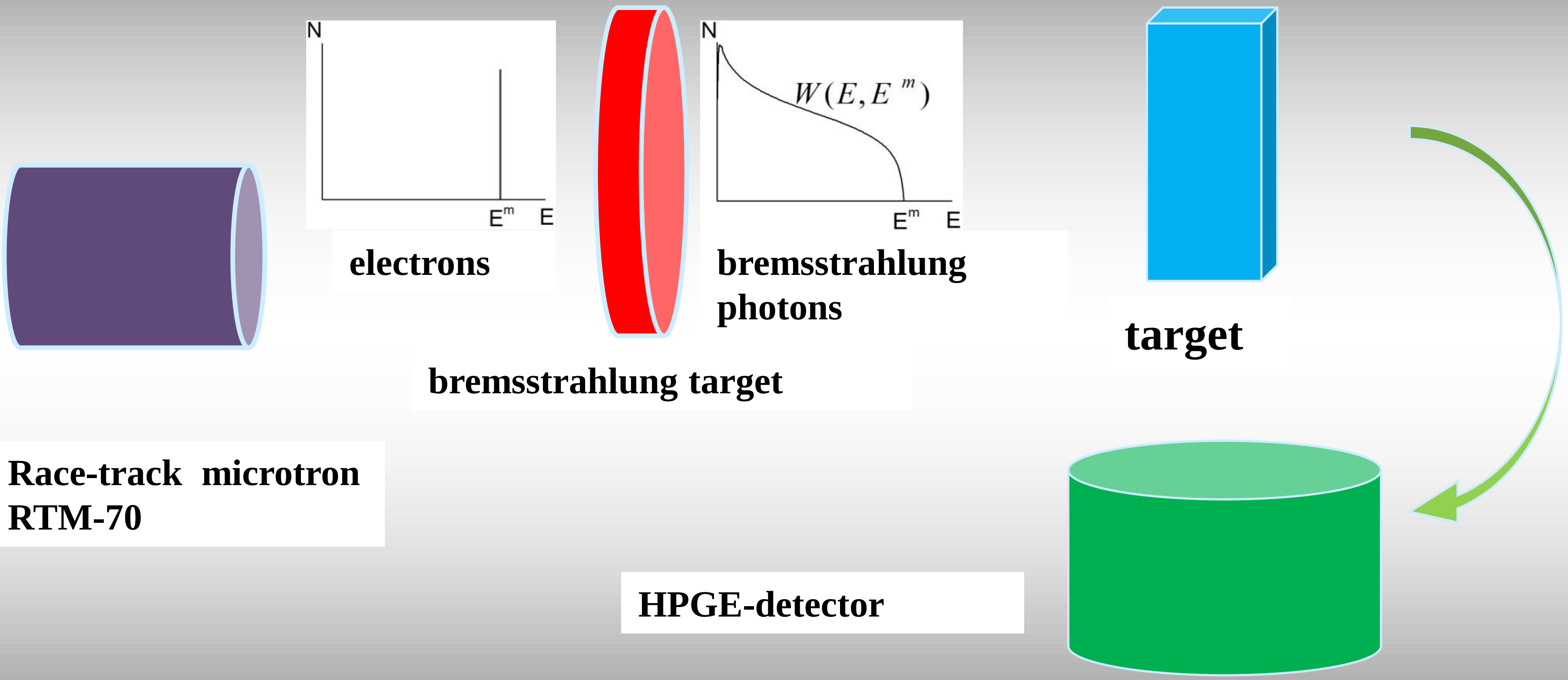
where

$$I(\xi) = \left[1 - 3\xi(1 + \pi^2\xi^2/3)/(1 + \pi^2\xi^2)\right] / (1 + \pi^2\xi^2)$$

W - decay probabilities (recurrent):

$$W(l, k, E; dp, dn, m) = \hbar \sum_{j=n,p} \sum_{\substack{m'=m \\ \Delta m'=2}}^{\bar{m}-2} \frac{D(m', E; dp, dn, m)}{\Gamma^{\uparrow}(E; dp, dn, m') + \Gamma^{\downarrow}(E; dp, dn, m')} \times \\ \times \int_0^{E-B_j} \lambda_j(\varepsilon_j, E; dp, dn, m') W(l_j, k_j, U_j; dp_j, dn_j, m') d\varepsilon_j + \\ + D(\bar{m}, E; dp, dn, m) P(l, k, E; dp, dn),$$

