A multiharmonic/large-order flow cumulant analysis for relativistic heavy-ion collisions

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SFT, Eur.Phys.J.C **81** (2021) 7,652

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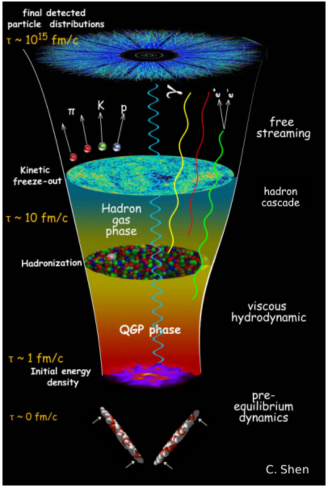

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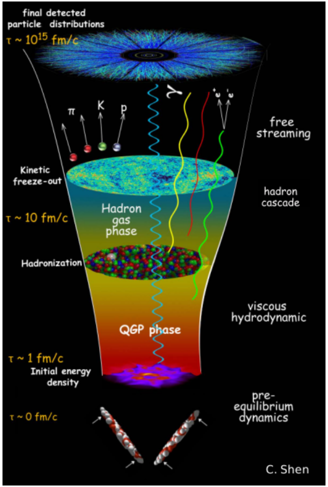

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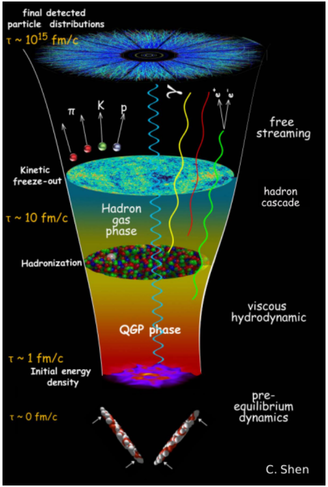

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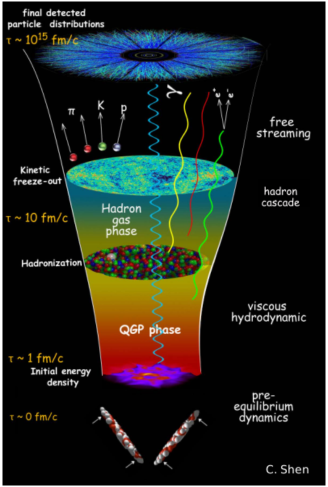

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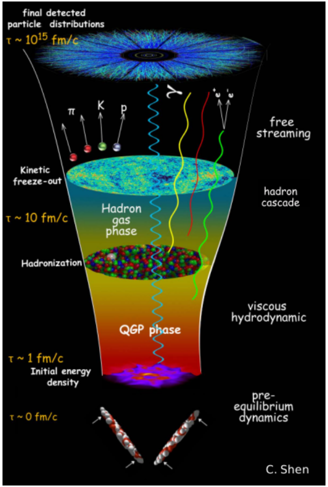

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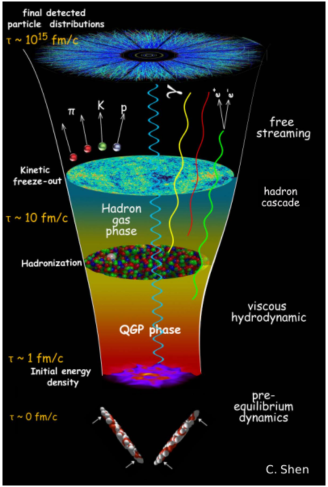
It is important to introduce experimental observables.



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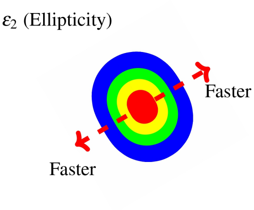

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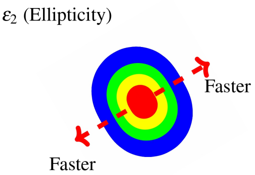

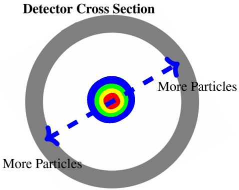

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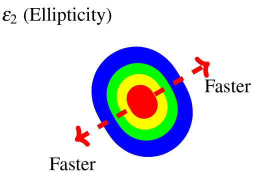

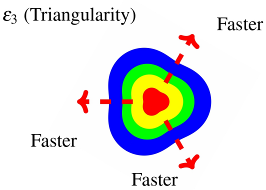

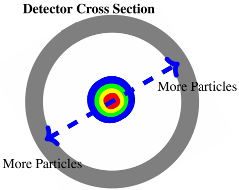

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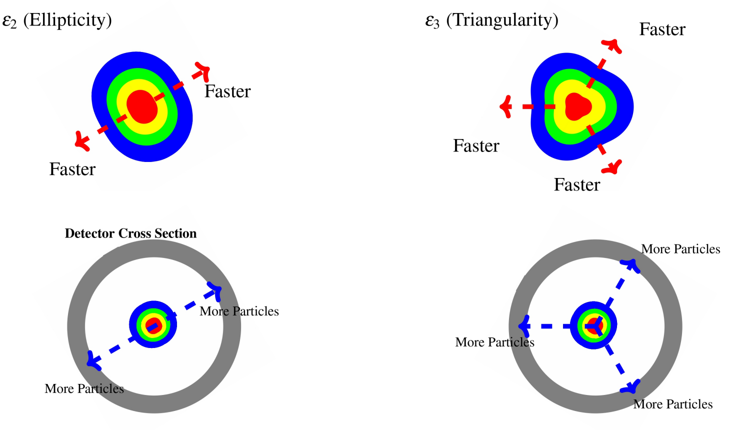

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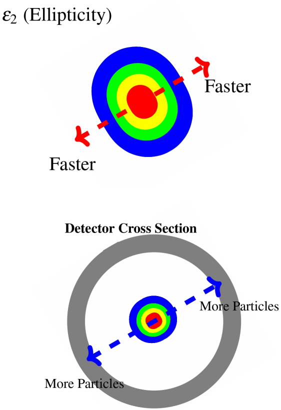

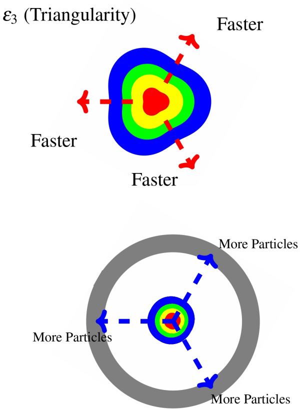

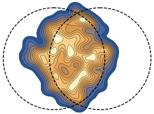

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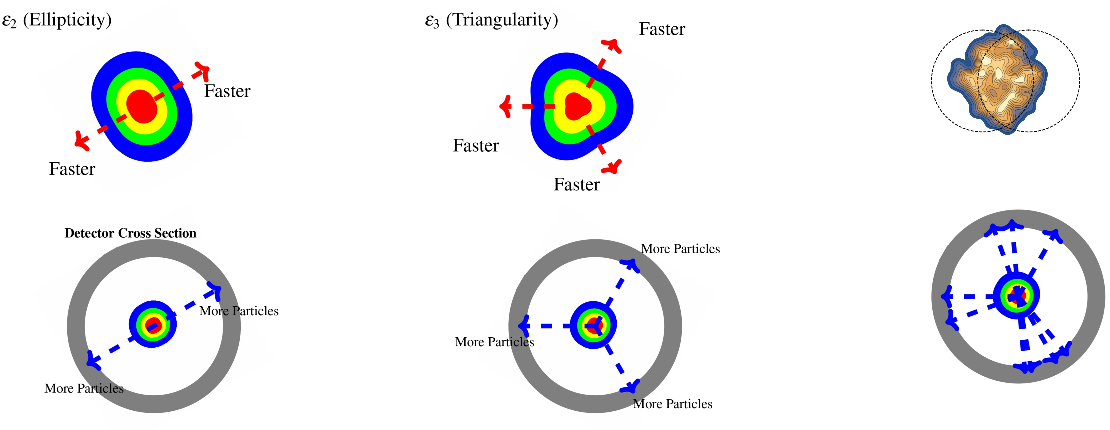

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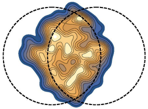
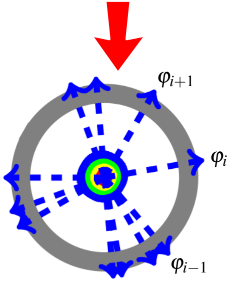


collective evolution



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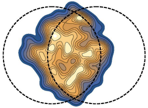

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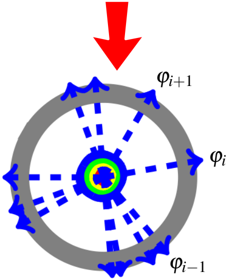

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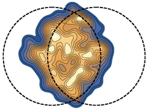

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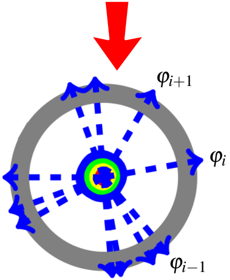

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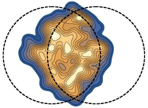

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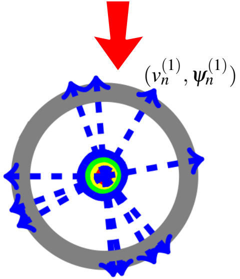

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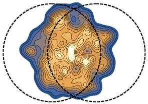

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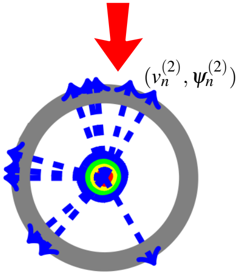

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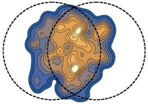

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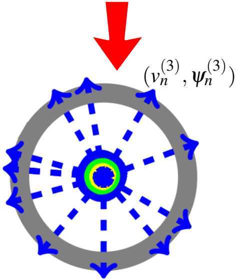

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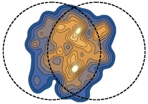

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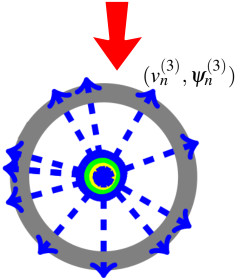

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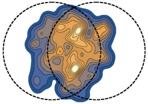

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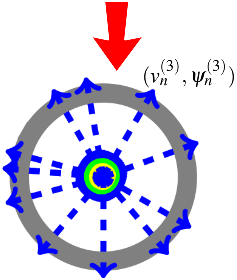

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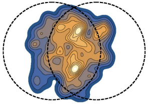

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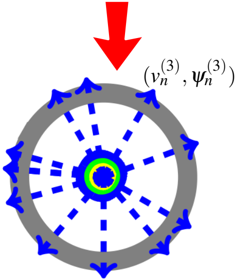

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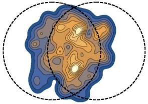

