



A multiharmonic/large-order flow cumulant analysis for relativistic heavy-ion collisions

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Dense & Strange Hadronic Matter Group, Technical University of Munich

SFT, Eur.Phys.J.C **81** (2021) 7,652

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20 September 2021



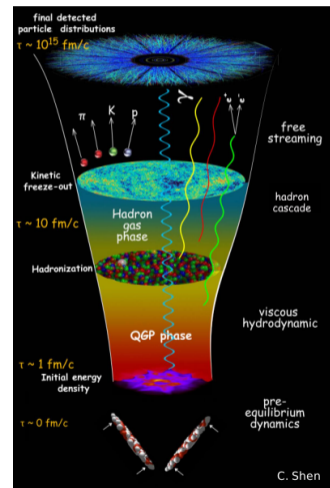
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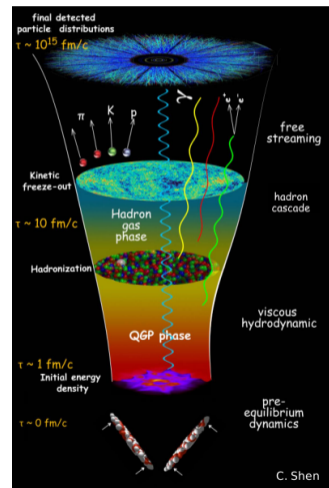
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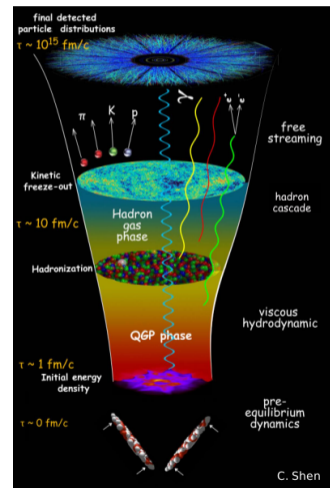


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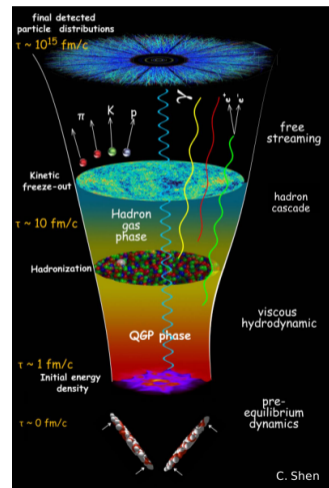


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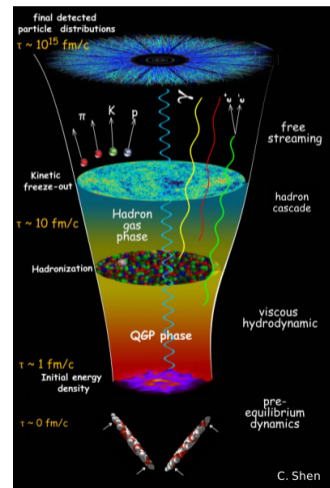


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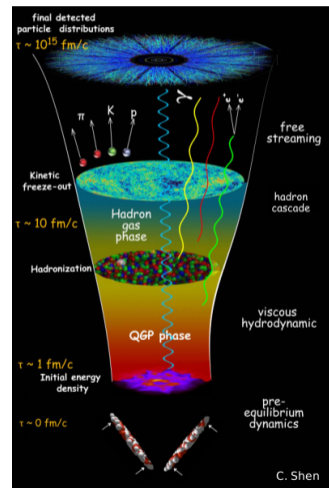
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It is important to introduce experimental observables.

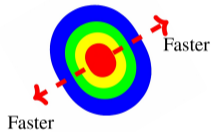




Flow harmonics in a nutshell!

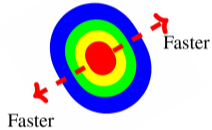
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ε_2 (Ellipticity)

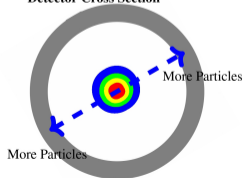


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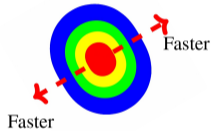


Detector Cross Section

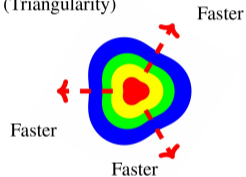


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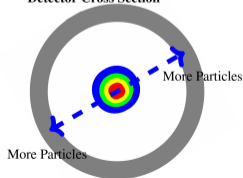
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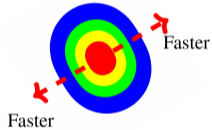


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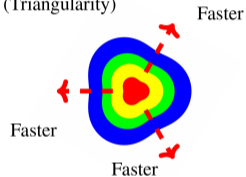


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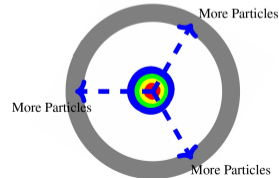
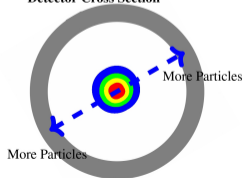
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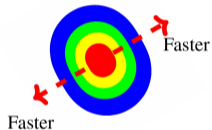


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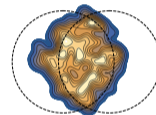
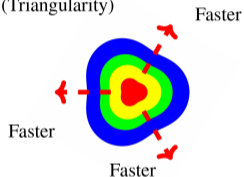


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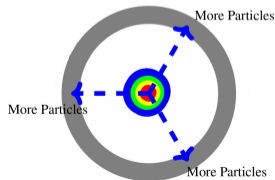
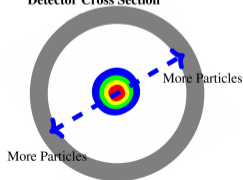
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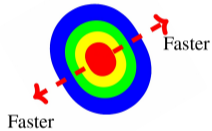


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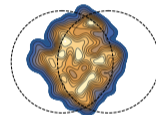
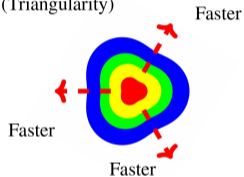


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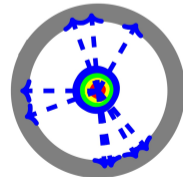
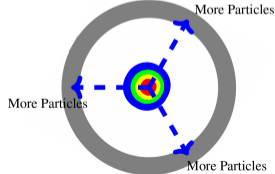
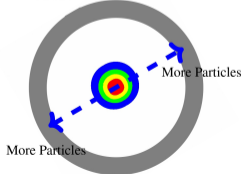
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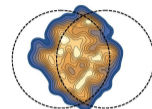
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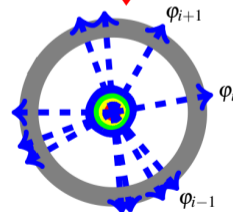
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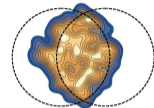
collective evolution



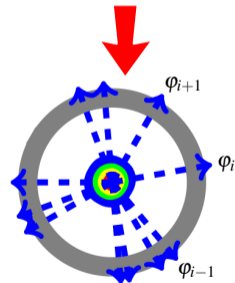
$$\frac{dN}{d\varphi} \propto 1 + \sum_{n=1}^{\infty} 2v_n \cos[n(\varphi - \psi_n)]$$

Flow harmonics in a nutshell!

The coefficient $v_n e^{in\psi_n}$ is called **Flow Harmonic**.



collective evolution

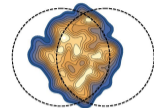


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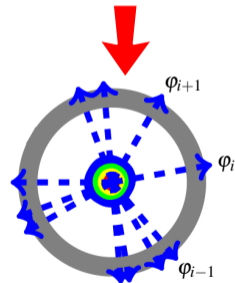
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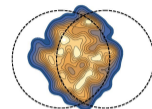


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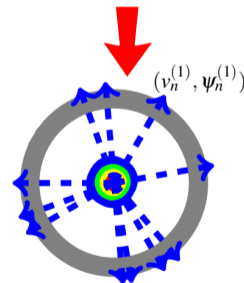
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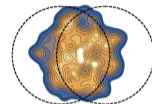
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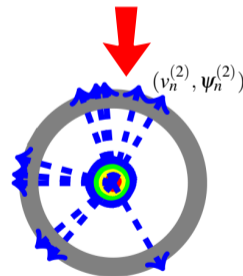
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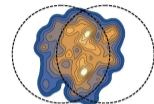
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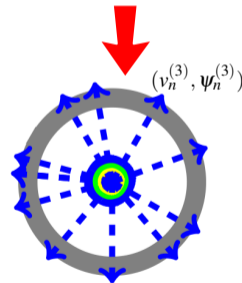
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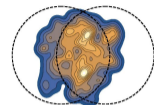
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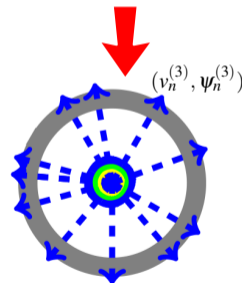
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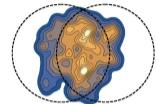
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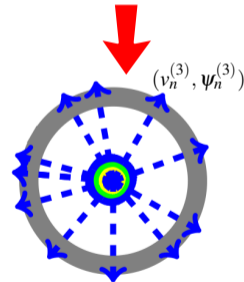
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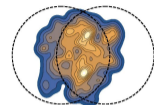
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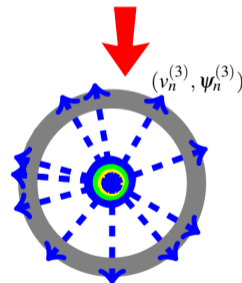
- ▶ Single-harmonic cumulants:

[Borghini, Dinh, Ollitrault, PRC, 2000, 2001]

$$c_n\{2\} = \langle v_n^2 \rangle, \quad c_n\{4\} = \langle v_n^4 \rangle - 2\langle v_n^2 \rangle^2, \quad \dots$$



collective evolution



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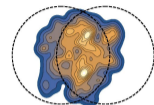
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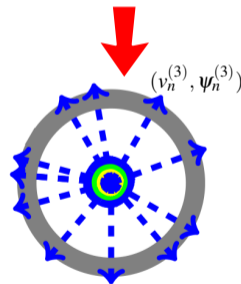
► Symmetric Cumulants:

[Bilandzic, Christensen, Gulbrandsen, Hansen, Zhou, PRC, 2013]

$$SC(n, m) = \langle v_n^2 v_m^2 \rangle - \langle v_n^2 \rangle \langle v_m^2 \rangle$$