Independent test – activity method: identification of reaction using not outgoing neutrons but final nucleus



electrons

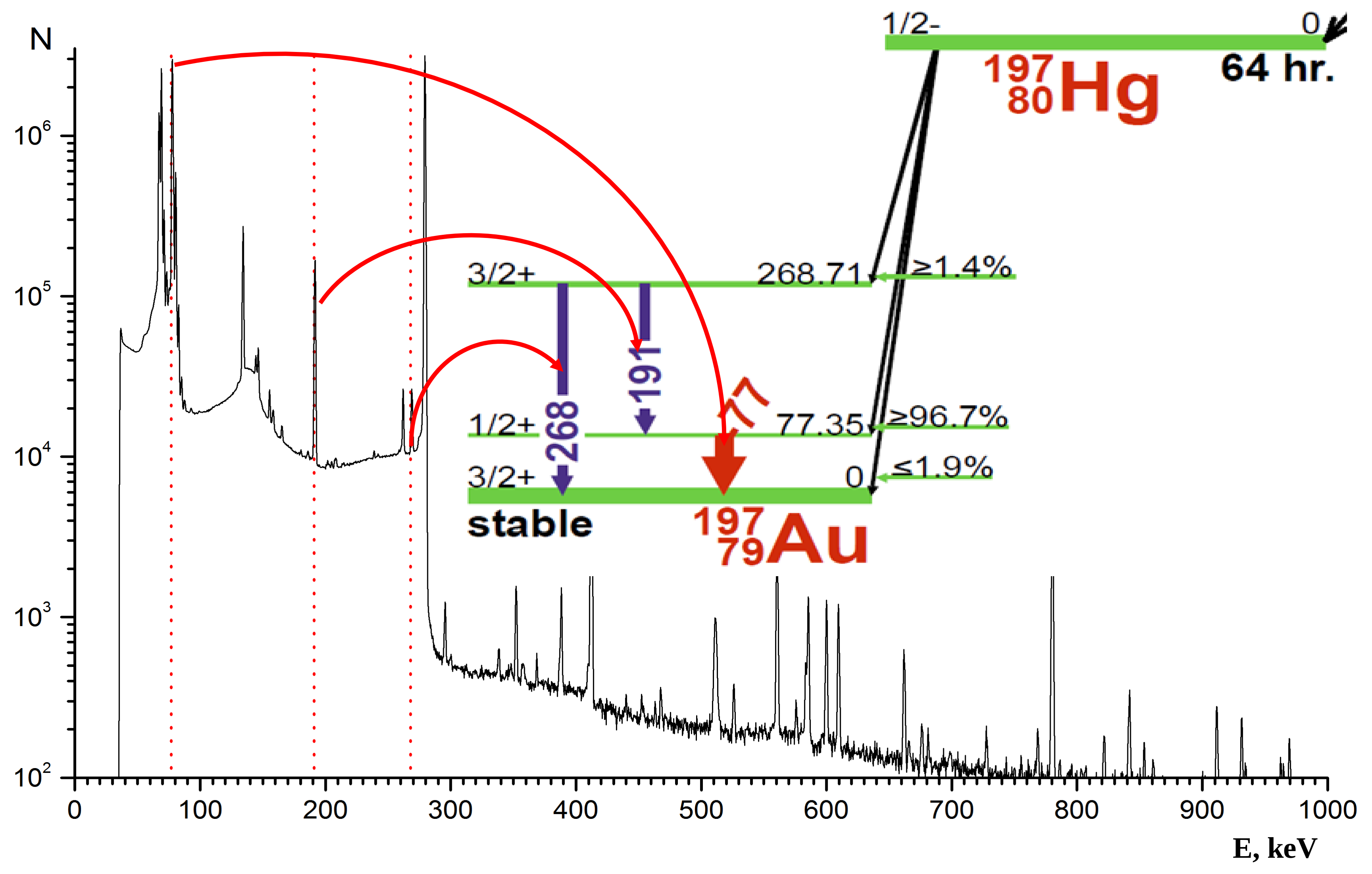
bremsstrahlung photons

target

bremsstrahlung target

Race-track microtron RTM-70

HPGE-detector


N
10^6
10^5
10^4
10^3
10^2
0
100
200
300
400
500
600
700
800
900
1000
E, keV
1/2-
0
$^{197}_{80}$Hg
64 hr.
3/2+
268.71
≥1.4%
268
191
1/2+
77.35
≥96.7%
77
3/2+
0
≤1.9%
stable
$^{197}_{79}$Au

The flat efficiency detector (FED) idea: special data treatment.

1. The numbers of neutrons from separate partial reactions N_i (i = 1, 2, 3, 4…) with cross sections $\sigma(\gamma, in)$

$$N_i = N_\gamma N_{target}\sigma(\gamma, in)$$

can not be measured directly (the main problem of cross section determination process).

2. The number of events with single neutron

$$N_{single} = N_1\varepsilon(E_1)+N_{22}C_1\varepsilon(E_2)(1-\varepsilon(E_2))+N_{33}C_1\varepsilon(E_3)(1-\varepsilon(E_3))^2.$$

3. Because there is no way to know the energies E1, E2, and E3 and the numbers N_1, N_2, and N_3:

$$N_{22}C_1\varepsilon(E_2)(1-\varepsilon(E_2)) = N_2\varepsilon(E_{21})(1-\varepsilon(E_{22}))+N_2\varepsilon(E_{22})(1-\varepsilon(E_{21}))$$

4. Therefore for the known ε = const: three equation for direct determination of N_1, N_2, and N_3

$$\begin{cases} N_{single} = N_1\varepsilon+N_{22}C_1\varepsilon(1-\varepsilon)+N_{33}C_1\varepsilon(1-\varepsilon)^2 \\ N_{double} = N_2\varepsilon^2+N_{22}C_2\varepsilon^2(1-\varepsilon) \\ N_{triple} = N_3\varepsilon^3 \end{cases}$$