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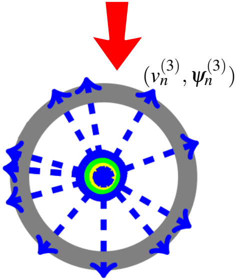

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  [C. Mordasini, A. Bilandzic, D. Karakoc, SFT, PRC, 2020],
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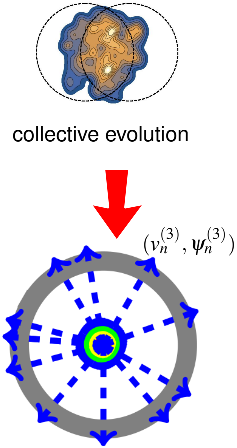

One package for all cumulants

[SFT, Eur.Phys.J.C 81 (2021) 7,652]

$$p_f(v_1, v_2, v_3, \dots, \psi_1 - \psi_2, \psi_2 - \psi_3, \dots)$$

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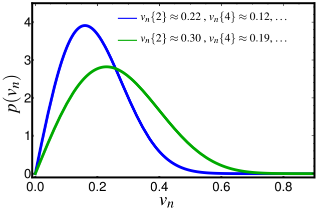

- Example: single harmonic distribution $p(v_n)$ and its cumulants $v_n\{2\}$, $v_n\{4\}$, $v_n\{6\}$, $v_n\{8\}, \ldots$ $(v_n\{2m\} \propto c_n^{1/2m}\{2m\})$

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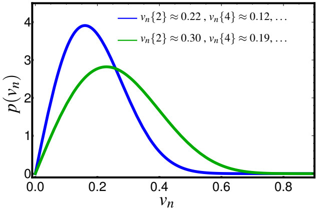

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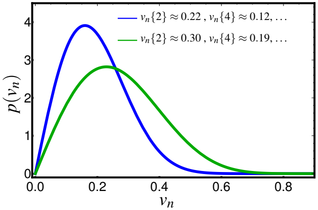

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  https://github.com/FaridTaghavi/MultiharmonicCumulants.git

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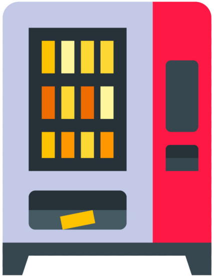

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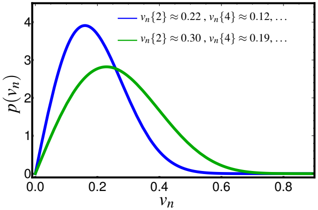

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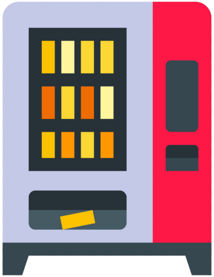

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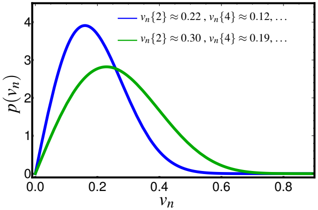

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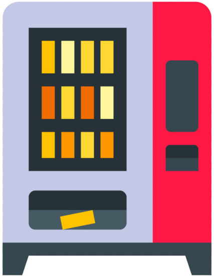
- A new method for extracting **statistical error** is implemented.

ORDER YOUR CUMULANT!



All cumulants involving harmonics 2,3,4,5 and orders 2,3,4,5

	cumulant	order	cumulant expression
1	$c_2\{2\}$	2	$\langle v_2^2\rangle$
2	$c_3\{2\}$	2	$\langle v_3^2\rangle$
3	$c_4\{2\}$	2	$\langle v_4^2\rangle$
4	$c_5\{2\}$	2	$\langle v_5^2\rangle$
5	$c_{2,4}^{\{4\}}\{2,1\}$	3	$\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
6	$c_{2,3,5}^{\{-3,5\}}\{1,1,1\}$	3	$\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
7	$c_2\{4\}$	4	$\langle v_2^4\rangle - 2\langle v_2^2\rangle^2$
8	$c_3\{4\}$	4	$\langle v_3^4\rangle - 2\langle v_3^2\rangle^2$
9	$c_4\{4\}$	4	$\langle v_4^4\rangle - 2\langle v_4^2\rangle^2$
10	$c_5\{4\}$	4	$\langle v_5^4\rangle - 2\langle v_5^2\rangle^2$
11	$c_{2,3}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_3^2\rangle - \langle v_2^2\rangle\langle v_3^2\rangle$
12	$c_{2,4}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_4^2\rangle - \langle v_2^2\rangle\langle v_4^2\rangle$
13	$c_{2,5}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_5^2\rangle - \langle v_2^2\rangle\langle v_5^2\rangle$
14	$c_{3,4}^{\{0\}}\{2,2\}$	4	$\langle v_3^2 v_4^2\rangle - \langle v_3^2\rangle\langle v_4^2\rangle$
15	$c_{3,5}^{\{0\}}\{2,2\}$	4	$\langle v_3^2 v_5^2\rangle - \langle v_3^2\rangle\langle v_5^2\rangle$
16	$c_{4,5}^{\{0\}}\{2,2\}$	4	$\langle v_4^2 v_5^2\rangle - \langle v_4^2\rangle\langle v_5^2\rangle$
17	$c_{2,3,4}^{\{6,-4\}}\{1,2,1\}$	4	$\langle v_3^2 v_2 v_4 \cos(2(\psi_2-3\psi_3+2\psi_4))\rangle$
18	$c_{3,4,5}^{\{8,-5\}}\{1,2,1\}$	4	$\langle v_4^2 v_3 v_5 \cos(3\psi_3-8\psi_4+5\psi_5)\rangle$
19	$c_{2,3,4,5}^{\{3,4,-5\}}\{1,1,1,1\}$	4	$\langle v_2 v_3 v_4 v_5 \cos(2\psi_2-3\psi_3-4\psi_4+5\psi_5)\rangle$
20	$c_{2,3}^{\{6\}}\{3,2\}$	5	$\langle v_2^3 v_3^2 \cos(6(\psi_2-\psi_3))\rangle$
21	$c_{2,4}^{\{4\}}\{2,3\}$	5	$\langle v_2^2 v_4^3 \cos(4(\psi_2-\psi_4))\rangle - 2\langle v_4^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
22	$c_{2,4}^{\{4\}}\{4,1\}$	5	$\langle v_2^4 v_4 \cos(4(\psi_2-\psi_4))\rangle - 3\langle v_2^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
23	$c_{2,3,4}^{\{-6,8\}}\{1,2,2\}$	5	$\langle v_4^2 v_3^2 v_2 \cos(2(\psi_2+3\psi_3-4\psi_4))\rangle$
24	$c_{2,3,4}^{\{0,4\}}\{2,2,1\}$	5	$\langle v_3^2 v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle - \langle v_3^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
25	$c_{2,3,5}^{\{-3,5\}}\{1,1,3\}$	5	$\langle v_5^3 v_2 v_3 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_5^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
26	$c_{2,3,5}^{\{-3,5\}}\{1,3,1\}$	5	$\langle v_3^3 v_2 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_3^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
27	$c_{2,3,5}^{\{-3,5\}}\{3,1,1\}$	5	$\langle v_2^3 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_2^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
28	$c_{2,4,5}^{\{-8,10\}}\{1,2,2\}$	5	$\langle v_5^2 v_4^2 v_2 \cos(2(\psi_2+4\psi_4-5\psi_5))\rangle$
29	$c_{2,4,5}^{\{4,0\}}\{2,1,2\}$	5	$\langle v_5^2 v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle - \langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle\langle v_5^2\rangle$
30	$c_{3,4,5}^{\{-4,10\}}\{2,1,2\}$	5	$\langle v_5^2 v_3^2 v_4 \cos(6\psi_3+4\psi_4-10\psi_5)\rangle$
31	$c_{3,4,5}^{\{4,5\}}\{3,1,1\}$	5	$\langle v_3^3 v_4 v_5 \cos(9\psi_3-4\psi_4-5\psi_5)\rangle$
32	$c_{2,3,4,5}^{\{-3,0,5\}}\{1,1,2,1\}$	5	$\langle v_4^2 v_2 v_5 v_3 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - \langle v_4^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
33	$c_{2,3,4,5}^{\{3,-4,5\}}\{2,1,1,1\}$	5	$\langle v_2^2 v_3 v_4 v_5 \cos(4\psi_2-3\psi_3+4\psi_4-5\psi_5)\rangle$

- There are 33 distinct cumulants.

