Multidimensional spinors and Dirac equation in the theory of superalgebraic spinors

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Presentation plan

Part 1: superalgebraic spinors, gamma-operators, T and C symmetry breaking. Already published in:

V. Monakhov. The Dirac Sea, T and C Symmetry Breaking, and the Spinor Vacuum of the Universe, *Universe,* 2021, v. 7(5), 124.

V. Monakhov. A.Kozhedub. Algebra of Superalgebraic Spinors as Algebra of Second Quantization of Fermions. *Geom. Integrability & Quantization*, 2021, vol. 22, p.165-187.

Part 2: matrix of generalized Dirac conjugation in n-dimensional spacetime and two forms of the Dirac equation.

4-component superalgebraic spinors – Grassmann densities and their derivatives

$$\Psi = \int d^3 p \left(\psi^{\alpha}(p) \frac{\partial}{\partial \theta^{\alpha}(p)} + \psi^{\tau}(p)\theta^{\tau}(p) \right)$$

$$\theta^{a}(p)^{+} = \frac{\partial}{\partial \theta^{a}(p)};\ \{\frac{\partial}{\partial \theta^{k}(p)}, \theta^{l}(p')\} = \delta^{l}_{k}\, \delta(p - p')$$

$$\frac{\partial}{\partial \theta^{1}(p)} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \frac{\partial}{\partial \theta^{2}(p)} \cong \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \theta^{3}(p) \cong \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \theta^{4}(p) \cong \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\theta^{1}(p) \cong \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}, \quad \theta^{2}(p) \cong \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial \theta^{3}(p)} \cong \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}, \frac{\partial}{\partial \theta^{4}(p)} \cong \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$$

Theory of superalgebraic spinors, part 1

1. M. Pavšič. A theory of quantized fields based on orthogonal and symplectic Clifford algebras. *Advances in Applied Clifford Algebras*, 2012, v.22, p.449-481.
2. V. Monakhov. Superalgebraic representation of Dirac matrices. *Theoretical and Mathematical Physics*. 2016. v. 186. p.70–82.
3. V. Monakhov. Dirac matrices as elements of superalgebraic matrix algebra. *Bulletin of the Russian Academy of Sciences: Physics*, 2016, v.80, p. 985–988.
4. V. Monakhov. Superalgebraic structure of Lorentz transformations. *J. of Physics: Conf. Series*, 2018, v.1051, 012023.
5. V.Monakhov. Generalization of Dirac conjugation in the superalgebraic theory of spinors *Theoretical and Mathematical Physics*, 2019, v.200, p.1026-1042.
6. V. Monakhov. Vacuum and spacetime signature in the theory of superalgebraic spinors. *Universe*, 2019, v.5(7), 162.

Theory of superalgebraic spinors, part 2

7. V. Monakhov. Spacetime and inner space of spinors in the theory of superalgebraic spinors. *Journal of Physics: Conference Series,* 2020, v.1557(1), 12031.
8. V. Monakhov. Generation of Electroweak Interaction by Analogs of Dirac Gamma Matrices Constructed from Operators of the Creation and Annihilation of Spinors. *Bulletin of the Russian Academy of Sciences: Physics*, 2020, v. 84(10), pp. 1216–1220.
9. V. Monakhov. The Dirac Sea, T and C Symmetry Breaking, and the Spinor Vacuum of the Universe, *Universe,* 2021, v. 7(5), 124.
10. V. Monakhov. A.Kozhedub. Algebra of Superalgebraic Spinors as Algebra of Second Quantization of Fermions. *Geom. Integrability & Quantization*, 2021, vol. 22, p.165-187.

Gamma-operators (analogs of matrices): two additional compared to Dirac theory

$$\hat{A} = [A, \bullet] \Rightarrow \hat{A}\Psi = [A, \Psi] = A\Psi - \Psi A$$

$$\hat{\gamma}^0 = \int \mathrm{d}^3 p \, [\frac{\partial}{\partial\theta^1(p)}\theta^1(p) + \frac{\partial}{\partial\theta^2(p)}\theta^2(p) + \frac{\partial}{\partial\theta^3(p)}\theta^3(p) + \frac{\partial}{\partial\theta^4(p)}\theta^4(p), \bullet]$$

$$\hat{\gamma}^1 = \int \mathrm{d}^3 p \, [\frac{\partial}{\partial\theta^1(p)}\frac{\partial}{\partial\theta^4(p)} - \theta^4(p)\theta^1(p) + \frac{\partial}{\partial\theta^2(p)}\frac{\partial}{\partial\theta^3(p)} - \theta^3(p)\theta^2(p), \bullet]$$

$$\hat{\gamma}^2 = i\int \mathrm{d}^3 p \, [-\frac{\partial}{\partial\theta^1(p)}\frac{\partial}{\partial\theta^4(p)} - \theta^4(p)\theta^1(p) + \frac{\partial}{\partial\theta^2(p)}\frac{\partial}{\partial\theta^3(p)} + \theta^3(p)\theta^2(p), \bullet]$$

$$\hat{\gamma}^3 = \int \mathrm{d}^3 p \, [\frac{\partial}{\partial\theta^1(p)}\frac{\partial}{\partial\theta^3(p)} - \theta^3(p)\theta^1(p) - \frac{\partial}{\partial\theta^2(p)}\frac{\partial}{\partial\theta^4(p)} + \theta^4(p)\theta^2(p), \bullet]$$