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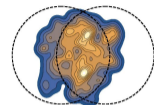
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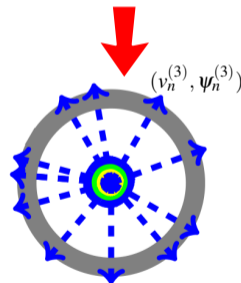
- ▶ Generalized symmetric cumulants:

[C. Mordasini, A. Bilandzic, D. Karakoc, SFT, PRC, 2020],

$$SC(k, l, m) = \langle v_k^2 v_l^2 v_m^2 \rangle - \langle v_k^2 v_l^2 \rangle \langle v_m^2 \rangle - \langle v_k^2 v_m^2 \rangle \langle v_l^2 \rangle - \langle v_l^2 v_m^2 \rangle \langle v_k^2 \rangle + 2\langle v_k^2 \rangle \langle v_l^2 \rangle \langle v_m^2 \rangle$$



collective evolution





One package for all cumulants

[SFT, Eur.Phys.J.C 81 (2021) 7,652]

$$p_f(v_1, v_2, v_3, \dots, \psi_1 - \psi_2, \psi_2 - \psi_3, \dots)$$

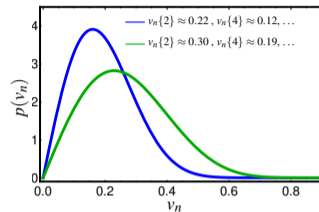


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- ▶ Example: single harmonic distribution $p(v_n)$ and its cumulants $v_n\{2\}$, $v_n\{4\}$, $v_n\{6\}$, $v_n\{8\}$, ... ($v_n\{2m\} \propto c_n^{1/2m}\{2m\}$)



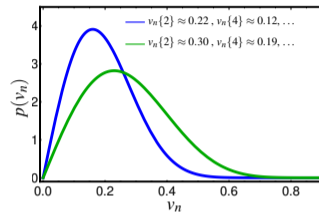


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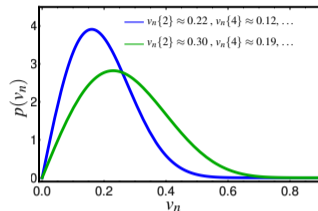


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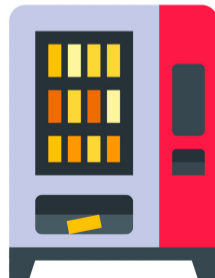
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- ▶ Mathematica package **MultiharmonicCumulants_v2_1.m**
<https://github.com/FaridTaghavi/MultiharmonicCumulants.git>



ORDER YOUR CUMULANT!

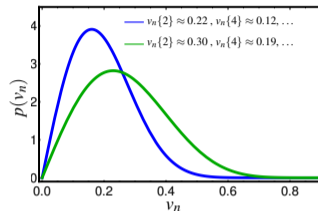


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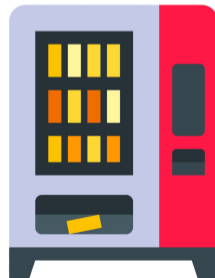
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- ▶ Returns the cumulants in terms of symbolic moments, correlation functions, and Q -vectors.



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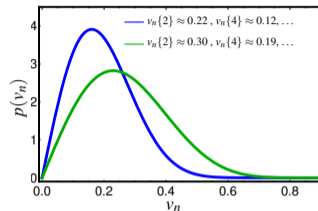


One package for all cumulants

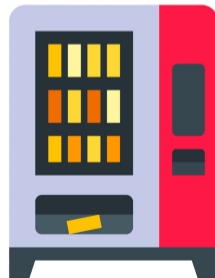
[SFT, Eur.Phys.J.C 81 (2021) 7,652]

$$p_f(v_1, v_2, v_3, \dots, \psi_1 - \psi_2, \psi_2 - \psi_3, \dots)$$

- ▶ Example: single harmonic distribution $p(v_n)$ and its cumulants $v_n\{2\}$, $v_n\{4\}$, $v_n\{6\}$, $v_n\{8\}$, ... ($v_n\{2m\} \propto c_n^{1/2m}\{2m\}$)
- ▶ We employ generating function method to extract the cumulants.
- ▶ Mathematica package **MultiharmonicCumulants_v2_1.m**
<https://github.com/FaridTaghavi/MultiharmonicCumulants.git>
- ▶ Returns the cumulants in terms of symbolic moments, correlation functions, and Q -vectors.
- ▶ A new method for extracting **statistical error** is implemented.



ORDER YOUR CUMULANT!





All cumulants involving harmonics 2,3,4,5 and orders 2,3,4,5

► There are 33 distinct cumulants.

cumulant	order	cumulant expression
1 $c_2\{2\}$	2	$\langle v_2^2 \rangle$
2 $c_3\{2\}$	2	$\langle v_3^2 \rangle$
3 $c_4\{2\}$	2	$\langle v_4^2 \rangle$
4 $c_5\{2\}$	2	$\langle v_5^2 \rangle$
5 $c_{2,3}^{(4)}\{2,1\}$	3	$\langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
6 $c_{2,3,5}^{(-3,5)}\{1,1,1\}$	3	$\langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
7 $c_2\{4\}$	4	$\langle v_2^4 \rangle - 2\langle v_2^2 \rangle^2$
8 $c_3\{4\}$	4	$\langle v_3^4 \rangle - 2\langle v_3^2 \rangle^2$
9 $c_4\{4\}$	4	$\langle v_4^4 \rangle - 2\langle v_4^2 \rangle^2$
10 $c_5\{4\}$	4	$\langle v_5^4 \rangle - 2\langle v_5^2 \rangle^2$
11 $c_{2,3}^{(0)}\{2,2\}$	4	$\langle v_2^2 v_3^2 \rangle - \langle v_2^2 \rangle \langle v_3^2 \rangle$
12 $c_{2,4}^{(0)}\{2,2\}$	4	$\langle v_2^2 v_4^2 \rangle - \langle v_2^2 \rangle \langle v_4^2 \rangle$
13 $c_{2,5}^{(0)}\{2,2\}$	4	$\langle v_2^2 v_5^2 \rangle - \langle v_2^2 \rangle \langle v_5^2 \rangle$
14 $c_{3,4}^{(0)}\{2,2\}$	4	$\langle v_3^2 v_4^2 \rangle - \langle v_3^2 \rangle \langle v_4^2 \rangle$
15 $c_{3,5}^{(0)}\{2,2\}$	4	$\langle v_3^2 v_5^2 \rangle - \langle v_3^2 \rangle \langle v_5^2 \rangle$
16 $c_{4,5}^{(0)}\{2,2\}$	4	$\langle v_4^2 v_5^2 \rangle - \langle v_4^2 \rangle \langle v_5^2 \rangle$
17 $c_{2,3,4}^{(6,-4)}\{1,2,1\}$	4	$\langle v_2^3 v_3 v_4 \cos(2(\psi_2 - 3\psi_3 + 2\psi_4)) \rangle$
18 $c_{2,3,5}^{(8,-5)}\{1,2,1\}$	4	$\langle v_2^3 v_3 v_5 \cos(3\psi_3 - 8\psi_4 + 5\psi_5) \rangle$
19 $c_{2,3,4,5}^{(5,4,-5)}\{1,1,1,1\}$	4	$\langle v_2 v_3 v_4 v_5 \cos(2\psi_2 - 3\psi_3 - 4\psi_4 + 5\psi_5) \rangle$
20 $c_{2,3}^{(0)}\{3,2\}$	5	$\langle v_2^3 v_3^2 \cos(6(\psi_2 - \psi_3)) \rangle$
21 $c_{2,4}^{(4)}\{2,3\}$	5	$\langle v_2^3 v_4^2 \cos(4(\psi_2 - \psi_4)) \rangle - 2\langle v_2^2 \rangle \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
22 $c_{2,4}^{(4)}\{4,1\}$	5	$\langle v_2^4 v_4 \cos(4(\psi_2 - \psi_4)) \rangle - 3\langle v_2^2 \rangle \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
23 $c_{2,3,4}^{(-6,8)}\{1,2,2\}$	5	$\langle v_2^3 v_3^2 v_4 \cos(2(\psi_2 + 3\psi_3 - 4\psi_4)) \rangle$
24 $c_{2,3,4}^{(0,4)}\{2,2,1\}$	5	$\langle v_2^3 v_3^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle - \langle v_2^2 \rangle \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
25 $c_{2,3,5}^{(-3,5)}\{1,1,3\}$	5	$\langle v_2^3 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
26 $c_{2,3,5}^{(-3,5)}\{1,3,1\}$	5	$\langle v_2^3 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
27 $c_{2,3,5}^{(-3,5)}\{3,1,1\}$	5	$\langle v_2^3 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
28 $c_{2,4,5}^{(-8,10)}\{1,2,2\}$	5	$\langle v_2^3 v_4^2 v_5 \cos(2(\psi_2 + 4\psi_4 - 5\psi_5)) \rangle$
29 $c_{2,4,5}^{(4,0)}\{2,1,2\}$	5	$\langle v_2^3 v_4^2 v_5 \cos(4(\psi_2 - \psi_4)) \rangle - \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle \langle v_2^2 \rangle$
30 $c_{2,4,5}^{(-4,10)}\{2,1,2\}$	5	$\langle v_2^3 v_4^2 v_5 \cos(6\psi_3 + 4\psi_4 - 10\psi_5) \rangle$
31 $c_{2,4,5}^{(4,0)}\{3,1,1\}$	5	$\langle v_2^3 v_4 v_5 \cos(3\psi_3 - 4\psi_4 - 5\psi_5) \rangle$
32 $c_{2,3,4,5}^{(-3,0,0)}\{1,1,2,1\}$	5	$\langle v_2^3 v_3 v_4 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - \langle v_2^2 \rangle \langle v_2 v_3 v_4 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
33 $c_{2,3,4,5}^{(3,-4,5)}\{2,1,1,1\}$	5	$\langle v_2^3 v_3 v_4 v_5 \cos(4\psi_2 - 3\psi_3 + 4\psi_4 - 5\psi_5) \rangle$

A systematic way to extract all multiharmonic flow cumulants



All cumulants involving harmonics 2,3,4,5 and orders 2,3,4,5

► There are 33 distinct cumulants.