All cumulants involving harmonics 2,3,4,5 and orders 2,3,4,5

	cumulant	order	cumulant expression
1	$c_2\{2\}$	2	$\langle v_2^2\rangle$
2	$c_3\{2\}$	2	$\langle v_3^2\rangle$
3	$c_4\{2\}$	2	$\langle v_4^2\rangle$
4	$c_5\{2\}$	2	$\langle v_5^2\rangle$
5	$c_{2,4}^{\{4\}}\{2,1\}$	3	$\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
6	$c_{2,3,5}^{\{-3,5\}}\{1,1,1\}$	3	$\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
7	$c_2\{4\}$	4	$\langle v_2^4\rangle - 2\langle v_2^2\rangle^2$
8	$c_3\{4\}$	4	$\langle v_3^4\rangle - 2\langle v_3^2\rangle^2$
9	$c_4\{4\}$	4	$\langle v_4^4\rangle - 2\langle v_4^2\rangle^2$
10	$c_5\{4\}$	4	$\langle v_5^4\rangle - 2\langle v_5^2\rangle^2$
11	$c_{2,3}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_3^2\rangle - \langle v_2^2\rangle\langle v_3^2\rangle$
12	$c_{2,4}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_4^2\rangle - \langle v_2^2\rangle\langle v_4^2\rangle$
13	$c_{2,5}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_5^2\rangle - \langle v_2^2\rangle\langle v_5^2\rangle$
14	$c_{3,4}^{\{0\}}\{2,2\}$	4	$\langle v_3^2 v_4^2\rangle - \langle v_3^2\rangle\langle v_4^2\rangle$
15	$c_{3,5}^{\{0\}}\{2,2\}$	4	$\langle v_3^2 v_5^2\rangle - \langle v_3^2\rangle\langle v_5^2\rangle$
16	$c_{4,5}^{\{0\}}\{2,2\}$	4	$\langle v_4^2 v_5^2\rangle - \langle v_4^2\rangle\langle v_5^2\rangle$
17	$c_{2,3,4}^{\{6,-4\}}\{1,2,1\}$	4	$\langle v_3^2 v_2 v_4 \cos(2(\psi_2-3\psi_3+2\psi_4))\rangle$
18	$c_{3,4,5}^{\{8,-5\}}\{1,2,1\}$	4	$\langle v_4^2 v_3 v_5 \cos(3\psi_3-8\psi_4+5\psi_5)\rangle$
19	$c_{2,3,4,5}^{\{3,4,-5\}}\{1,1,1,1\}$	4	$\langle v_2 v_3 v_4 v_5 \cos(2\psi_2-3\psi_3-4\psi_4+5\psi_5)\rangle$
20	$c_{2,3}^{\{6\}}\{3,2\}$	5	$\langle v_2^3 v_3^2 \cos(6(\psi_2-\psi_3))\rangle$
21	$c_{2,4}^{\{4\}}\{2,3\}$	5	$\langle v_2^2 v_4^3 \cos(4(\psi_2-\psi_4))\rangle - 2\langle v_4^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
22	$c_{2,4}^{\{4\}}\{4,1\}$	5	$\langle v_2^4 v_4 \cos(4(\psi_2-\psi_4))\rangle - 3\langle v_2^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
23	$c_{2,3,4}^{\{-6,8\}}\{1,2,2\}$	5	$\langle v_4^2 v_3^2 v_2 \cos(2(\psi_2+3\psi_3-4\psi_4))\rangle$
24	$c_{2,3,4}^{\{0,4\}}\{2,2,1\}$	5	$\langle v_3^2 v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle - \langle v_3^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
25	$c_{2,3,5}^{\{-3,5\}}\{1,1,3\}$	5	$\langle v_5^3 v_2 v_3 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_5^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
26	$c_{2,3,5}^{\{-3,5\}}\{1,3,1\}$	5	$\langle v_3^3 v_2 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_3^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
27	$c_{2,3,5}^{\{-3,5\}}\{3,1,1\}$	5	$\langle v_2^3 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_2^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
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31	$c_{3,4,5}^{\{4,5\}}\{3,1,1\}$	5	$\langle v_3^3 v_4 v_5 \cos(9\psi_3-4\psi_4-5\psi_5)\rangle$
32	$c_{2,3,4,5}^{\{-3,0,5\}}\{1,1,2,1\}$	5	$\langle v_4^2 v_2 v_5 v_3 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - \langle v_4^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
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6	$c_{2,3,5}^{\{-3,5\}}\{1,1,1\}$	3	$\langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
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10	$c_5\{4\}$	4	$\langle v_5^4 \rangle - 2\langle v_5^2 \rangle^2$
11	$c_{2,3}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_3^2 \rangle - \langle v_2^2 \rangle \langle v_3^2 \rangle$
12	$c_{2,4}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_4^2 \rangle - \langle v_2^2 \rangle \langle v_4^2 \rangle$
13	$c_{2,5}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_5^2 \rangle - \langle v_2^2 \rangle \langle v_5^2 \rangle$
14	$c_{3,4}^{\{0\}}\{2,2\}$	4	$\langle v_3^2 v_4^2 \rangle - \langle v_3^2 \rangle \langle v_4^2 \rangle$
15	$c_{3,5}^{\{0\}}\{2,2\}$	4	$\langle v_3^2 v_5^2 \rangle - \langle v_3^2 \rangle \langle v_5^2 \rangle$
16	$c_{4,5}^{\{0\}}\{2,2\}$	4	$\langle v_4^2 v_5^2 \rangle - \langle v_4^2 \rangle \langle v_5^2 \rangle$
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19	$c_{2,3,4,5}^{\{3,4,-5\}}\{1,1,1,1\}$	4	$\langle v_2 v_3 v_4 v_5 \cos(2\psi_2 - 3\psi_3 - 4\psi_4 + 5\psi_5) \rangle$
20	$c_{2,3}^{\{6\}}\{3,2\}$	5	$\langle v_2^3 v_3^2 \cos(6(\psi_2 - \psi_3)) \rangle$
21	$c_{2,4}^{\{4\}}\{2,3\}$	5	$\langle v_2^2 v_4^3 \cos(4(\psi_2 - \psi_4)) \rangle - 2\langle v_4^2 \rangle \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
22	$c_{2,4}^{\{4\}}\{4,1\}$	5	$\langle v_2^4 v_4 \cos(4(\psi_2 - \psi_4)) \rangle - 3\langle v_2^2 \rangle \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
23	$c_{2,3,4}^{\{-6,8\}}\{1,2,2\}$	5	$\langle v_4^2 v_3^2 v_2 \cos(2(\psi_2 + 3\psi_3 - 4\psi_4)) \rangle$
24	$c_{2,3,4}^{\{0,4\}}\{2,2,1\}$	5	$\langle v_3^2 v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle - \langle v_3^2 \rangle \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle$
25	$c_{2,3,5}^{\{-3,5\}}\{1,1,3\}$	5	$\langle v_5^3 v_2 v_3 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_5^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
26	$c_{2,3,5}^{\{-3,5\}}\{1,3,1\}$	5	$\langle v_3^3 v_2 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_3^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
27	$c_{2,3,5}^{\{-3,5\}}\{3,1,1\}$	5	$\langle v_2^3 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - 2\langle v_2^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
28	$c_{2,4,5}^{\{-8,10\}}\{1,2,2\}$	5	$\langle v_5^2 v_4^2 v_2 \cos(2(\psi_2 + 4\psi_4 - 5\psi_5)) \rangle$
29	$c_{2,4,5}^{\{4,0\}}\{2,1,2\}$	5	$\langle v_5^2 v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle - \langle v_2^2 v_4 \cos(4(\psi_2 - \psi_4)) \rangle \langle v_5^2 \rangle$
30	$c_{3,4,5}^{\{-4,10\}}\{2,1,2\}$	5	$\langle v_5^2 v_3^2 v_4 \cos(6\psi_3 + 4\psi_4 - 10\psi_5) \rangle$
31	$c_{3,4,5}^{\{4,5\}}\{3,1,1\}$	5	$\langle v_3^3 v_4 v_5 \cos(9\psi_3 - 4\psi_4 - 5\psi_5) \rangle$
32	$c_{2,3,4,5}^{\{-3,0,5\}}\{1,1,2,1\}$	5	$\langle v_4^2 v_2 v_5 v_3 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle - \langle v_4^2 \rangle \langle v_2 v_3 v_5 \cos(2\psi_2 + 3\psi_3 - 5\psi_5) \rangle$
33	$c_{2,3,4,5}^{\{3,-4,5\}}\{2,1,1,1\}$	5	$\langle v_2^2 v_3 v_4 v_5 \cos(4\psi_2 - 3\psi_3 + 4\psi_4 - 5\psi_5) \rangle$

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4	$c_5\{2\}$	2	$\langle v_5^2\rangle$
5	$c_{2,4}^{\{4\}}\{2,1\}$	3	$\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
6	$c_{2,3,5}^{\{-3,5\}}\{1,1,1\}$	3	$\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
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9	$c_4\{4\}$	4	$\langle v_4^4\rangle - 2\langle v_4^2\rangle^2$
10	$c_5\{4\}$	4	$\langle v_5^4\rangle - 2\langle v_5^2\rangle^2$
11	$c_{2,3}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_3^2\rangle - \langle v_2^2\rangle\langle v_3^2\rangle$
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15	$c_{3,5}^{\{0\}}\{2,2\}$	4	$\langle v_3^2 v_5^2\rangle - \langle v_3^2\rangle\langle v_5^2\rangle$
16	$c_{4,5}^{\{0\}}\{2,2\}$	4	$\langle v_4^2 v_5^2\rangle - \langle v_4^2\rangle\langle v_5^2\rangle$
17	$c_{2,3,4}^{\{6,-4\}}\{1,2,1\}$	4	$\langle v_3^2 v_2 v_4 \cos(2(\psi_2-3\psi_3+2\psi_4))\rangle$
18	$c_{3,4,5}^{\{8,-5\}}\{1,2,1\}$	4	$\langle v_4^2 v_3 v_5 \cos(3\psi_3-8\psi_4+5\psi_5)\rangle$
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22	$c_{2,4}^{\{4\}}\{4,1\}$	5	$\langle v_2^4 v_4 \cos(4(\psi_2-\psi_4))\rangle - 3\langle v_2^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
23	$c_{2,3,4}^{\{-6,8\}}\{1,2,2\}$	5	$\langle v_4^2 v_3^2 v_2 \cos(2(\psi_2+3\psi_3-4\psi_4))\rangle$
24	$c_{2,3,4}^{\{0,4\}}\{2,2,1\}$	5	$\langle v_3^2 v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle - \langle v_3^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
25	$c_{2,3,5}^{\{-3,5\}}\{1,1,3\}$	5	$\langle v_5^3 v_2 v_3 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_5^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
26	$c_{2,3,5}^{\{-3,5\}}\{1,3,1\}$	5	$\langle v_3^3 v_2 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_3^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
27	$c_{2,3,5}^{\{-3,5\}}\{3,1,1\}$	5	$\langle v_2^3 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_2^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
28	$c_{2,4,5}^{\{-8,10\}}\{1,2,2\}$	5	$\langle v_5^2 v_4^2 v_2 \cos(2(\psi_2+4\psi_4-5\psi_5))\rangle$
29	$c_{2,4,5}^{\{4,0\}}\{2,1,2\}$	5	$\langle v_5^2 v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle - \langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle\langle v_5^2\rangle$
30	$c_{3,4,5}^{\{-4,10\}}\{2,1,2\}$	5	$\langle v_5^2 v_3^2 v_4 \cos(6\psi_3+4\psi_4-10\psi_5)\rangle$
31	$c_{3,4,5}^{\{4,5\}}\{3,1,1\}$	5	$\langle v_3^3 v_4 v_5 \cos(9\psi_3-4\psi_4-5\psi_5)\rangle$
32	$c_{2,3,4,5}^{\{-3,0,5\}}\{1,1,2,1\}$	5	$\langle v_4^2 v_2 v_5 v_3 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - \langle v_4^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
33	$c_{2,3,4,5}^{\{3,-4,5\}}\{2,1,1,1\}$	5	$\langle v_2^2 v_3 v_4 v_5 \cos(4\psi_2-3\psi_3+4\psi_4-5\psi_5)\rangle$

- There are 33 distinct cumulants.
- Seven of them have been missed in previous theoretical and experimental studies.