cumulant	order	cumulant expression
1	2	$\langle v_2^2 \rangle$
2	2	$\langle v_3^2 \rangle$
3	2	$\langle v_4^2 \rangle$
4	2	$\langle v_5^2 \rangle$
5	3	$\langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
6	3	$\langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
7	4	$\langle v_2^4 \rangle - 2\langle v_2^2 \rangle^2$
8	4	$\langle v_3^4 \rangle - 2\langle v_3^2 \rangle^2$
9	4	$\langle v_4^4 \rangle - 2\langle v_4^2 \rangle^2$
10	4	$\langle v_5^4 \rangle - 2\langle v_5^2 \rangle^2$
11	4	$\langle v_2^2 v_3^2 \rangle - \langle v_2^2 \rangle \langle v_3^2 \rangle$
12	4	$\langle v_2^2 v_4^2 \rangle - \langle v_2^2 \rangle \langle v_4^2 \rangle$
13	4	$\langle v_2^2 v_5^2 \rangle - \langle v_2^2 \rangle \langle v_5^2 \rangle$
14	4	$\langle v_3^2 v_4^2 \rangle - \langle v_3^2 \rangle \langle v_4^2 \rangle$
15	4	$\langle v_3^2 v_5^2 \rangle - \langle v_3^2 \rangle \langle v_5^2 \rangle$
16	4	$\langle v_4^2 v_5^2 \rangle - \langle v_4^2 \rangle \langle v_5^2 \rangle$
17	4	$\langle v_2^2 v_4 v_4 \cos(2(\psi_2 - 3\psi_3 + 2\psi_4)) \rangle$
18	4	$\langle v_2^2 v_3 v_5 \cos(3\psi_3 - 8\psi_4 + 5\psi_5) \rangle$
19	4	$\langle v_2 v_3 v_4 v_5 \cos(2\psi_2 - 3\psi_3 - 4\psi_4 + 5\psi_5) \rangle$
20	5	$\langle v_2^3 v_3^2 \cos(6(\psi_2 - \psi_3)) \rangle$
21	5	$\langle v_2^3 v_4^2 \cos(4(\psi_2 - \psi_4)) \rangle - 2\langle v_2^2 \rangle \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
22	5	$\langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle - 3\langle v_2^2 \rangle \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
23	5	$\langle v_2^2 v_3^2 v_2 \cos(2(\psi_2 + 3\psi_3 - 4\psi_4)) \rangle$
24	5	$\langle v_2^2 v_4^2 \cos(4(\psi_2 - \psi_4)) \rangle - \langle v_2^2 \rangle \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
25	5	$\langle v_2^2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
26	5	$\langle v_2^2 v_2 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_2 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
27	5	$\langle v_2^2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
28	5	$\langle v_2^2 v_2^2 v_2 \cos(2(\psi_2 + 4\psi_4 - 5\psi_5)) \rangle$
29	5	$\langle v_2^2 v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle - \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle \langle v_2^2 \rangle$
30	5	$\langle v_2^2 v_4^2 \cos(6\psi_3 + 4\psi_4 - 10\psi_5) \rangle$
31	5	$\langle v_2^2 v_4 v_5 \cos(6\psi_3 - 4\psi_4 - 5\psi_5) \rangle$
32	5	$\langle v_2^2 v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - \langle v_2^2 \rangle \langle v_2 v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
33	5	$\langle v_2^2 v_3 v_4 v_5 \cos(4\psi_2 - 3\psi_3 + 4\psi_4 - 5\psi_5) \rangle$

A systematic way to extract all multiharmonic flow cumulants



All cumulants involving harmonics 2,3,4,5 and orders 2,3,4,5

► There are 33 distinct cumulants.

cumulant	order	cumulant expression
1	2	$\langle v_2^2 \rangle$
2	2	$\langle v_3^2 \rangle$
3	2	$\langle v_4^2 \rangle$
4	2	$\langle v_5^2 \rangle$
5	3	$\langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
6	3	$\langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
7	4	$\langle v_2^4 \rangle - 2\langle v_2^2 \rangle^2$
8	4	$\langle v_3^4 \rangle - 2\langle v_3^2 \rangle^2$
9	4	$\langle v_4^4 \rangle - 2\langle v_4^2 \rangle^2$
10	4	$\langle v_5^4 \rangle - 2\langle v_5^2 \rangle^2$
11	4	$\langle v_2^2 v_4^2 \rangle - \langle v_2^2 \rangle \langle v_4^2 \rangle$
12	4	$\langle v_2^2 v_4^2 \rangle - \langle v_2^2 \rangle \langle v_4^2 \rangle$
13	4	$\langle v_2^2 v_5^2 \rangle - \langle v_2^2 \rangle \langle v_5^2 \rangle$
14	4	$\langle v_3^2 v_4^2 \rangle - \langle v_3^2 \rangle \langle v_4^2 \rangle$
15	4	$\langle v_3^2 v_5^2 \rangle - \langle v_3^2 \rangle \langle v_5^2 \rangle$
16	4	$\langle v_4^2 v_5^2 \rangle - \langle v_4^2 \rangle \langle v_5^2 \rangle$
17	4	$\langle v_2^2 v_4 \cos(2(\psi_2 - 3\psi_3 + 2\psi_4)) \rangle$
18	4	$\langle v_2^2 v_4 \cos(3\psi_3 - 8\psi_4 + 5\psi_5) \rangle$
19	4	$\langle v_2 v_3 v_4 v_5 \cos(2\psi_2 - 3\psi_3 - 4\psi_4 + 5\psi_5) \rangle$
20	5	$\langle v_2^3 v_3^2 \cos(6(\psi_2 - \psi_3)) \rangle$
21	5	$\langle v_2^3 v_3^2 \cos(4(\psi_2 - \psi_4)) \rangle - 2\langle v_2^2 \rangle \langle v_3^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
22	5	$\langle v_2^3 v_4 \cos(4(\psi_2 - \psi_4)) \rangle - 3\langle v_2^2 \rangle \langle v_3^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
23	5	$\langle v_2^2 v_3^2 v_2 \cos(2(\psi_2 + 3\psi_3 - 4\psi_4)) \rangle$
24	5	$\langle v_2^2 v_3^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle - \langle v_2^2 \rangle \langle v_3^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
25	5	$\langle v_2^2 v_3 v_2 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
26	5	$\langle v_2^2 v_2 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
27	5	$\langle v_2^2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
28	5	$\langle v_2^2 v_2^2 v_2 \cos(2(\psi_2 + 4\psi_4 - 5\psi_5)) \rangle$
29	5	$\langle v_2^2 v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle - \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle \langle v_2^2 \rangle$
30	5	$\langle v_2^2 v_2^2 v_4 \cos(6\psi_3 + 4\psi_4 - 10\psi_5) \rangle$
31	5	$\langle v_2^2 v_4 v_5 \cos(6\psi_3 - 4\psi_4 - 5\psi_5) \rangle$
32	5	$\langle v_2^2 v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - \langle v_2^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
33	5	$\langle v_2^2 v_3 v_4 v_5 \cos(4\psi_2 - 3\psi_3 + 4\psi_4 - 5\psi_5) \rangle$

A systematic way to extract all multiharmonic flow cumulants



All cumulants involving harmonics 2,3,4,5 and orders 2,3,4,5

cumulant	order	cumulant expression
1 $c_2(2)$	2	$\langle v_2^2 \rangle$
2 $c_3(2)$	2	$\langle v_3^2 \rangle$
3 $c_4(2)$	2	$\langle v_4^2 \rangle$
4 $c_5(2)$	2	$\langle v_5^2 \rangle$
5 $c_{2,4}^{(4)}(2, 1)$	3	$\langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
6 $c_{2,3,5}^{(-3,5)}(1, 1, 1)$	3	$\langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
7 $c_2(4)$	4	$\langle v_2^4 \rangle - 2\langle v_2^2 \rangle^2$
8 $c_3(4)$	4	$\langle v_3^4 \rangle - 2\langle v_3^2 \rangle^2$
9 $c_4(4)$	4	$\langle v_4^4 \rangle - 2\langle v_4^2 \rangle^2$
10 $c_5(4)$	4	$\langle v_5^4 \rangle - 2\langle v_5^2 \rangle^2$
11 $c_{2,4}^{(0)}(2, 2)$	4	$\langle v_2^2 v_4^2 \rangle - \langle v_2^2 \rangle \langle v_4^2 \rangle$
12 $c_{2,4}^{(0)}(2, 2)$	4	$\langle v_2^2 v_4^2 \rangle - \langle v_2^2 \rangle \langle v_4^2 \rangle$
13 $c_{2,5}^{(0)}(2, 2)$	4	$\langle v_2^2 v_5^2 \rangle - \langle v_2^2 \rangle \langle v_5^2 \rangle$
14 $c_{3,4}^{(0)}(2, 2)$	4	$\langle v_3^2 v_4^2 \rangle - \langle v_3^2 \rangle \langle v_4^2 \rangle$
15 $c_{3,5}^{(0)}(2, 2)$	4	$\langle v_3^2 v_5^2 \rangle - \langle v_3^2 \rangle \langle v_5^2 \rangle$
16 $c_{4,5}^{(0)}(2, 2)$	4	$\langle v_4^2 v_5^2 \rangle - \langle v_4^2 \rangle \langle v_5^2 \rangle$
17 $c_{2,3,4}^{(6,-4)}(1, 2, 1)$	4	$\langle v_2^3 v_3 v_4 \cos(2(\psi_2 - 3\psi_3 + 2\psi_4)) \rangle$
18 $c_{2,3,4}^{(8,-5)}(1, 2, 1)$	4	$\langle v_2^3 v_3 v_4 \cos(3\psi_3 - 8\psi_4 + 5\psi_5) \rangle$
19 $c_{2,3,4,5}^{(5,4,-5)}(1, 1, 1, 1)$	4	$\langle v_2 v_3 v_4 v_5 \cos(2\psi_2 - 4\psi_4 + 5\psi_5) \rangle$
20 $c_{3,4}^{(0)}(3, 2)$	5	$\langle v_3^3 v_4^2 \cos(6(\psi_3 - \psi_4)) \rangle$
21 $c_{3,4}^{(4)}(2, 3)$	5	$\langle v_3^3 v_4^2 \cos(4(\psi_2 - \psi_4)) \rangle - 2\langle v_2^2 \rangle \langle v_3^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
22 $c_{3,4}^{(0)}(4, 1)$	5	$\langle v_3^4 v_4 \cos(4(\psi_2 - \psi_4)) \rangle - 3\langle v_2^2 \rangle \langle v_3^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
23 $c_{3,4}^{(-6,3)}(1, 2, 2)$	5	$\langle v_3^4 v_4^2 \cos(2(\psi_3 + 3\psi_4 - 4\psi_1)) \rangle$
24 $c_{3,4}^{(0,4)}(2, 2, 1)$	5	$\langle v_3^3 v_4^2 \cos(4(\psi_2 - \psi_4)) \rangle - \langle v_2^2 \rangle \langle v_3^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
25 $c_{2,3,4}^{(-3,5)}(1, 1, 3)$	5	$\langle v_2^3 v_3 v_4 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_4 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
26 $c_{2,3,5}^{(-3,5)}(1, 3, 1)$	5	$\langle v_2^3 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
27 $c_{3,4,5}^{(-3,5)}(3, 1, 1)$	5	$\langle v_3^3 v_4 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_4 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
28 $c_{3,4}^{(-9,10)}(1, 2, 2)$	5	$\langle v_3^4 v_4^2 v_5 \cos(2(\psi_3 + 4\psi_4 - 5\psi_5)) \rangle$
29 $c_{2,4,5}^{(4,0)}(2, 1, 2)$	5	$\langle v_2^4 v_4^2 \cos(4(\psi_2 - \psi_4)) \rangle - \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle \langle v_2^2 \rangle$
30 $c_{2,4,5}^{(-10,10)}(2, 1, 2)$	5	$\langle v_2^5 v_4^2 \cos(6\psi_4 + 4\psi_5 - 10\psi_5) \rangle$
31 $c_{3,4,5}^{(4,0)}(3, 1, 1)$	5	$\langle v_3^4 v_4 v_5 \cos(6\psi_4 - 4\psi_5 - 5\psi_5) \rangle$
32 $c_{2,3,4,5}^{(-3,0,5)}(1, 1, 2, 1)$	5	$\langle v_2^4 v_3 v_4 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - \langle v_2^2 \rangle \langle v_2 v_3 v_4 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
33 $c_{2,3,4,5}^{(10,-10)}(2, 1, 1, 1)$	5	$\langle v_2^5 v_3 v_4 v_5 \cos(4\psi_2 - 3\psi_3 + 4\psi_4 - 5\psi_5) \rangle$

- ▶ There are 33 distinct cumulants.
- ▶ Seven of them have been missed in previous theoretical and experimental studies.



All cumulants involving harmonics 2,3,4,5 and orders 2,3,4,5

cumulant	order	cumulant expression	
1	$c_2\{2\}$	2	$\langle v_2^2 \rangle$
2	$c_3\{2\}$	2	$\langle v_3^2 \rangle$
3	$c_4\{2\}$	2	$\langle v_4^2 \rangle$
4	$c_5\{2\}$	2	$\langle v_5^2 \rangle$
5	$c_{2,4}^{(4)}\{2,1\}$	3	$\langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
6	$c_{2,3,5}^{(-3,5)}\{1,1,1\}$	3	$\langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
7	$c_2\{4\}$	4	$\langle v_2^4 \rangle - 2\langle v_2^2 \rangle^2$
8	$c_3\{4\}$	4	$\langle v_3^4 \rangle - 2\langle v_3^2 \rangle^2$
9	$c_4\{4\}$	4	$\langle v_4^4 \rangle - 2\langle v_4^2 \rangle^2$
10	$c_5\{4\}$	4	$\langle v_5^4 \rangle - 2\langle v_5^2 \rangle^2$
11	$c_{2,4}^{(0)}\{2,2\}$	4	$\langle v_2^2 v_4^2 \rangle - \langle v_2^2 \rangle \langle v_4^2 \rangle$
12	$c_{2,4}^{(0)}\{2,2\}$	4	$\langle v_2^2 v_4^2 \rangle - \langle v_2^2 \rangle \langle v_4^2 \rangle$
13	$c_{2,5}^{(0)}\{2,2\}$	4	$\langle v_2^2 v_5^2 \rangle - \langle v_2^2 \rangle \langle v_5^2 \rangle$
14	$c_{3,4}^{(0)}\{2,2\}$	4	$\langle v_3^2 v_4^2 \rangle - \langle v_3^2 \rangle \langle v_4^2 \rangle$
15	$c_{3,5}^{(0)}\{2,2\}$	4	$\langle v_3^2 v_5^2 \rangle - \langle v_3^2 \rangle \langle v_5^2 \rangle$
16	$c_{4,5}^{(0)}\{2,2\}$	4	$\langle v_4^2 v_5^2 \rangle - \langle v_4^2 \rangle \langle v_5^2 \rangle$
17	$c_{2,3,4}^{(6,-4)}\{1,2,1\}$	4	$\langle v_2^3 v_3 v_4 \cos(2(\psi_2 - 3\psi_3 + 2\psi_4)) \rangle$
18	$c_{2,3,4}^{(8,-5)}\{1,2,1\}$	4	$\langle v_2^3 v_3 v_4 \cos(3\psi_2 - 8\psi_3 + 5\psi_4) \rangle$
19	$c_{2,3,4,5}^{(5,4,-5)}\{1,1,1,1\}$	4	$\langle v_2 v_3 v_4 v_5 \cos(2\psi_2 - 3\psi_3 - 4\psi_4 + 5\psi_5) \rangle$
20	$c_{3,4}^{(0)}\{3,2\}$	5	$\langle v_3^3 v_4^2 \cos(6(\psi_3 - \psi_4)) \rangle$
21	$c_{2,4}^{(4)}\{2,3\}$	5	$\langle v_2^3 v_4^3 \cos(4(\psi_2 - \psi_4)) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
22	$c_{2,4}^{(4)}\{4,1\}$	5	$\langle v_2^3 v_4 \cos(4(\psi_2 - \psi_4)) \rangle - 3\langle v_2^2 \rangle \langle v_2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
23	$c_{2,4}^{(-6,3)}\{1,2,2\}$	5	$\langle v_2^3 v_2^2 v_4 \cos(2(\psi_2 + 3\psi_3 - 4\psi_4)) \rangle$
24	$c_{2,3,4}^{(0,4)}\{2,2,1\}$	5	$\langle v_2^3 v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle - \langle v_2^2 \rangle \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
25	$c_{2,3,4,5}^{(-3,5)}\{1,1,3\}$	5	$\langle v_2^3 v_2 v_3 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
26	$c_{2,3,5}^{(-3,5)}\{1,3,1\}$	5	$\langle v_2^3 v_2 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
27	$c_{2,3,5}^{(-3,5)}\{3,1,1\}$	5	$\langle v_2^3 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
28	$c_{2,4}^{(-8,10)}\{1,2,2\}$	5	$\langle v_2^3 v_2^2 v_4 \cos(2(\psi_2 + 4\psi_3 - 5\psi_4)) \rangle$
29	$c_{2,4,2}^{(4,0)}\{2,1,2\}$	5	$\langle v_2^3 v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle - \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle \langle v_2^2 \rangle$
30	$c_{2,4,5}^{(-8,10)}\{2,1,2\}$	5	$\langle v_2^3 v_2^2 v_4 \cos(6\psi_2 + 4\psi_4 - 10\psi_5) \rangle$
31	$c_{2,4}^{(4,0)}\{3,1,1\}$	5	$\langle v_2^3 v_2 v_4 \cos(8\psi_2 - 4\psi_3 - 5\psi_5) \rangle$
32	$c_{2,3,4,5}^{(-3,0,5)}\{1,1,2,1\}$	5	$\langle v_2^3 v_2 v_3 v_4 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - \langle v_2^2 \rangle \langle v_2 v_3 v_4 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
33	$c_{2,3,4,5}^{(8,-10)}\{2,1,1,1\}$	5	$\langle v_2^3 v_3 v_4 v_5 \cos(4\psi_2 - 3\psi_3 + 4\psi_4 - 5\psi_5) \rangle$

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n_i : the involving harmonics in the cumulant.

m_i : the power of the flow amplitude v_{n_i} in the cumulant.