All cumulants involving harmonics 2,3,4,5 and orders 2,3,4,5

	cumulant	order	cumulant expression
1	$c_2\{2\}$	2	$\langle v_2^2\rangle$
2	$c_3\{2\}$	2	$\langle v_3^2\rangle$
3	$c_4\{2\}$	2	$\langle v_4^2\rangle$
4	$c_5\{2\}$	2	$\langle v_5^2\rangle$
5	$c_{2,4}^{\{4\}}\{2,1\}$	3	$\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
6	$c_{2,3,5}^{\{-3,5\}}\{1,1,1\}$	3	$\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
7	$c_2\{4\}$	4	$\langle v_2^4\rangle-2\langle v_2^2\rangle^2$
8	$c_3\{4\}$	4	$\langle v_3^4\rangle-2\langle v_3^2\rangle^2$
9	$c_4\{4\}$	4	$\langle v_4^4\rangle-2\langle v_4^2\rangle^2$
10	$c_5\{4\}$	4	$\langle v_5^4\rangle-2\langle v_5^2\rangle^2$
11	$c_{2,3}^{\{0\}}\{2,2\}$	4	$\langle v_2^2v_3^2\rangle-\langle v_2^2\rangle\langle v_3^2\rangle$
12	$c_{2,4}^{\{0\}}\{2,2\}$	4	$\langle v_2^2v_4^2\rangle-\langle v_2^2\rangle\langle v_4^2\rangle$
13	$c_{2,5}^{\{0\}}\{2,2\}$	4	$\langle v_2^2v_5^2\rangle-\langle v_2^2\rangle\langle v_5^2\rangle$
14	$c_{3,4}^{\{0\}}\{2,2\}$	4	$\langle v_3^2v_4^2\rangle-\langle v_3^2\rangle\langle v_4^2\rangle$
15	$c_{3,5}^{\{0\}}\{2,2\}$	4	$\langle v_3^2v_5^2\rangle-\langle v_3^2\rangle\langle v_5^2\rangle$
16	$c_{4,5}^{\{0\}}\{2,2\}$	4	$\langle v_4^2v_5^2\rangle-\langle v_4^2\rangle\langle v_5^2\rangle$
17	$c_{2,3,4}^{\{6,-4\}}\{1,2,1\}$	4	$\langle v_3^2 v_2 v_4 \cos(2(\psi_2-3\psi_3+2\psi_4))\rangle$
18	$c_{3,4,5}^{\{8,-5\}}\{1,2,1\}$	4	$\langle v_4^2 v_3 v_5 \cos(3\psi_3-8\psi_4+5\psi_5)\rangle$
19	$c_{2,3,4,5}^{\{3,4,-5\}}\{1,1,1,1\}$	4	$\langle v_2v_3v_4v_5 \cos(2\psi_2-3\psi_3-4\psi_4+5\psi_5)\rangle$
20	$c_{2,3}^{\{6\}}\{3,2\}$	5	$\langle v_2^3v_3^2 \cos(6(\psi_2-\psi_3))\rangle$
21	$c_{2,4}^{\{4\}}\{2,3\}$	5	$\langle v_2^2v_4^3 \cos(4(\psi_2-\psi_4))\rangle-2\langle v_4^2\rangle\langle v_2^2v_4\cos(4(\psi_2-\psi_4))\rangle$
22	$c_{2,4}^{\{4\}}\{4,1\}$	5	$\langle v_2^4v_4 \cos(4(\psi_2-\psi_4))\rangle-3\langle v_2^2\rangle\langle v_2^2v_4\cos(4(\psi_2-\psi_4))\rangle$
23	$c_{2,3,4}^{\{-6,8\}}\{1,2,2\}$	5	$\langle v_4^2v_3^2v_2 \cos(2(\psi_2+3\psi_3-4\psi_4))\rangle$
24	$c_{2,3,4}^{\{0,4\}}\{2,2,1\}$	5	$\langle v_3^2v_2^2v_4 \cos(4(\psi_2-\psi_4))\rangle-\langle v_3^2\rangle\langle v_2^2v_4\cos(4(\psi_2-\psi_4))\rangle$
25	$c_{2,3,5}^{\{-3,5\}}\{1,1,3\}$	5	$\langle v_5^3v_2v_3 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle-2\langle v_5^2\rangle\langle v_2v_3v_5\cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
26	$c_{2,3,5}^{\{-3,5\}}\{1,3,1\}$	5	$\langle v_3^3v_2v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle-2\langle v_3^2\rangle\langle v_2v_3v_5\cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
27	$c_{2,3,5}^{\{-3,5\}}\{3,1,1\}$	5	$\langle v_2^3v_3v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle-2\langle v_2^2\rangle\langle v_2v_3v_5\cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
28	$c_{2,4,5}^{\{-8,10\}}\{1,2,2\}$	5	$\langle v_5^2v_4^2v_2 \cos(2(\psi_2+4\psi_4-5\psi_5))\rangle$
29	$c_{2,4,5}^{\{4,0\}}\{2,1,2\}$	5	$\langle v_5^2v_2^2v_4 \cos(4(\psi_2-\psi_4))\rangle-\langle v_2^2v_4\cos(4(\psi_2-\psi_4))\rangle\langle v_5^2\rangle$
30	$c_{3,4,5}^{\{-4,10\}}\{2,1,2\}$	5	$\langle v_5^2v_3^2v_4 \cos(6\psi_3+4\psi_4-10\psi_5)\rangle$
31	$c_{3,4,5}^{\{4,5\}}\{3,1,1\}$	5	$\langle v_3^3v_4v_5 \cos(9\psi_3-4\psi_4-5\psi_5)\rangle$
32	$c_{2,3,4,5}^{\{-3,0,5\}}\{1,1,2,1\}$	5	$\langle v_4^2v_2v_5v_3 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle-\langle v_4^2\rangle\langle v_2v_3v_5\cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
33	$c_{2,3,4,5}^{\{3,-4,5\}}\{2,1,1,1\}$	5	$\langle v_2^2v_3v_4v_5 \cos(4\psi_2-3\psi_3+4\psi_4-5\psi_5)\rangle$

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$$\text{normalized cumulant} = \frac{\text{cumulant}}{\sqrt{c_{n_1}^{m_1}\{2\}\cdots c_{n_k}^{m_k}\{2\}}}.$$

n_i : the involving harmonics in the cumulant.
m_i : the power of the flow amplitude v_{n_i} in the cumulant.

All cumulants involving harmonics 2,3,4,5 and orders 2,3,4,5

	cumulant	order	cumulant expression
1	$c_2\{2\}$	2	$\langle v_2^2\rangle$
2	$c_3\{2\}$	2	$\langle v_3^2\rangle$
3	$c_4\{2\}$	2	$\langle v_4^2\rangle$
4	$c_5\{2\}$	2	$\langle v_5^2\rangle$
5	$c_{2,4}^{\{4\}}\{2,1\}$	3	$\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
6	$c_{2,3,5}^{\{-3,5\}}\{1,1,1\}$	3	$\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
7	$c_2\{4\}$	4	$\langle v_2^4\rangle - 2\langle v_2^2\rangle^2$
8	$c_3\{4\}$	4	$\langle v_3^4\rangle - 2\langle v_3^2\rangle^2$
9	$c_4\{4\}$	4	$\langle v_4^4\rangle - 2\langle v_4^2\rangle^2$
10	$c_5\{4\}$	4	$\langle v_5^4\rangle - 2\langle v_5^2\rangle^2$
11	$c_{2,3}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_3^2\rangle - \langle v_2^2\rangle\langle v_3^2\rangle$
12	$c_{2,4}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_4^2\rangle - \langle v_2^2\rangle\langle v_4^2\rangle$
13	$c_{2,5}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_5^2\rangle - \langle v_2^2\rangle\langle v_5^2\rangle$
14	$c_{3,4}^{\{0\}}\{2,2\}$	4	$\langle v_3^2 v_4^2\rangle - \langle v_3^2\rangle\langle v_4^2\rangle$
15	$c_{3,5}^{\{0\}}\{2,2\}$	4	$\langle v_3^2 v_5^2\rangle - \langle v_3^2\rangle\langle v_5^2\rangle$
16	$c_{4,5}^{\{0\}}\{2,2\}$	4	$\langle v_4^2 v_5^2\rangle - \langle v_4^2\rangle\langle v_5^2\rangle$
17	$c_{2,3,4}^{\{6,-4\}}\{1,2,1\}$	4	$\langle v_3^2 v_2 v_4 \cos(2(\psi_2-3\psi_3+2\psi_4))\rangle$
18	$c_{3,4,5}^{\{8,-5\}}\{1,2,1\}$	4	$\langle v_4^2 v_3 v_5 \cos(3\psi_3-8\psi_4+5\psi_5)\rangle$
19	$c_{2,3,4,5}^{\{3,4,-5\}}\{1,1,1,1\}$	4	$\langle v_2 v_3 v_4 v_5 \cos(2\psi_2-3\psi_3-4\psi_4+5\psi_5)\rangle$
20	$c_{2,3}^{\{6\}}\{3,2\}$	5	$\langle v_2^3 v_3^2 \cos(6(\psi_2-\psi_3))\rangle$
21	$c_{2,4}^{\{4\}}\{2,3\}$	5	$\langle v_2^2 v_4^3 \cos(4(\psi_2-\psi_4))\rangle - 2\langle v_4^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
22	$c_{2,4}^{\{4\}}\{4,1\}$	5	$\langle v_2^4 v_4 \cos(4(\psi_2-\psi_4))\rangle - 3\langle v_2^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
23	$c_{2,3,4}^{\{-6,8\}}\{1,2,2\}$	5	$\langle v_4^2 v_3^2 v_2 \cos(2(\psi_2+3\psi_3-4\psi_4))\rangle$
24	$c_{2,3,4}^{\{0,4\}}\{2,2,1\}$	5	$\langle v_3^2 v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle - \langle v_3^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
25	$c_{2,3,5}^{\{-3,5\}}\{1,1,3\}$	5	$\langle v_5^3 v_2 v_3 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_5^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
26	$c_{2,3,5}^{\{-3,5\}}\{1,3,1\}$	5	$\langle v_3^3 v_2 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_3^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
27	$c_{2,3,5}^{\{-3,5\}}\{3,1,1\}$	5	$\langle v_2^3 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_2^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
28	$c_{2,4,5}^{\{-8,10\}}\{1,2,2\}$	5	$\langle v_5^2 v_4^2 v_2 \cos(2(\psi_2+4\psi_4-5\psi_5))\rangle$
29	$c_{2,4,5}^{\{4,0\}}\{2,1,2\}$	5	$\langle v_5^2 v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle - \langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle\langle v_5^2\rangle$
30	$c_{3,4,5}^{\{-4,10\}}\{2,1,2\}$	5	$\langle v_5^2 v_3^2 v_4 \cos(6\psi_3+4\psi_4-10\psi_5)\rangle$
31	$c_{3,4,5}^{\{4,5\}}\{3,1,1\}$	5	$\langle v_3^3 v_4 v_5 \cos(9\psi_3-4\psi_4-5\psi_5)\rangle$
32	$c_{2,3,4,5}^{\{-3,0,5\}}\{1,1,2,1\}$	5	$\langle v_4^2 v_2 v_5 v_3 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - \langle v_4^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
33	$c_{2,3,4,5}^{\{3,-4,5\}}\{2,1,1,1\}$	5	$\langle v_2^2 v_3 v_4 v_5 \cos(4\psi_2-3\psi_3+4\psi_4-5\psi_5)\rangle$

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$$\text{normalized cumulant} = \frac{\text{cumulant}}{\sqrt{c_{n_1}^{m_1}\{2\}\cdots c_{n_k}^{m_k}\{2\}}}.$$

n_i : the involving harmonics in the cumulant.
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- We have 29 distinct normalized cumulants.

All cumulants involving harmonics 2,3,4,5 and orders 2,3,4,5

	cumulant	order	cumulant expression
1	$c_2\{2\}$	2	$\langle v_2^2\rangle$
2	$c_3\{2\}$	2	$\langle v_3^2\rangle$
3	$c_4\{2\}$	2	$\langle v_4^2\rangle$
4	$c_5\{2\}$	2	$\langle v_5^2\rangle$
5	$c_{2,4}^{\{4\}}\{2,1\}$	3	$\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
6	$c_{2,3,5}^{\{-3,5\}}\{1,1,1\}$	3	$\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
7	$c_2\{4\}$	4	$\langle v_2^4\rangle - 2\langle v_2^2\rangle^2$
8	$c_3\{4\}$	4	$\langle v_3^4\rangle - 2\langle v_3^2\rangle^2$
9	$c_4\{4\}$	4	$\langle v_4^4\rangle - 2\langle v_4^2\rangle^2$
10	$c_5\{4\}$	4	$\langle v_5^4\rangle - 2\langle v_5^2\rangle^2$
11	$c_{2,3}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_3^2\rangle - \langle v_2^2\rangle\langle v_3^2\rangle$
12	$c_{2,4}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_4^2\rangle - \langle v_2^2\rangle\langle v_4^2\rangle$
13	$c_{2,5}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_5^2\rangle - \langle v_2^2\rangle\langle v_5^2\rangle$
14	$c_{3,4}^{\{0\}}\{2,2\}$	4	$\langle v_3^2 v_4^2\rangle - \langle v_3^2\rangle\langle v_4^2\rangle$
15	$c_{3,5}^{\{0\}}\{2,2\}$	4	$\langle v_3^2 v_5^2\rangle - \langle v_3^2\rangle\langle v_5^2\rangle$
16	$c_{4,5}^{\{0\}}\{2,2\}$	4	$\langle v_4^2 v_5^2\rangle - \langle v_4^2\rangle\langle v_5^2\rangle$
17	$c_{2,3,4}^{\{6,-4\}}\{1,2,1\}$	4	$\langle v_3^2 v_2 v_4 \cos(2(\psi_2-3\psi_3+2\psi_4))\rangle$
18	$c_{3,4,5}^{\{8,-5\}}\{1,2,1\}$	4	$\langle v_4^2 v_3 v_5 \cos(3\psi_3-8\psi_4+5\psi_5)\rangle$
19	$c_{2,3,4,5}^{\{3,4,-5\}}\{1,1,1,1\}$	4	$\langle v_2 v_3 v_4 v_5 \cos(2\psi_2-3\psi_3-4\psi_4+5\psi_5)\rangle$
20	$c_{2,3}^{\{6\}}\{3,2\}$	5	$\langle v_2^3 v_3^2 \cos(6(\psi_2-\psi_3))\rangle$
21	$c_{2,4}^{\{4\}}\{2,3\}$	5	$\langle v_2^2 v_4^3 \cos(4(\psi_2-\psi_4))\rangle - 2\langle v_4^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
22	$c_{2,4}^{\{4\}}\{4,1\}$	5	$\langle v_2^4 v_4 \cos(4(\psi_2-\psi_4))\rangle - 3\langle v_2^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
23	$c_{2,3,4}^{\{-6,8\}}\{1,2,2\}$	5	$\langle v_4^2 v_3^2 v_2 \cos(2(\psi_2+3\psi_3-4\psi_4))\rangle$
24	$c_{2,3,4}^{\{0,4\}}\{2,2,1\}$	5	$\langle v_3^2 v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle - \langle v_3^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
25	$c_{2,3,5}^{\{-3,5\}}\{1,1,3\}$	5	$\langle v_5^3 v_2 v_3 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_5^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
26	$c_{2,3,5}^{\{-3,5\}}\{1,3,1\}$	5	$\langle v_3^3 v_2 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_3^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
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28	$c_{2,4,5}^{\{-8,10\}}\{1,2,2\}$	5	$\langle v_5^2 v_4^2 v_2 \cos(2(\psi_2+4\psi_4-5\psi_5))\rangle$
29	$c_{2,4,5}^{\{4,0\}}\{2,1,2\}$	5	$\langle v_5^2 v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle - \langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle\langle v_5^2\rangle$
30	$c_{3,4,5}^{\{-4,10\}}\{2,1,2\}$	5	$\langle v_5^2 v_3^2 v_4 \cos(6\psi_3+4\psi_4-10\psi_5)\rangle$
31	$c_{3,4,5}^{\{4,5\}}\{3,1,1\}$	5	$\langle v_3^3 v_4 v_5 \cos(9\psi_3-4\psi_4-5\psi_5)\rangle$
32	$c_{2,3,4,5}^{\{-3,0,5\}}\{1,1,2,1\}$	5	$\langle v_4^2 v_2 v_5 v_3 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - \langle v_4^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
33	$c_{2,3,4,5}^{\{3,-4,5\}}\{2,1,1,1\}$	5	$\langle v_2^2 v_3 v_4 v_5 \cos(4\psi_2-3\psi_3+4\psi_4-5\psi_5)\rangle$

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- $\mathrm{T_RENTo}$ + free streaming + VISH(2+1) + UrQMD Maximum A Posteriori (MAP) tuning.
[Bernhard et al, Nature Phys., 2019]

All cumulants involving harmonics 2,3,4,5 and orders 2,3,4,5

	cumulant	order	cumulant expression
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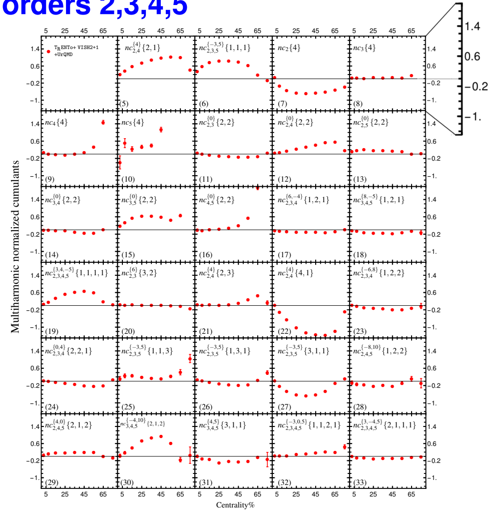
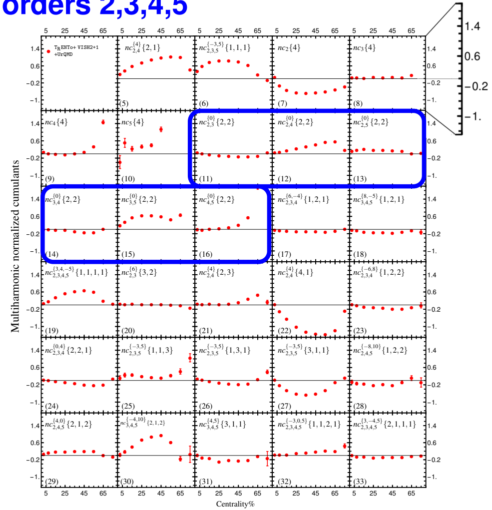
4	$c_5\{2\}$	2	$\langle v_5^2\rangle$
5	$c_{2,4}^{\{4\}}\{2,1\}$	3	$\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
6	$c_{2,3,5}^{\{-3,5\}}\{1,1,1\}$	3	$\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
7	$c_2\{4\}$	4	$\langle v_2^4\rangle - 2\langle v_2^2\rangle^2$
8	$c_3\{4\}$	4	$\langle v_3^4\rangle - 2\langle v_3^2\rangle^2$
9	$c_4\{4\}$	4	$\langle v_4^4\rangle - 2\langle v_4^2\rangle^2$
10	$c_5\{4\}$	4	$\langle v_5^4\rangle - 2\langle v_5^2\rangle^2$
11	$c_{2,3}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_3^2\rangle - \langle v_2^2\rangle\langle v_3^2\rangle$
12	$c_{2,4}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_4^2\rangle - \langle v_2^2\rangle\langle v_4^2\rangle$
13	$c_{2,5}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_5^2\rangle - \langle v_2^2\rangle\langle v_5^2\rangle$
14	$c_{3,4}^{\{0\}}\{2,2\}$	4	$\langle v_3^2 v_4^2\rangle - \langle v_3^2\rangle\langle v_4^2\rangle$
15	$c_{3,5}^{\{0\}}\{2,2\}$	4	$\langle v_3^2 v_5^2\rangle - \langle v_3^2\rangle\langle v_5^2\rangle$
16	$c_{4,5}^{\{0\}}\{2,2\}$	4	$\langle v_4^2 v_5^2\rangle - \langle v_4^2\rangle\langle v_5^2\rangle$
17	$c_{2,3,4}^{\{6,-4\}}\{1,2,1\}$	4	$\langle v_3^2 v_2 v_4 \cos(2(\psi_2-3\psi_3+2\psi_4))\rangle$
18	$c_{3,4,5}^{\{8,-5\}}\{1,2,1\}$	4	$\langle v_4^2 v_3 v_5 \cos(3\psi_3-8\psi_4+5\psi_5)\rangle$
19	$c_{2,3,4,5}^{\{3,4,-5\}}\{1,1,1,1\}$	4	$\langle v_2 v_3 v_4 v_5 \cos(2\psi_2-3\psi_3-4\psi_4+5\psi_5)\rangle$
20	$c_{2,3}^{\{6\}}\{3,2\}$	5	$\langle v_2^3 v_3^2 \cos(6(\psi_2-\psi_3))\rangle$
21	$c_{2,4}^{\{4\}}\{2,3\}$	5	$\langle v_2^2 v_4^3 \cos(4(\psi_2-\psi_4))\rangle - 2\langle v_4^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
22	$c_{2,4}^{\{4\}}\{4,1\}$	5	$\langle v_2^4 v_4 \cos(4(\psi_2-\psi_4))\rangle - 3\langle v_2^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
23	$c_{2,3,4}^{\{-6,8\}}\{1,2,2\}$	5	$\langle v_4^2 v_3^2 v_2 \cos(2(\psi_2+3\psi_3-4\psi_4))\rangle$
24	$c_{2,3,4}^{\{0,4\}}\{2,2,1\}$	5	$\langle v_3^2 v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle - \langle v_3^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
25	$c_{2,3,5}^{\{-3,5\}}\{1,1,3\}$	5	$\langle v_5^3 v_2 v_3 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_5^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
26	$c_{2,3,5}^{\{-3,5\}}\{1,3,1\}$	5	$\langle v_3^3 v_2 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_3^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
27	$c_{2,3,5}^{\{-3,5\}}\{3,1,1\}$	5	$\langle v_2^3 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_2^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
28	$c_{2,4,5}^{\{-8,10\}}\{1,2,2\}$	5	$\langle v_5^2 v_4^2 v_2 \cos(2(\psi_2+4\psi_4-5\psi_5))\rangle$
29	$c_{2,4,5}^{\{4,0\}}\{2,1,2\}$	5	$\langle v_5^2 v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle - \langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle\langle v_5^2\rangle$
30	$c_{3,4,5}^{\{-4,10\}}\{2,1,2\}$	5	$\langle v_5^2 v_3^2 v_4 \cos(6\psi_3+4\psi_4-10\psi_5)\rangle$
31	$c_{3,4,5}^{\{4,5\}}\{3,1,1\}$	5	$\langle v_3^3 v_4 v_5 \cos(9\psi_3-4\psi_4-5\psi_5)\rangle$
32	$c_{2,3,4,5}^{\{-3,0,5\}}\{1,1,2,1\}$	5	$\langle v_4^2 v_2 v_5 v_3 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - \langle v_4^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
33	$c_{2,3,4,5}^{\{3,-4,5\}}\{2,1,1,1\}$	5	$\langle v_2^2 v_3 v_4 v_5 \cos(4\psi_2-3\psi_3+4\psi_4-5\psi_5)\rangle$