All cumulants involving harmonics 2,3,4,5 and orders 2,3,4,5

cumulant	order	cumulant expression	
1	$c_2\{2\}$	2	$\langle v_2^2 \rangle$
2	$c_3\{2\}$	2	$\langle v_3^2 \rangle$
3	$c_4\{2\}$	2	$\langle v_4^2 \rangle$
4	$c_5\{2\}$	2	$\langle v_5^2 \rangle$
5	$c_{2,4}^{(4)}\{2,1\}$	3	$\langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
6	$c_{2,3,5}^{(-3,5)}\{1,1,1\}$	3	$\langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
7	$c_2\{4\}$	4	$\langle v_2^4 \rangle - 2\langle v_2^2 \rangle^2$
8	$c_3\{4\}$	4	$\langle v_3^4 \rangle - 2\langle v_3^2 \rangle^2$
9	$c_4\{4\}$	4	$\langle v_4^4 \rangle - 2\langle v_4^2 \rangle^2$
10	$c_5\{4\}$	4	$\langle v_5^4 \rangle - 2\langle v_5^2 \rangle^2$
11	$c_{2,4}^{(0)}\{2,2\}$	4	$\langle v_2^2 v_4^2 \rangle - \langle v_2^2 \rangle \langle v_4^2 \rangle$
12	$c_{2,4}^{(0)}\{2,2\}$	4	$\langle v_2^2 v_4^2 \rangle - \langle v_2^2 \rangle \langle v_4^2 \rangle$
13	$c_{2,5}^{(0)}\{2,2\}$	4	$\langle v_2^2 v_5^2 \rangle - \langle v_2^2 \rangle \langle v_5^2 \rangle$
14	$c_{3,4}^{(0)}\{2,2\}$	4	$\langle v_3^2 v_4^2 \rangle - \langle v_3^2 \rangle \langle v_4^2 \rangle$
15	$c_{3,5}^{(0)}\{2,2\}$	4	$\langle v_3^2 v_5^2 \rangle - \langle v_3^2 \rangle \langle v_5^2 \rangle$
16	$c_{4,5}^{(0)}\{2,2\}$	4	$\langle v_4^2 v_5^2 \rangle - \langle v_4^2 \rangle \langle v_5^2 \rangle$
17	$c_{2,3,4}^{(6,-4)}\{1,2,1\}$	4	$\langle v_2^3 v_3 v_4 \cos(2(\psi_2 - 3\psi_3 + 2\psi_4)) \rangle$
18	$c_{2,3,4}^{(8,-5)}\{1,2,1\}$	4	$\langle v_2^3 v_3 v_4 \cos(3\psi_2 - 8\psi_3 + 5\psi_4) \rangle$
19	$c_{2,3,4,5}^{(5,4,-5)}\{1,1,1,1\}$	4	$\langle v_2 v_3 v_4 v_5 \cos(2\psi_2 - 3\psi_3 - 4\psi_4 + 5\psi_5) \rangle$
20	$c_{2,4}^{(0)}\{3,2\}$	5	$\langle v_2^3 v_4^2 \cos(6(\psi_2 - \psi_4)) \rangle$
21	$c_{2,4}^{(4)}\{2,3\}$	5	$\langle v_2^3 v_4^2 \cos(4(\psi_2 - \psi_4)) \rangle - 2\langle v_2^2 \rangle \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
22	$c_{2,4}^{(4)}\{4,1\}$	5	$\langle v_2^3 v_4 \cos(4(\psi_2 - \psi_4)) \rangle - 3\langle v_2^2 \rangle \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
23	$c_{2,4}^{(-6,3)}\{1,2,2\}$	5	$\langle v_2^3 v_4^2 \cos(2(\psi_2 + 3\psi_3 - 4\psi_4)) \rangle$
24	$c_{2,4}^{(0,4)}\{2,2,1\}$	5	$\langle v_2^3 v_4^2 \cos(4(\psi_2 - \psi_4)) \rangle - \langle v_2^2 \rangle \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
25	$c_{2,3,5}^{(-3,5)}\{1,1,3\}$	5	$\langle v_2^3 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
26	$c_{2,3,5}^{(-3,5)}\{1,3,1\}$	5	$\langle v_2^3 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
27	$c_{2,3,5}^{(-3,5)}\{3,1,1\}$	5	$\langle v_2^3 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
28	$c_{2,4}^{(-8,10)}\{1,2,2\}$	5	$\langle v_2^3 v_4^2 \cos(2(\psi_2 + 4\psi_3 - 5\psi_4)) \rangle$
29	$c_{2,4,2}^{(4,0)}\{2,1,2\}$	5	$\langle v_2^3 v_4^2 \cos(4(\psi_2 - \psi_4)) \rangle - \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle \langle v_2^2 \rangle$
30	$c_{2,4,5}^{(-8,10)}\{2,1,2\}$	5	$\langle v_2^3 v_4^2 \cos(6\psi_2 + 4\psi_3 - 10\psi_5) \rangle$
31	$c_{2,4}^{(4,0)}\{3,1,1\}$	5	$\langle v_2^3 v_4 v_5 \cos(3\psi_3 - 4\psi_4 - 5\psi_5) \rangle$
32	$c_{2,3,4,5}^{(-3,0,5)}\{1,1,2,1\}$	5	$\langle v_2^3 v_3 v_4 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - \langle v_2^2 \rangle \langle v_2 v_3 v_4 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
33	$c_{2,3,4,5}^{(-8,-10)}\{2,1,1,1\}$	5	$\langle v_2^3 v_3 v_4 v_5 \cos(4\psi_2 - 3\psi_3 + 4\psi_4 - 5\psi_5) \rangle$

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$$\text{normalized cumulant} = \frac{\text{cumulant}}{\sqrt{c_{n_1}^{m_1}\{2\} \cdots c_{n_k}^{m_k}\{2\}}}$$

n_i : the involving harmonics in the cumulant.

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All cumulants involving harmonics 2,3,4,5 and orders 2,3,4,5

cumulant	order	cumulant expression	
1	$c_2\{2\}$	2	$\langle v_2^2 \rangle$
2	$c_3\{2\}$	2	$\langle v_3^2 \rangle$
3	$c_4\{2\}$	2	$\langle v_4^2 \rangle$
4	$c_5\{2\}$	2	$\langle v_5^2 \rangle$
5	$c_{2,4}^{[4]}\{2,1\}$	3	$\langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
6	$c_{2,3,5}^{[-3,5]}\{1,1,1\}$	3	$\langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
7	$c_2\{4\}$	4	$\langle v_2^4 \rangle - 2\langle v_2^2 \rangle^2$
8	$c_3\{4\}$	4	$\langle v_3^4 \rangle - 2\langle v_3^2 \rangle^2$
9	$c_4\{4\}$	4	$\langle v_4^4 \rangle - 2\langle v_4^2 \rangle^2$
10	$c_5\{4\}$	4	$\langle v_5^4 \rangle - 2\langle v_5^2 \rangle^2$
11	$c_{2,4}^{[0]}\{2,2\}$	4	$\langle v_2^2 v_4^2 \rangle - \langle v_2^2 \rangle \langle v_4^2 \rangle$
12	$c_{2,4}^{[0]}\{2,2\}$	4	$\langle v_2^2 v_4^2 \rangle - \langle v_2^2 \rangle \langle v_4^2 \rangle$
13	$c_{2,5}^{[0]}\{2,2\}$	4	$\langle v_2^2 v_5^2 \rangle - \langle v_2^2 \rangle \langle v_5^2 \rangle$
14	$c_{3,4}^{[0]}\{2,2\}$	4	$\langle v_3^2 v_4^2 \rangle - \langle v_3^2 \rangle \langle v_4^2 \rangle$
15	$c_{3,5}^{[0]}\{2,2\}$	4	$\langle v_3^2 v_5^2 \rangle - \langle v_3^2 \rangle \langle v_5^2 \rangle$
16	$c_{4,5}^{[0]}\{2,2\}$	4	$\langle v_4^2 v_5^2 \rangle - \langle v_4^2 \rangle \langle v_5^2 \rangle$
17	$c_{2,3,4}^{[6,-4]}\{1,2,1\}$	4	$\langle v_2^3 v_3 v_4 \cos(2(\psi_2 - 3\psi_3 + 2\psi_4)) \rangle$
18	$c_{2,3,4}^{[8,-5]}\{1,2,1\}$	4	$\langle v_2^3 v_3 v_4 \cos(3\psi_2 - 8\psi_3 + 5\psi_4) \rangle$
19	$c_{2,3,4,5}^{[5,4,-5]}\{1,1,1,1\}$	4	$\langle v_2 v_3 v_4 v_5 \cos(2\psi_2 - 3\psi_3 - 4\psi_4 + 5\psi_5) \rangle$
20	$c_{3,4}^{[0]}\{3,2\}$	5	$\langle v_3^3 v_4^2 \cos(6(\psi_3 - \psi_4)) \rangle$
21	$c_{2,4}^{[4]}\{2,3\}$	5	$\langle v_2^3 v_4^3 \cos(4(\psi_2 - \psi_4)) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
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23	$c_{2,3,4}^{[-6,3]}\{1,2,2\}$	5	$\langle v_2^3 v_3^2 v_4 \cos(2(\psi_2 + 3\psi_3 - 4\psi_4)) \rangle$
24	$c_{2,3,4}^{[0,4]}\{2,2,1\}$	5	$\langle v_2^3 v_3^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle - \langle v_2^2 \rangle \langle v_2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
25	$c_{2,3,5}^{[-3,5]}\{1,1,3\}$	5	$\langle v_2^3 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
26	$c_{2,3,5}^{[-3,5]}\{1,3,1\}$	5	$\langle v_2^3 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
27	$c_{2,3,5}^{[-3,5]}\{3,1,1\}$	5	$\langle v_2^3 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
28	$c_{2,3,4}^{[-8,10]}\{1,2,2\}$	5	$\langle v_2^3 v_3^2 v_4 \cos(2(\psi_2 + 4\psi_3 - 5\psi_4)) \rangle$
29	$c_{2,4,2}^{[4,0]}\{2,1,2\}$	5	$\langle v_2^3 v_4^2 \cos(4(\psi_2 - \psi_4)) \rangle - \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle \langle v_2^2 \rangle$
30	$c_{2,4,5}^{[-4,10]}\{2,1,2\}$	5	$\langle v_2^3 v_4^2 \cos(6\psi_2 + 4\psi_4 - 10\psi_5) \rangle$
31	$c_{3,4}^{[4,0]}\{3,1,1\}$	5	$\langle v_3^3 v_4 v_5 \cos(6\psi_3 - 4\psi_4 - 5\psi_5) \rangle$
32	$c_{2,3,4,5}^{[-3,0,5]}\{1,1,2,1\}$	5	$\langle v_2^3 v_3 v_4 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - \langle v_2^2 \rangle \langle v_2 v_3 v_4 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
33	$c_{2,3,4,5}^{[8,-10]}\{2,1,1,1\}$	5	$\langle v_2^3 v_3 v_4 v_5 \cos(4\psi_2 - 3\psi_3 + 4\psi_4 - 5\psi_5) \rangle$

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n_i : the involving harmonics in the cumulant.

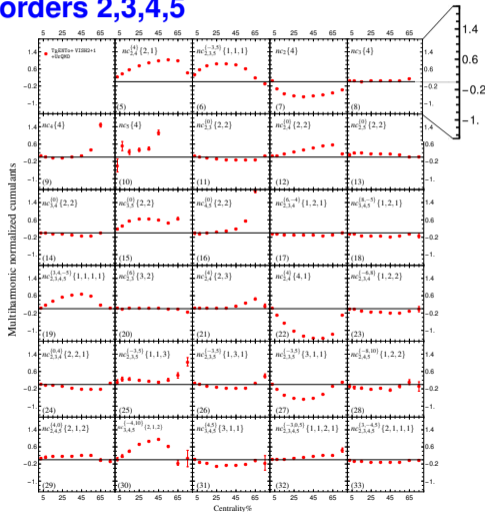
m_i : the power of the flow amplitude v_{n_i} in the cumulant.