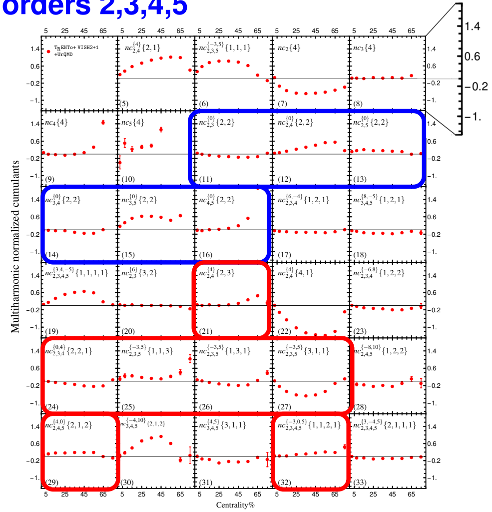
All cumulants involving harmonics 2,3,4,5 and orders 2,3,4,5

	cumulant	order	cumulant expression
1	$c_2\{2\}$	2	$\langle v_2^2\rangle$
2	$c_3\{2\}$	2	$\langle v_3^2\rangle$
3	$c_4\{2\}$	2	$\langle v_4^2\rangle$
4	$c_5\{2\}$	2	$\langle v_5^2\rangle$
5	$c_{2,4}^{\{4\}}\{2,1\}$	3	$\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
6	$c_{2,3,5}^{\{-3,5\}}\{1,1,1\}$	3	$\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
7	$c_2\{4\}$	4	$\langle v_2^4\rangle - 2\langle v_2^2\rangle^2$
8	$c_3\{4\}$	4	$\langle v_3^4\rangle - 2\langle v_3^2\rangle^2$
9	$c_4\{4\}$	4	$\langle v_4^4\rangle - 2\langle v_4^2\rangle^2$
10	$c_5\{4\}$	4	$\langle v_5^4\rangle - 2\langle v_5^2\rangle^2$
11	$c_{2,3}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_3^2\rangle - \langle v_2^2\rangle\langle v_3^2\rangle$
12	$c_{2,4}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_4^2\rangle - \langle v_2^2\rangle\langle v_4^2\rangle$
13	$c_{2,5}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_5^2\rangle - \langle v_2^2\rangle\langle v_5^2\rangle$
14	$c_{3,4}^{\{0\}}\{2,2\}$	4	$\langle v_3^2 v_4^2\rangle - \langle v_3^2\rangle\langle v_4^2\rangle$
15	$c_{3,5}^{\{0\}}\{2,2\}$	4	$\langle v_3^2 v_5^2\rangle - \langle v_3^2\rangle\langle v_5^2\rangle$
16	$c_{4,5}^{\{0\}}\{2,2\}$	4	$\langle v_4^2 v_5^2\rangle - \langle v_4^2\rangle\langle v_5^2\rangle$
17	$c_{2,3,4}^{\{6,-4\}}\{1,2,1\}$	4	$\langle v_3^2 v_2 v_4 \cos(2(\psi_2-3\psi_3+2\psi_4))\rangle$
18	$c_{3,4,5}^{\{8,-5\}}\{1,2,1\}$	4	$\langle v_4^2 v_3 v_5 \cos(3\psi_3-8\psi_4+5\psi_5)\rangle$
19	$c_{2,3,4,5}^{\{3,4,-5\}}\{1,1,1,1\}$	4	$\langle v_2 v_3 v_4 v_5 \cos(2\psi_2-3\psi_3-4\psi_4+5\psi_5)\rangle$
20	$c_{2,3}^{\{6\}}\{3,2\}$	5	$\langle v_2^3 v_3^2 \cos(6(\psi_2-\psi_3))\rangle$
21	$c_{2,4}^{\{4\}}\{2,3\}$	5	$\langle v_2^2 v_4^3 \cos(4(\psi_2-\psi_4))\rangle - 2\langle v_4^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
22	$c_{2,4}^{\{4\}}\{4,1\}$	5	$\langle v_2^4 v_4 \cos(4(\psi_2-\psi_4))\rangle - 3\langle v_2^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
23	$c_{2,3,4}^{\{-6,8\}}\{1,2,2\}$	5	$\langle v_4^2 v_3^2 v_2 \cos(2(\psi_2+3\psi_3-4\psi_4))\rangle$
24	$c_{2,3,4}^{\{0,4\}}\{2,2,1\}$	5	$\langle v_3^2 v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle - \langle v_3^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
25	$c_{2,3,5}^{\{-3,5\}}\{1,1,3\}$	5	$\langle v_5^3 v_2 v_3 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_5^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
26	$c_{2,3,5}^{\{-3,5\}}\{1,3,1\}$	5	$\langle v_3^3 v_2 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_3^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
27	$c_{2,3,5}^{\{-3,5\}}\{3,1,1\}$	5	$\langle v_2^3 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_2^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
28	$c_{2,4,5}^{\{-8,10\}}\{1,2,2\}$	5	$\langle v_5^2 v_4^2 v_2 \cos(2(\psi_2+4\psi_4-5\psi_5))\rangle$
29	$c_{2,4,5}^{\{4,0\}}\{2,1,2\}$	5	$\langle v_5^2 v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle - \langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle\langle v_5^2\rangle$
30	$c_{3,4,5}^{\{-4,10\}}\{2,1,2\}$	5	$\langle v_5^2 v_3^2 v_4 \cos(6\psi_3+4\psi_4-10\psi_5)\rangle$
31	$c_{3,4,5}^{\{4,5\}}\{3,1,1\}$	5	$\langle v_3^3 v_4 v_5 \cos(9\psi_3-4\psi_4-5\psi_5)\rangle$
32	$c_{2,3,4,5}^{\{-3,0,5\}}\{1,1,2,1\}$	5	$\langle v_4^2 v_2 v_5 v_3 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - \langle v_4^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
33	$c_{2,3,4,5}^{\{3,-4,5\}}\{2,1,1,1\}$	5	$\langle v_2^2 v_3 v_4 v_5 \cos(4\psi_2-3\psi_3+4\psi_4-5\psi_5)\rangle$

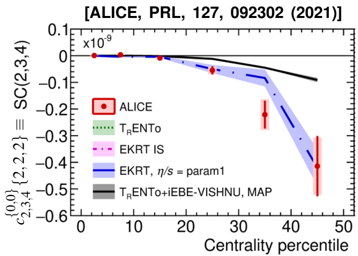


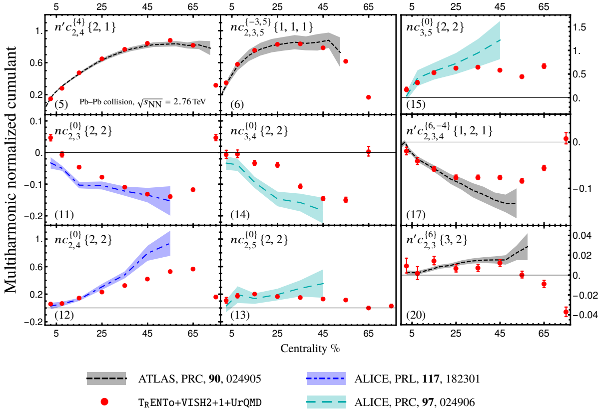
All cumulants involving harmonics 2,3,4,5 and orders 2,3,4,5