- ▶ We have 29 distinct normalized cumulants.
- ▶ T_RENTo + free streaming + VISH(2+1) + UrQMD Maximum A Posteriori (MAP) tuning. [Bernhard et al, Nature Phys., 2019]



All cumulants involving harmonics 2,3,4,5 and orders 2,3,4,5

cumulant	order	cumulant expression
1	$c_2\{2\}$	$\langle v_2^2 \rangle$
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4	$c_5\{2\}$	$\langle v_5^2 \rangle$
5	$c_{2,4}^{[4]}\{2,1\}$	$\langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
6	$c_{2,3,5}^{[-3,5]}\{1,1,1\}$	$\langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
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11	$c_{2,4}^{[0]}\{2,2\}$	$\langle v_2^2 v_4^2 \rangle - \langle v_2^2 \rangle \langle v_4^2 \rangle$
12	$c_{2,4}^{[0]}\{2,2\}$	$\langle v_2^2 v_4^2 \rangle - \langle v_2^2 \rangle \langle v_4^2 \rangle$
13	$c_{2,5}^{[0]}\{2,2\}$	$\langle v_2^2 v_5^2 \rangle - \langle v_2^2 \rangle \langle v_5^2 \rangle$
14	$c_{3,4}^{[0]}\{2,2\}$	$\langle v_3^2 v_4^2 \rangle - \langle v_3^2 \rangle \langle v_4^2 \rangle$
15	$c_{3,5}^{[0]}\{2,2\}$	$\langle v_3^2 v_5^2 \rangle - \langle v_3^2 \rangle \langle v_5^2 \rangle$
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17	$c_{2,3,4}^{[6,-4]}\{1,2,1\}$	$\langle v_2^3 v_3 v_4 \cos(2\psi_2 - 3\psi_3 + 2\psi_4) \rangle$
18	$c_{2,3,5}^{[8,-5]}\{1,2,1\}$	$\langle v_2^3 v_3 v_5 \cos(3\psi_3 - 8\psi_4 + 5\psi_5) \rangle$
19	$c_{2,3,4,5}^{[5,4,-5]}\{1,1,1,1\}$	$\langle v_2 v_3 v_4 v_5 \cos(2\psi_2 - 3\psi_3 - 4\psi_4 + 5\psi_5) \rangle$
20	$c_{3,4}^{[4]}\{3,2\}$	$\langle v_3^2 v_4^2 \cos(6(\psi_3 - \psi_4)) \rangle$
21	$c_{3,4}^{[4]}\{2,3\}$	$\langle v_3^2 v_4^2 \cos(4(\psi_2 - \psi_4)) \rangle - 2\langle v_2^2 \rangle \langle v_3^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
22	$c_{3,4}^{[4]}\{4,1\}$	$\langle v_3^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle - 3\langle v_2^2 \rangle \langle v_3^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
23	$c_{3,4,5}^{[-3,3]}\{1,2,2\}$	$\langle v_3^2 v_4^2 v_5 \cos(2(\psi_3 + 3\psi_4 - 4\psi_5)) \rangle$
24	$c_{2,3,4}^{[0,4]}\{2,2,1\}$	$\langle v_2^3 v_3^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle - \langle v_2^2 \rangle \langle v_3^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
25	$c_{2,3,4,5}^{[-3,5]}\{1,1,3\}$	$\langle v_2^3 v_3 v_4 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_4 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
26	$c_{2,3,5}^{[-3,5]}\{1,3,1\}$	$\langle v_2^3 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
27	$c_{2,3,4,5}^{[-3,5]}\{3,1,1\}$	$\langle v_2^3 v_3 v_4 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_4 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
28	$c_{3,4,5}^{[-3,10]}\{1,2,2\}$	$\langle v_3^2 v_4^2 v_5 \cos(2(\psi_3 + 4\psi_4 - 5\psi_5)) \rangle$
29	$c_{2,4,5}^{[4,0]}\{2,1,2\}$	$\langle v_2^3 v_4^2 v_5 \cos(4(\psi_2 - \psi_4)) \rangle - \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle \langle v_5^2 \rangle$
30	$c_{2,4,5}^{[4,0]}\{2,1,2\}$	$\langle v_2^3 v_4^2 v_5 \cos(6\psi_3 + 4\psi_4 - 10\psi_5) \rangle$
31	$c_{3,4,5}^{[4,5]}\{3,1,1\}$	$\langle v_3^2 v_4 v_5 \cos(6\psi_3 - 4\psi_4 - 5\psi_5) \rangle$
32	$c_{2,3,4,5}^{[-3,0,5]}\{1,1,2,1\}$	$\langle v_2^3 v_3 v_4 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - \langle v_2^2 \rangle \langle v_2 v_3 v_4 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
33	$c_{2,3,4,5}^{[4,0,-5]}\{2,1,1,1\}$	$\langle v_2^3 v_3 v_4 v_5 \cos(4\psi_2 - 3\psi_3 + 4\psi_4 - 5\psi_5) \rangle$

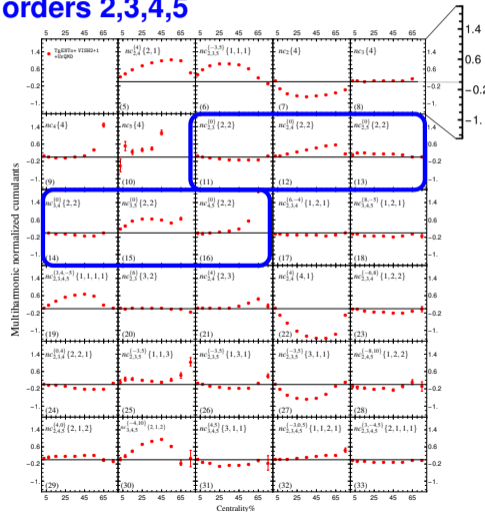


A systematic way to extract all multiharmonic flow cumulants



All cumulants involving harmonics 2,3,4,5 and orders 2,3,4,5

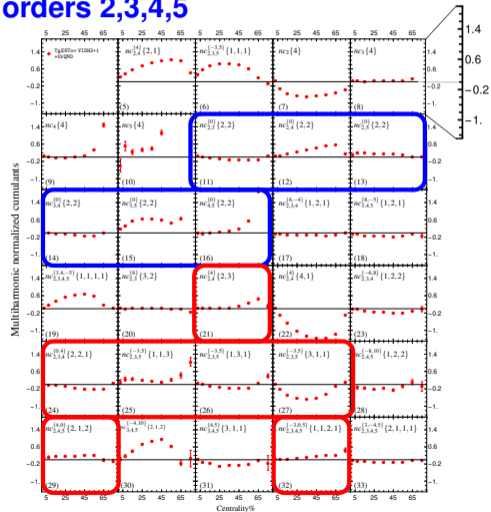
cumulant	order	cumulant expression
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3	$c_4\{2\}$	$\langle v_4^2 \rangle$
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5	$c_{2,4}^{[4]}\{2,1\}$	$\langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
6	$c_{2,3,5}^{[-3,5]}\{1,1,1\}$	$\langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
7	$c_2\{4\}$	$\langle v_2^2 \rangle - 2\langle v_2^2 \rangle^2$
8	$c_3\{4\}$	$\langle v_3^2 \rangle - 2\langle v_3^2 \rangle^2$
9	$c_4\{4\}$	$\langle v_4^2 \rangle - 2\langle v_4^2 \rangle^2$
10	$c_5\{4\}$	$\langle v_5^2 \rangle - 2\langle v_5^2 \rangle^2$
11	$c_{2,4}^{[0]}\{2,2\}$	$\langle v_2^2 v_4^2 \rangle - \langle v_2^2 \rangle \langle v_4^2 \rangle$
12	$c_{2,4}^{[0]}\{2,2\}$	$\langle v_2^2 v_4^2 \rangle - \langle v_2^2 \rangle \langle v_4^2 \rangle$
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14	$c_{3,4}^{[0]}\{2,2\}$	$\langle v_3^2 v_4^2 \rangle - \langle v_3^2 \rangle \langle v_4^2 \rangle$
15	$c_{3,5}^{[0]}\{2,2\}$	$\langle v_3^2 v_5^2 \rangle - \langle v_3^2 \rangle \langle v_5^2 \rangle$
16	$c_{4,5}^{[0]}\{2,2\}$	$\langle v_4^2 v_5^2 \rangle - \langle v_4^2 \rangle \langle v_5^2 \rangle$
17	$c_{2,3,4}^{[6,-4]}\{1,2,1\}$	$\langle v_2^2 v_3 v_4 \cos(2\psi_2 - 3\psi_3 + 2\psi_4) \rangle$
18	$c_{2,3,4}^{[8,-5]}\{1,2,1\}$	$\langle v_2^2 v_3 v_4 \cos(3\psi_2 - 8\psi_3 + 5\psi_4) \rangle$
19	$c_{2,3,4,5}^{[5,4,-5]}\{1,1,1,1\}$	$\langle v_2 v_3 v_4 v_5 \cos(2\psi_2 - 3\psi_3 - 4\psi_4 + 5\psi_5) \rangle$
20	$c_{3,4}^{[0]}\{3,2\}$	$\langle v_3^2 v_4^2 \cos(6(\psi_3 - \psi_4)) \rangle$
21	$c_{3,4}^{[4]}\{2,3\}$	$\langle v_3^2 v_4^2 \cos(4(\psi_2 - \psi_4)) \rangle - 2\langle v_2^2 \rangle \langle v_3^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
22	$c_{3,4}^{[4]}\{4,1\}$	$\langle v_3^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle - 3\langle v_2^2 \rangle \langle v_3^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
23	$c_{3,4}^{[-6,3]}\{1,2,2\}$	$\langle v_3^2 v_2 v_4 \cos(2(\psi_3 + 3\psi_4 - 4\psi_2)) \rangle$
24	$c_{2,3,4}^{[0,4]}\{2,2,1\}$	$\langle v_2^2 v_3^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle - \langle v_2^2 \rangle \langle v_3^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
25	$c_{2,3,4,5}^{[-3,5]}\{1,1,3\}$	$\langle v_2^2 v_3 v_4 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_4 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
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28	$c_{3,4}^{[-8,10]}\{1,2,2\}$	$\langle v_3^2 v_2 v_4 \cos(2(\psi_3 + 4\psi_4 - 5\psi_2)) \rangle$
29	$c_{2,4,5}^{[4,0]}\{2,1,2\}$	$\langle v_2^2 v_4^2 v_5 \cos(4(\psi_2 - \psi_4)) \rangle - \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle \langle v_5^2 \rangle$
30	$c_{2,4,5}^{[4,0]}\{2,1,2\}$	$\langle v_2^2 v_4^2 v_5 \cos(6\psi_2 + 4\psi_4 - 10\psi_5) \rangle$
31	$c_{3,4,5}^{[4,5]}\{3,1,1\}$	$\langle v_3^2 v_4 v_5 \cos(3\psi_3 - 4\psi_4 - 5\psi_5) \rangle$
32	$c_{2,3,4,5}^{[-3,0,5]}\{1,1,2,1\}$	$\langle v_2^2 v_3 v_4 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - \langle v_2^2 \rangle \langle v_2 v_3 v_4 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
33	$c_{2,3,4,5}^{[4,0,-5]}\{2,1,1,1\}$	$\langle v_2^2 v_3 v_4 v_5 \cos(4\psi_2 - 3\psi_3 + 4\psi_4 - 5\psi_5) \rangle$





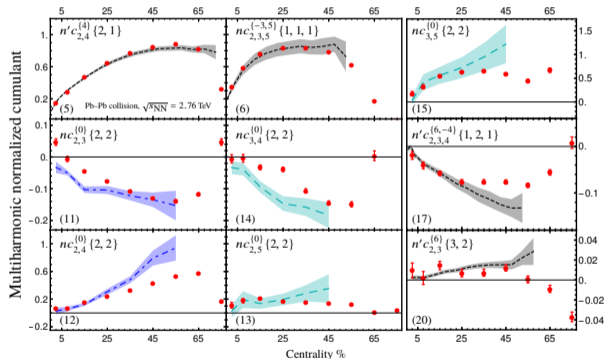
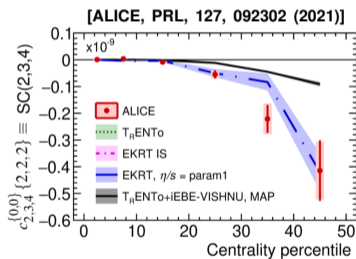
All cumulants involving harmonics 2,3,4,5 and orders 2,3,4,5

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6	$c_{2,3,5}^{[-3,5]}\{1,1,1\}$	$\langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
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20	$c_{3,2}^{[4]}\{3,2\}$	$\langle v_3^3 v_2^2 \cos(6(\psi_3 - \psi_2)) \rangle$
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29	$c_{4,2,5}^{[4,0]}\{2,1,2\}$	$\langle v_4^2 v_2^2 v_5 \cos(4(\psi_2 - \psi_4)) \rangle - \langle v_4^2 v_5 \cos(4(\psi_2 - \psi_4)) \rangle \langle v_2^2 \rangle$
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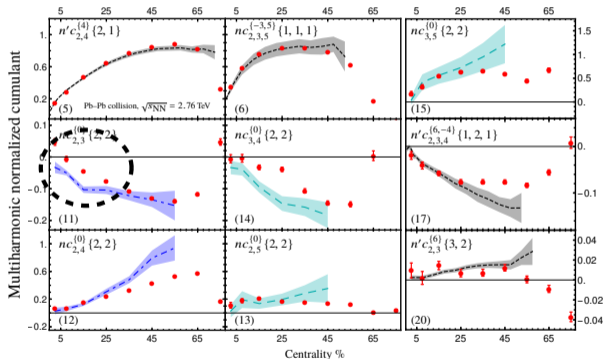
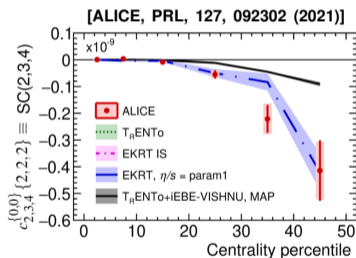


(Normalized) multiharmonic cumulants at the LHC



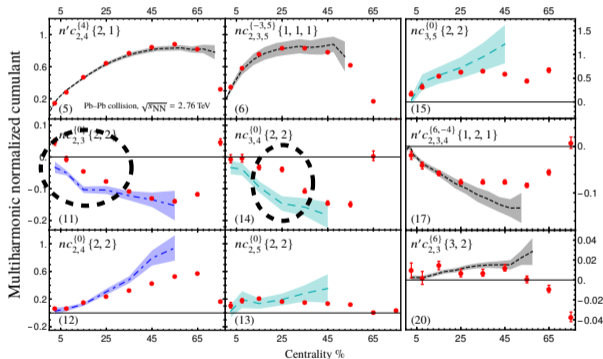
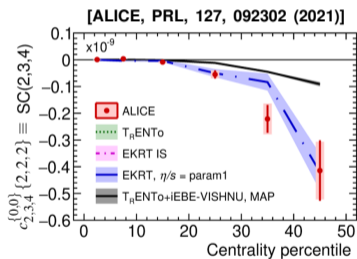


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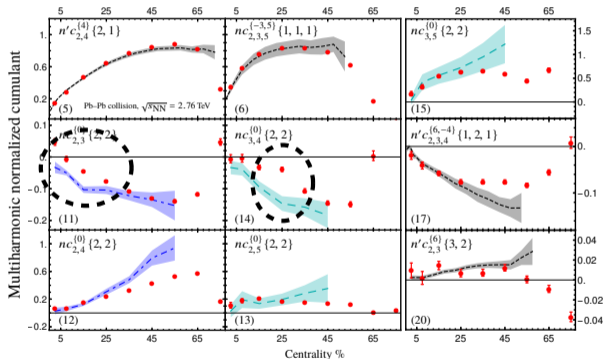
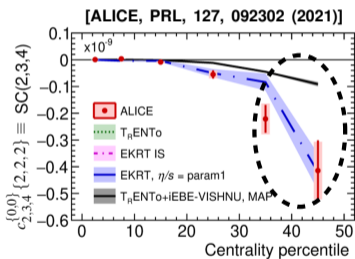
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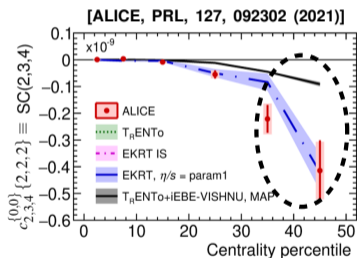
↳ A systematic way to extract all multiharmonic flow cumulants



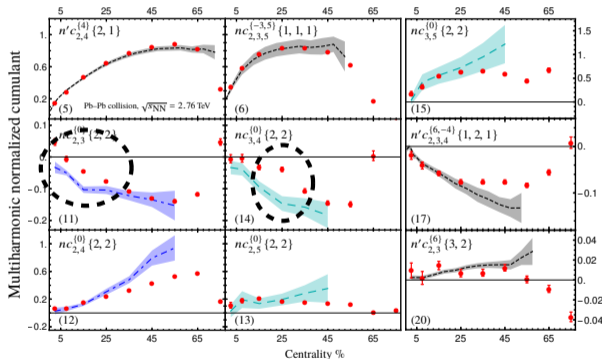
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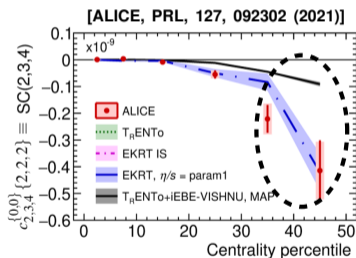
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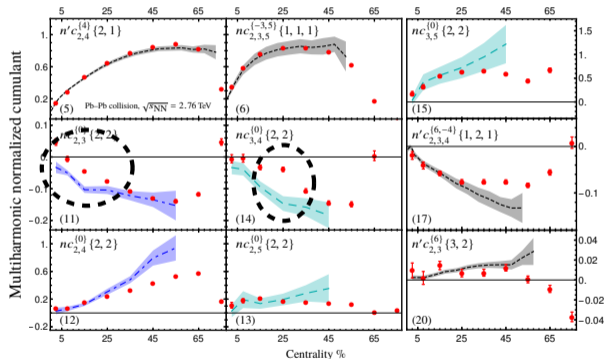
- ▶ The simulations are all predictions from a model tuned by Bayesian analysis.



(Normalized) multiharmonic cumulants at the LHC



- ▶ The simulations are all predictions from a model tuned by Bayesian analysis.
- ▶ How would the inferred parameters change if we use the observables as inputs for a Bayesian analysis?





Linear and nonlinear hydrodynamic response from distribution

[SFT, Eur.Phys.J.C 81 (2021) 7,652]



Linear and nonlinear hydrodynamic response from distribution

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$$v_2 e^{i2\psi_2} \simeq w_2 \varepsilon_2 e^{i2\phi_2},$$

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$$v_4 e^{i4\psi_4} \simeq w_4 \varepsilon_4 e^{i4\phi_4} + w_{4(22)} \varepsilon_2^2 e^{i4\phi_2},$$

$$v_5 e^{i5\psi_4} \simeq w_5 \varepsilon_5 e^{i5\phi_5} + w_{5(23)} \varepsilon_2 \varepsilon_3 e^{i2\phi_2 + i3\phi_3}.$$



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Initial state fluctuation $\rightarrow p_i(\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots, \phi_1 - \phi_2, \phi_2 - \phi_3, \dots)$

$\xrightarrow{\text{collec. evol.}}$ $p_f(v_1, v_2, v_3, \dots, \Psi_1 - \Psi_2, \Psi_2 - \Psi_3, \dots)$



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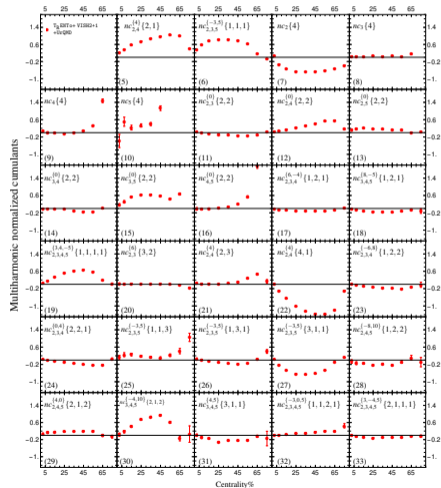
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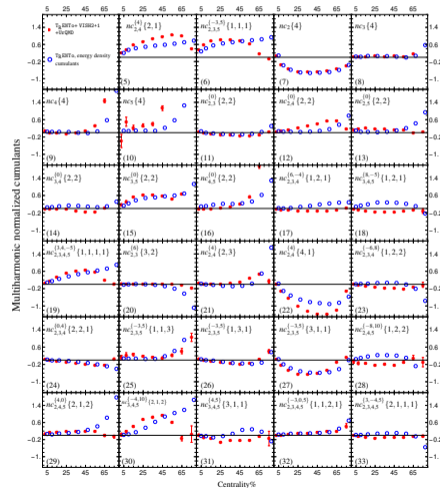
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[SFT, Eur.Phys.J.C 81 (2021) 7,652]

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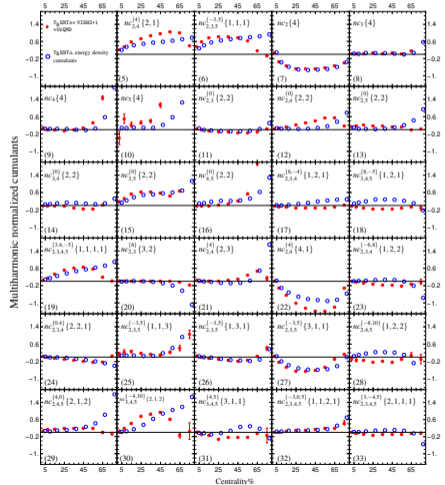
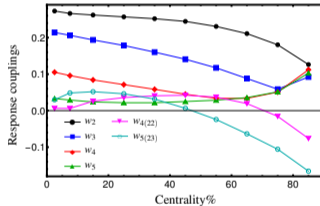
[Teaney, Yan, PRC, 2010, 2012, 2013] [Noronha-Hostler, Gardim, Yan, Luzum, Ollitrault, PRC, 2012, 2015]