	cumulant	order	cumulant expression
1	$c_2\{2\}$	2	$\langle v_2^2 \rangle$
2	$c_3\{2\}$	2	$\langle v_3^2 \rangle$
3	$c_4\{2\}$	2	$\langle v_4^2 \rangle$
4	$c_5\{2\}$	2	$\langle v_5^2 \rangle$
5	$c_{2,4}^{\{4\}}\{2,1\}$	3	$\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
6	$c_{2,3,5}^{\{-3,5\}}\{1,1,1\}$	3	$\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
7	$c_2\{4\}$	4	$\langle v_2^4 \rangle - 2\langle v_2^2\rangle^2$
8	$c_3\{4\}$	4	$\langle v_3^4 \rangle - 2\langle v_3^2\rangle^2$
9	$c_4\{4\}$	4	$\langle v_4^4 \rangle - 2\langle v_4^2\rangle^2$
10	$c_5\{4\}$	4	$\langle v_5^4 \rangle - 2\langle v_5^2\rangle^2$
11	$c_{2,3}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_3^2\rangle - \langle v_2^2\rangle\langle v_3^2\rangle$
12	$c_{2,4}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_4^2\rangle - \langle v_2^2\rangle\langle v_4^2\rangle$
13	$c_{2,5}^{\{0\}}\{2,2\}$	4	$\langle v_2^2 v_5^2\rangle - \langle v_2^2\rangle\langle v_5^2\rangle$
14	$c_{3,4}^{\{0\}}\{2,2\}$	4	$\langle v_3^2 v_4^2\rangle - \langle v_3^2\rangle\langle v_4^2\rangle$
15	$c_{3,5}^{\{0\}}\{2,2\}$	4	$\langle v_3^2 v_5^2\rangle - \langle v_3^2\rangle\langle v_5^2\rangle$
16	$c_{4,5}^{\{0\}}\{2,2\}$	4	$\langle v_4^2 v_5^2\rangle - \langle v_4^2\rangle\langle v_5^2\rangle$
17	$c_{2,3,4}^{\{6,-4\}}\{1,2,1\}$	4	$\langle v_3^2 v_2 v_4 \cos(2(\psi_2-3\psi_3+2\psi_4))\rangle$
18	$c_{3,4,5}^{\{8,-5\}}\{1,2,1\}$	4	$\langle v_4^2 v_3 v_5 \cos(3\psi_3-8\psi_4+5\psi_5)\rangle$
19	$c_{2,3,4,5}^{\{3,4,-5\}}\{1,1,1,1\}$	4	$\langle v_2 v_3 v_4 v_5 \cos(2\psi_2-3\psi_3-4\psi_4+5\psi_5)\rangle$
20	$c_{2,3}^{\{6\}}\{3,2\}$	5	$\langle v_2^3 v_3^2 \cos(6(\psi_2-\psi_3))\rangle$
21	$c_{2,4}^{\{4\}}\{2,3\}$	5	$\langle v_2^2 v_4^3 \cos(4(\psi_2-\psi_4))\rangle - 2\langle v_4^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
22	$c_{2,4}^{\{4\}}\{4,1\}$	5	$\langle v_2^4 v_4 \cos(4(\psi_2-\psi_4))\rangle - 3\langle v_2^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
23	$c_{2,3,4}^{\{-6,8\}}\{1,2,2\}$	5	$\langle v_4^2 v_3^2 v_2 \cos(2(\psi_2+3\psi_3-4\psi_4))\rangle$
24	$c_{2,3,4}^{\{0,4\}}\{2,2,1\}$	5	$\langle v_3^2 v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle - \langle v_3^2\rangle\langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle$
25	$c_{2,3,5}^{\{-3,5\}}\{1,1,3\}$	5	$\langle v_5^3 v_2 v_3 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_5^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
26	$c_{2,3,5}^{\{-3,5\}}\{1,3,1\}$	5	$\langle v_3^3 v_2 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_3^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
27	$c_{2,3,5}^{\{-3,5\}}\{3,1,1\}$	5	$\langle v_2^3 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - 2\langle v_2^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
28	$c_{2,4,5}^{\{-8,10\}}\{1,2,2\}$	5	$\langle v_5^2 v_4^2 v_2 \cos(2(\psi_2+4\psi_4-5\psi_5))\rangle$
29	$c_{2,4,5}^{\{4,0\}}\{2,1,2\}$	5	$\langle v_5^2 v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle - \langle v_2^2 v_4 \cos(4(\psi_2-\psi_4))\rangle\langle v_5^2\rangle$
30	$c_{3,4,5}^{\{-4,10\}}\{2,1,2\}$	5	$\langle v_5^2 v_3^2 v_4 \cos(6\psi_3+4\psi_4-10\psi_5)\rangle$
31	$c_{3,4,5}^{\{4,5\}}\{3,1,1\}$	5	$\langle v_3^3 v_4 v_5 \cos(9\psi_3-4\psi_4-5\psi_5)\rangle$
32	$c_{2,3,4,5}^{\{-3,0,5\}}\{1,1,2,1\}$	5	$\langle v_4^2 v_2 v_5 v_3 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle - \langle v_4^2\rangle\langle v_2 v_3 v_5 \cos(2\psi_2+3\psi_3-5\psi_5)\rangle$
33	$c_{2,3,4,5}^{\{3,-4,5\}}\{2,1,1,1\}$	5	$\langle v_2^2 v_3 v_4 v_5 \cos(4\psi_2-3\psi_3+4\psi_4-5\psi_5)\rangle$



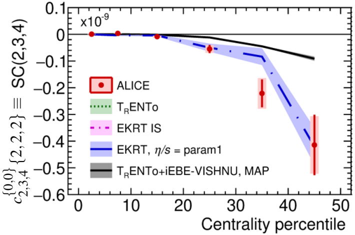
(Normalized) multiharmonic cumulants at the LHC

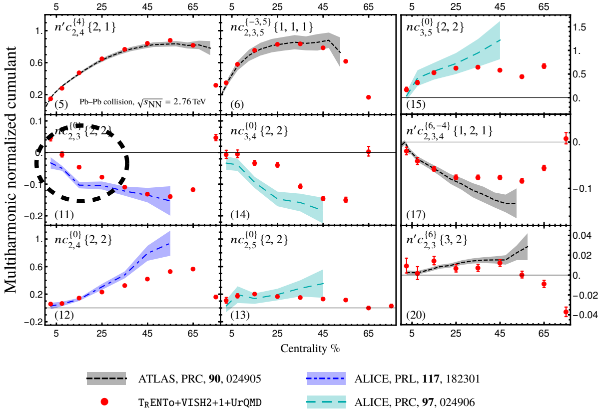




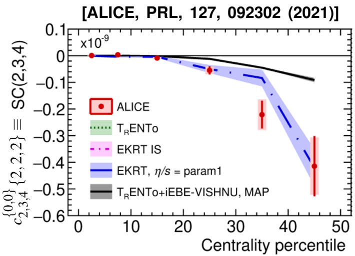
(Normalized) multiharmonic cumulants at the LHC

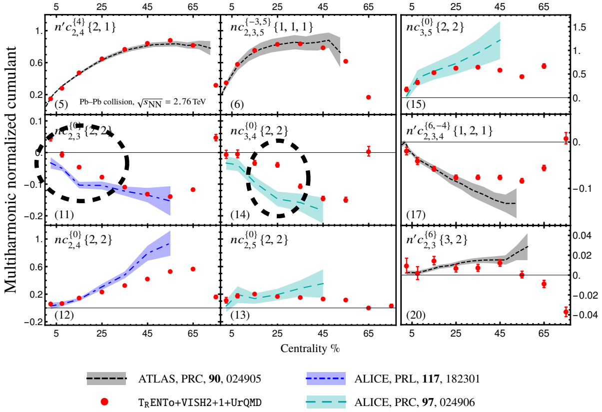
[ALICE, PRL, 127, 092302 (2021)]



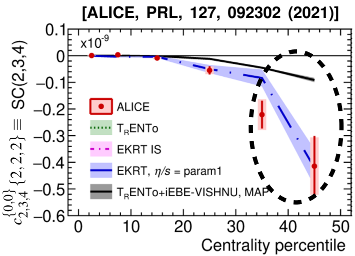


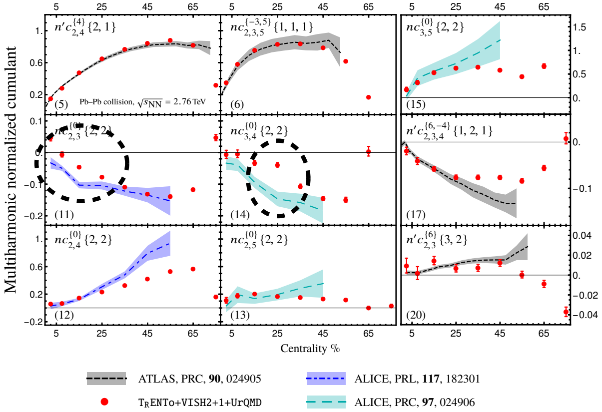
(Normalized) multiharmonic cumulants at the LHC



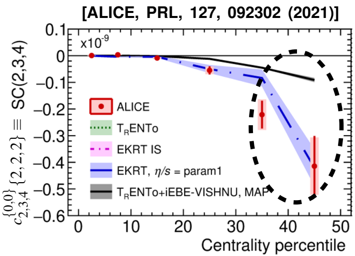


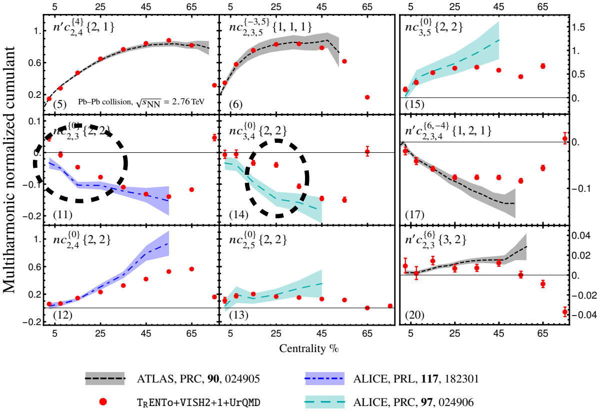
(Normalized) multiharmonic cumulants at the LHC





(Normalized) multiharmonic cumulants at the LHC

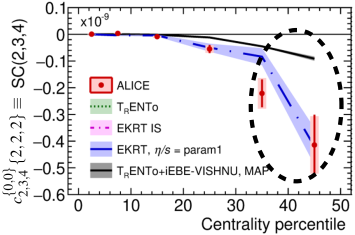


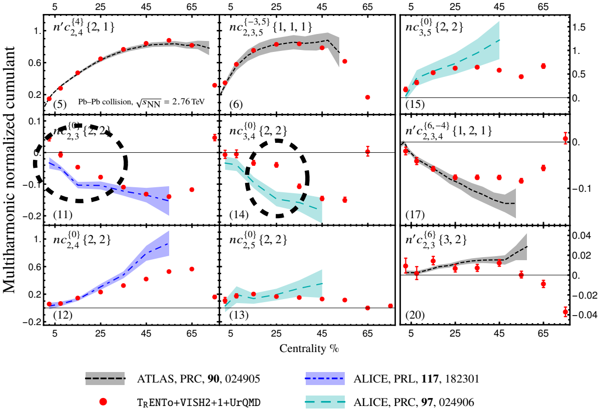


- The simulations are all predictions from a model tuned by Bayesian analysis.

(Normalized) multiharmonic cumulants at the LHC

[ALICE, PRL, 127, 092302 (2021)]





- The simulations are all predictions from a model tuned by Bayesian analysis.
- How would the inferred parameters change if we use the observables as inputs for a Bayesian analysis?