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Initial state fluctuation $\rightarrow p_i(\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots, \phi_1 - \phi_2, \phi_2 - \phi_3, \dots)$

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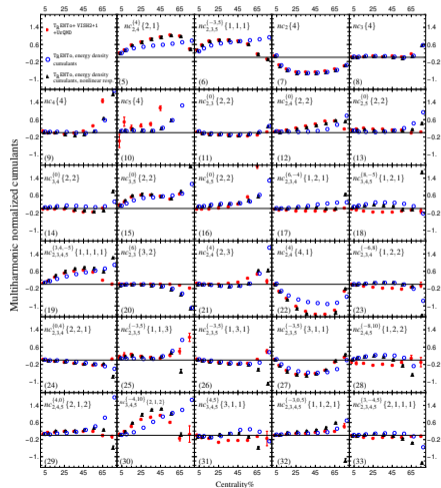
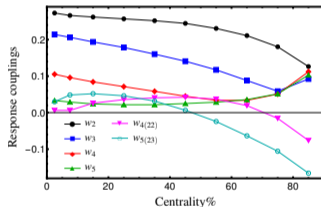
Linear and nonlinear hydrodynamic response from distribution

[SFT, Eur.Phys.J.C 81 (2021) 7,652]

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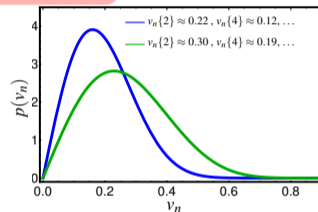


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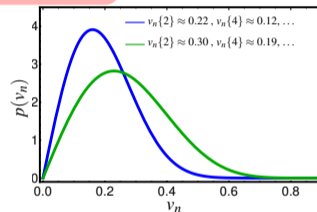


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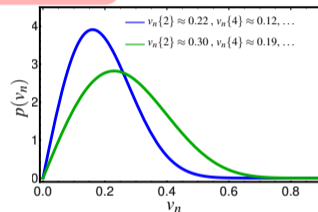


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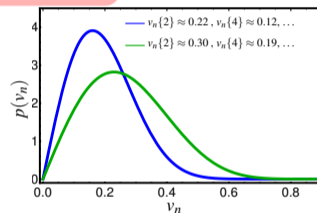
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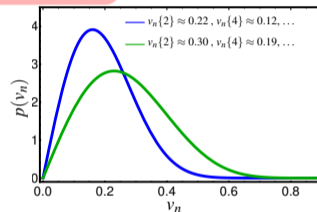
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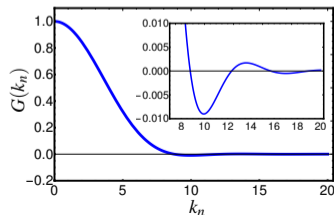
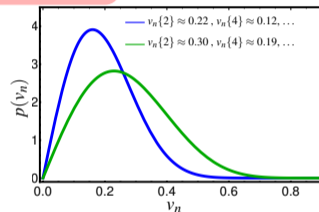
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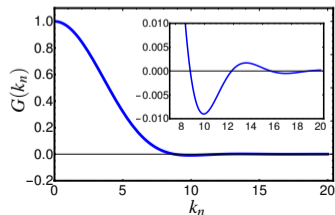
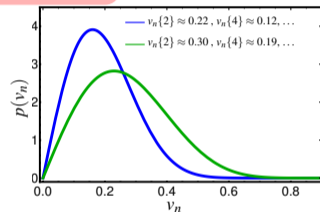
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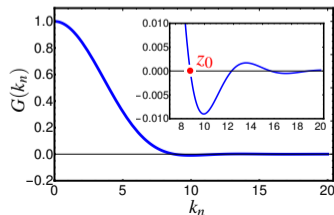
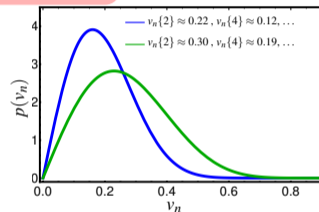
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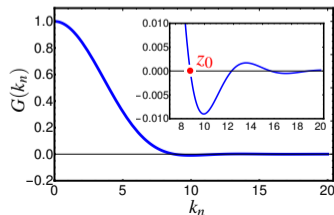
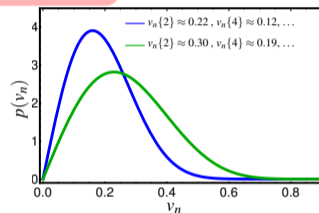
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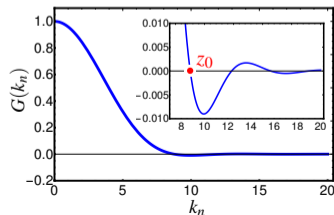
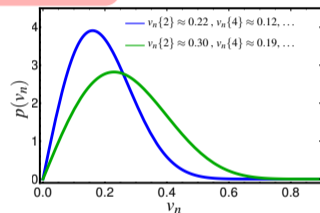
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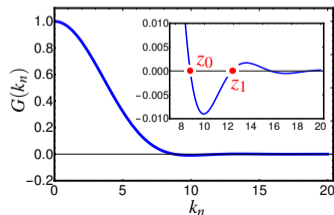
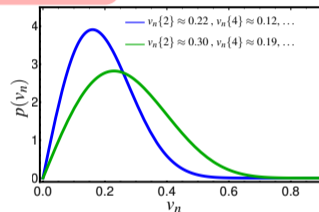
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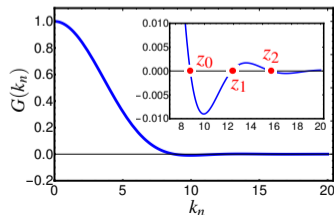
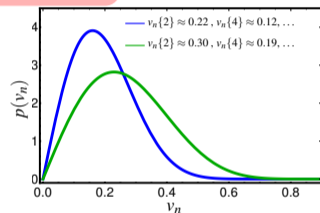
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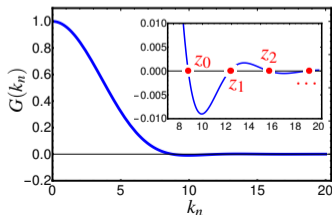
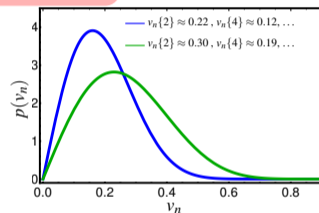
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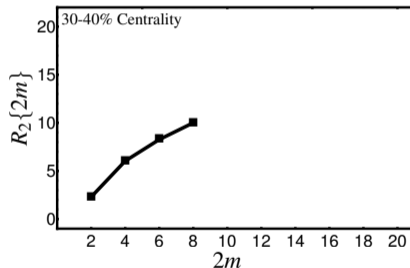


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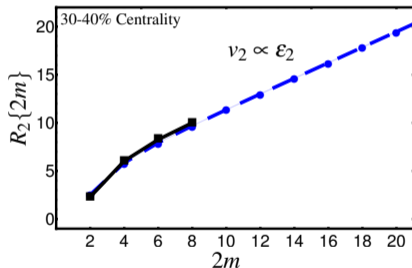


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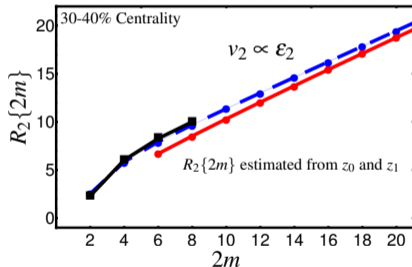


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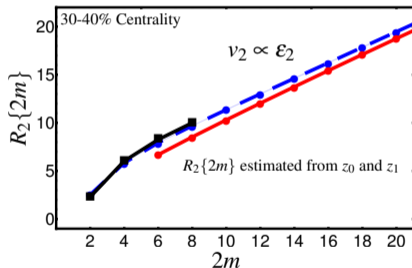


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Lee-Yang zeros expansion is a complementary study for the conventional multiparticle technique.



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- ▶ A new method is employed to calculate statistical uncertainties.
- ▶ Mathematica package **MultiharmonicCumulants_v2_1.m**
<https://github.com/FaridTaghavi/MultiharmonicCumulants.git>
- ▶ The unmeasured cumulants contain an independent piece of information about the heavy-ion models.
- ▶ One would ask whether the new cumulants help the estimations in the Bayesian analysis?
- ▶ We obtained the linear and nonlinear hydrodynamic response couplings by comparing fluctuation distributions.



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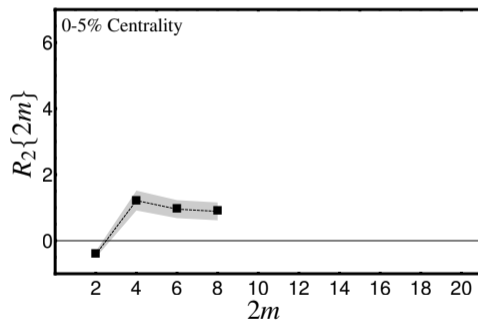
Thank you for your attention!

Backup slides



Large order single harmonic cumulants [J. Jia, SFT, (in progress)]

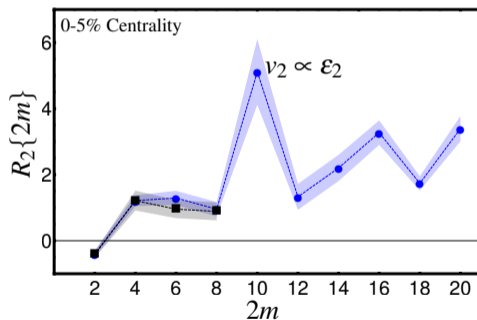
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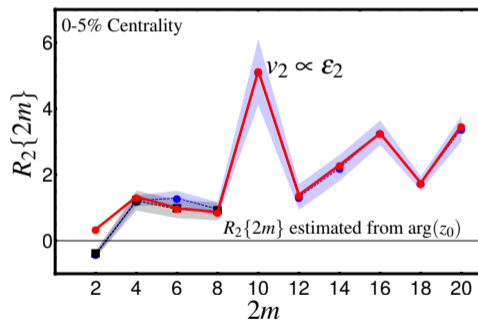
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- ▶ Here, we have two zeros $z_0 e^{i\alpha_0}$ and

$$z_0 e^{-i\alpha_0}$$