Linear and nonlinear hydrodynamic response from distribution

[SFT, Eur.Phys.J.C 81 (2021) 7,652]

Linear and nonlinear hydrodynamic response from distribution

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Hydrodynamic response coefficients:

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$$v_2 e^{i2\psi_2} \simeq w_2\, \varepsilon_2 e^{i2\phi_2},$$
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[SFT, Eur.Phys.J.C 81 (2021) 7,652]

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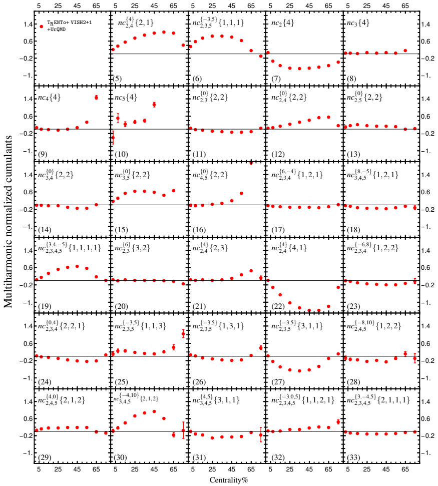

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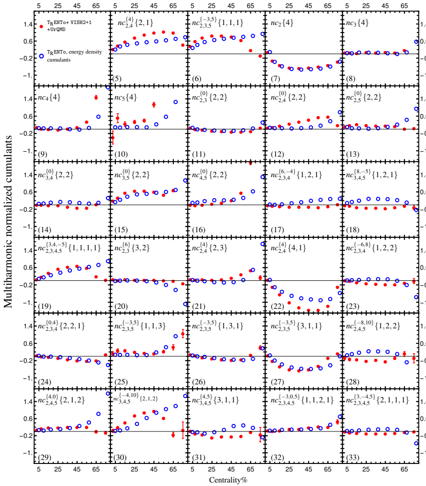

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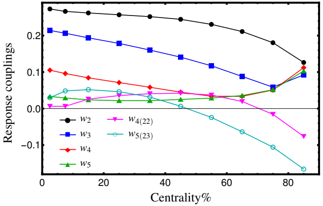

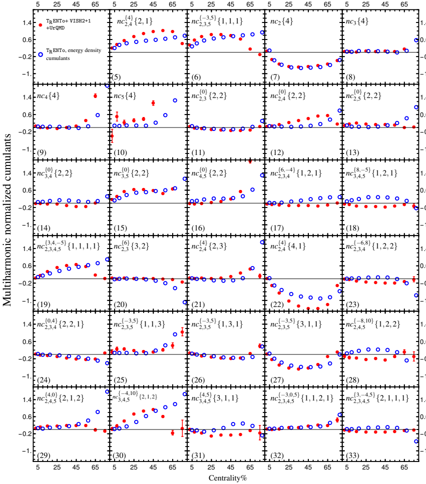

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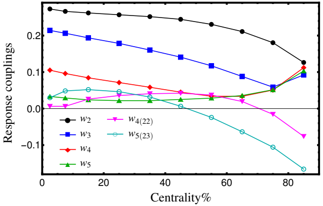

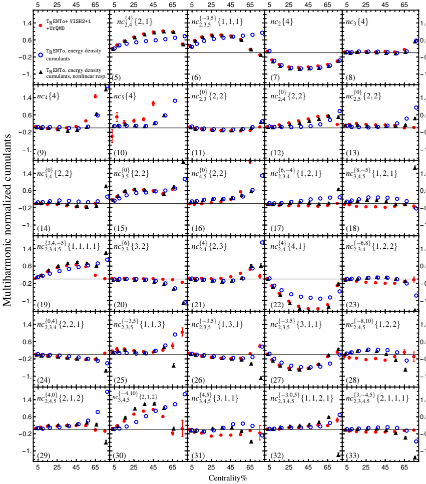

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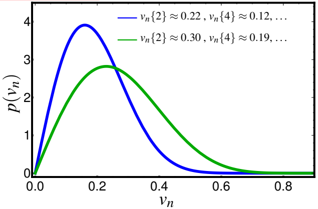

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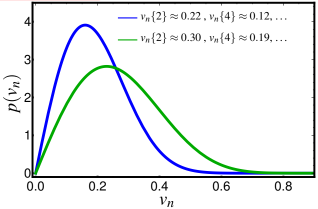

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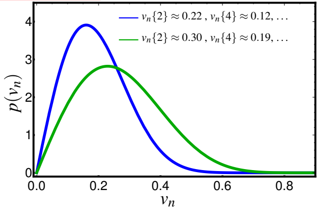

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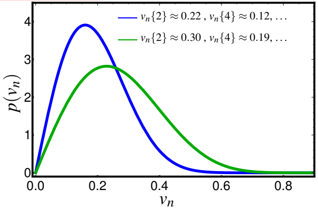

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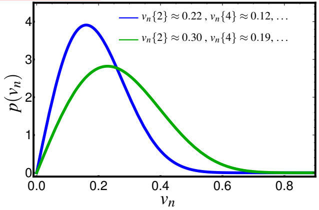

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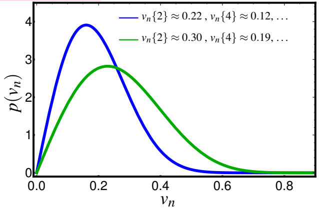

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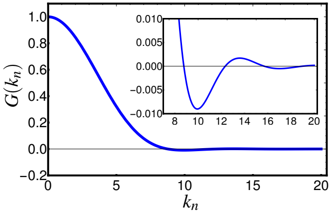

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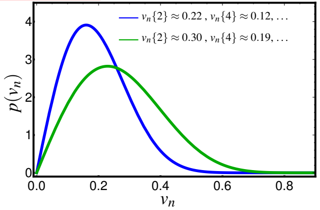

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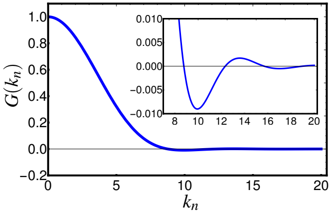

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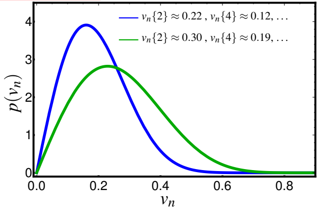

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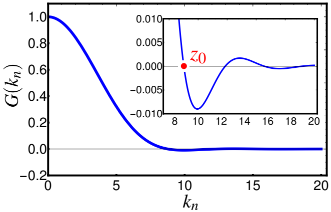

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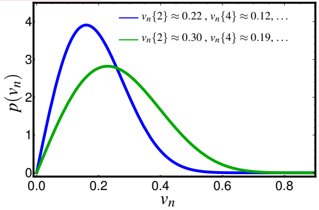

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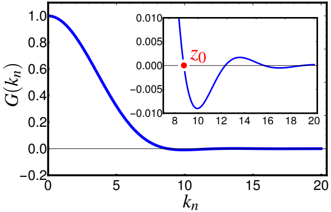

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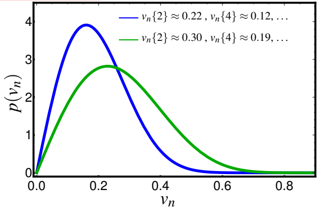

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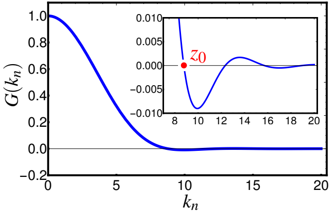

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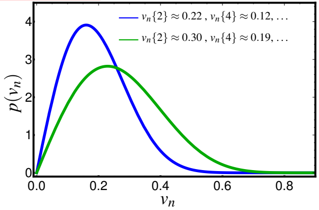

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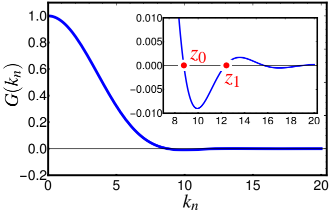

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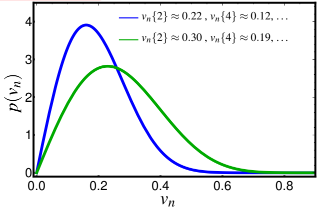

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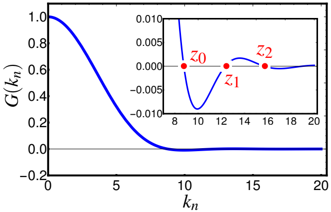

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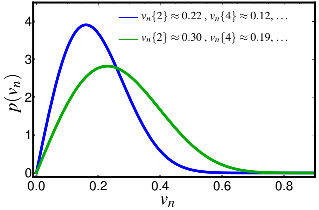

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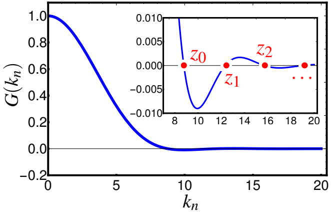

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Large order single harmonic cumulants [J. Jia, SFT, (in progress)]

Increasing cumulant order $(2m)$

Mid-central Pb–Pb collisions

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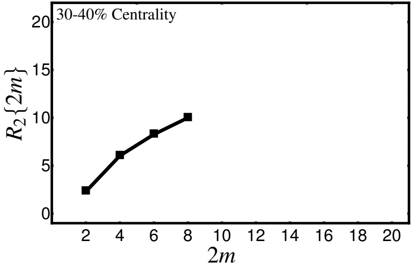

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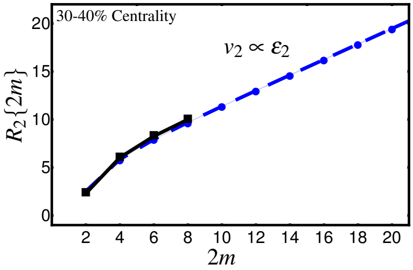

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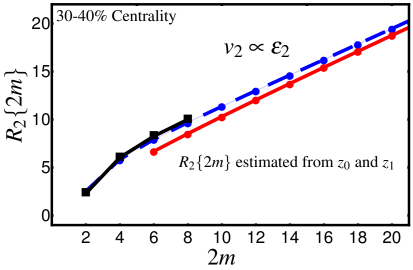

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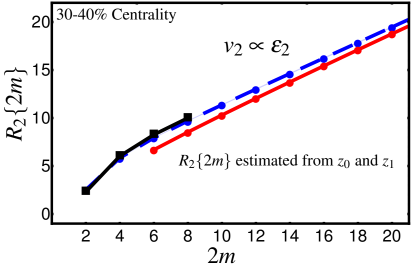

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Lee-Yang zeros expansion is a complementary study for the conventional multiparticle technique.

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Thank you for your attention!

Backup slides

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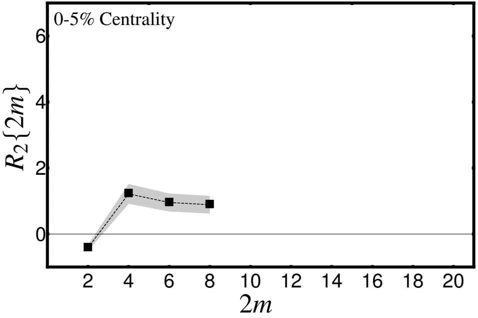

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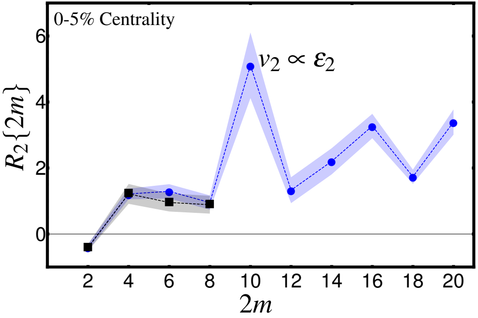

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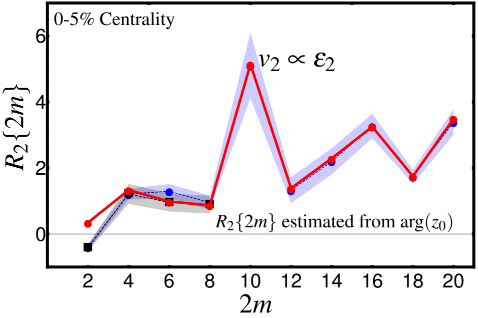

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- Here, we have two zeros $z_0 e^{i\alpha_0}$ and $z_0 e^{-i\alpha_0}$