$$\hat{\gamma}^4 = i\hat{\gamma}^5 = i\int \mathrm{d}^3 p \, [\frac{\partial}{\partial\theta^1(p)}\frac{\partial}{\partial\theta^3(p)} + \theta^3(p)\theta^1(p) + \frac{\partial}{\partial\theta^2(p)}\frac{\partial}{\partial\theta^4(p)} + \theta^4(p)\theta^2(p), \bullet]$$

$$\hat{\gamma}^6 = i\int \mathrm{d}^3 p \, [\frac{\partial}{\partial\theta^1(p)}\frac{\partial}{\partial\theta^2(p)} + \theta^2(p)\theta^1(p) - \frac{\partial}{\partial\theta^3(p)}\frac{\partial}{\partial\theta^4(p)} - \theta^4(p)\theta^3(p), \bullet]$$

$$\hat{\gamma}^7 = \int \mathrm{d}^3 p \, [\frac{\partial}{\partial\theta^1(p)}\frac{\partial}{\partial\theta^2(p)} - \theta^2(p)\theta^1(p) + \frac{\partial}{\partial\theta^3(p)}\frac{\partial}{\partial\theta^4(p)} - \theta^4(p)\theta^3(p), \bullet]$$

Operators of annihilation and creation of spinor. Generalized Dirac conjugation

$$b_\alpha(p_i) = \exp(\hat{\gamma}^{0k}\varphi_k)\frac{\partial}{\partial\theta^\alpha(0)}|_{p=0\to p=p_i}$$

$$\overline{b_\alpha}(p_i) = \exp(\hat{\gamma}^{0k}\varphi_k)\theta^l(0)|_{p=0\to p=p_i}$$

$$\hat{\gamma}^{0k}=[\hat{\gamma}^0,\hat{\gamma}^k]/2$$

$$\overline{\Psi} = (M\Psi)^+; \text{Signature} = (+---) \Rightarrow M = \hat{\gamma}^0$$

$$\overline{b_\alpha}(p) = (\hat{\gamma}^0 b_\alpha(p))^+$$

$$\overline{\Psi} = (\hat{\gamma}^0\Psi)^+ = (\bullet)^+\hat{\gamma}^0\Psi$$

Discretization of momentum space. Spinor vacuum

$$\{\frac{\partial}{\partial\theta^k(p_i)},\theta^l(p_j)\} = \delta^l_k \frac{1}{\Delta^3 p_i}\delta^i_j; \quad \delta(p_i - p_j) = \frac{1}{\Delta^3 p_i}\delta^i_j$$

$$\{\frac{\partial}{\partial\theta^k(p_i)},\frac{\partial}{\partial\theta^l(p_j)}\} = \{\theta^k(p_i),\theta^l(p_j)\} = 0$$

$$\Psi_{\mathrm{V}} = \prod_i \Psi_{\mathrm{V}}(p_i)$$

$$\Psi_{\mathrm{V}}(0) = (\Delta^3 p|_{p=0})^4 \frac{\partial}{\partial\theta^1(0)}\theta^1(0)\frac{\partial}{\partial\theta^2(0)}\theta^2(0)\frac{\partial}{\partial\theta^3(0)}\theta^3(0)\frac{\partial}{\partial\theta^4(0)}\theta^4(0)$$

$$\Psi_{\mathrm{V}}(p_i) = (\Delta^3 p_i)^4\, b_1(p_i)\bar{b}_1(p_i) b_2(p_i)\bar{b}_2(p_i) b_3(p_i)\bar{b}_3(p_i) b_4(p_i)\bar{b}_4(p_i)$$

Alternative spinor vacuum

$\hat{\gamma}^1, \hat{\gamma}^2, \hat{\gamma}^3, \hat{\gamma}^5, \hat{\gamma}^6, \hat{\gamma}^7$ – change Ψ_{V} to Ψ_{alt}

$\hat{\gamma}^0$ – keeps Ψ_{V}

$$\Psi_{\mathrm{altV}}(0) = (\Delta^3 p\,|_{p=0})^4\, \theta^1(0)\frac{\partial}{\partial\theta^1(0)}\theta^2(0)\frac{\partial}{\partial\theta^2(0)}\theta^3(0)\frac{\partial}{\partial\theta^3(0)}\theta^4(0)\frac{\partial}{\partial\theta^4(0)}$$

$$\Psi_{\mathrm{altV}}(p_i) = (\Delta^3 p_i)^4\, \bar{b}_1(p_i) b_1(p_i) \bar{b}_2(p_i) b_2(p_i) \bar{b}_3(p_i) b_3(p_i) \bar{b}_4(p_i) b_4(p_i)$$

$$\Psi_{\mathrm{altV}} = \prod_i \Psi_{\mathrm{altV}}(p_i)$$

$\bar{b}_k(p_i)$ – annihilation operator

$b_k(p_i)$ – creation operator

R-operators: automorphisms and antiautomorphisms

$$d\hat{G} = [dG, \bullet]$$

$$(1 + d\hat{G})\Psi_1\Psi_2 \ldots \Psi_k = 1 + [dG, \Psi_1]\Psi_2 \ldots \Psi_k + \Psi_1[dG, \Psi_2] \ldots \Psi_k + \ldots =$$

$$= (e^{d\hat{G}}\Psi_1)(e^{d\hat{G}}\Psi_2) \ldots (e^{d\hat{G}}\Psi_k)$$

$$e^{\hat{G}}\Psi_1\Psi_2 \ldots \Psi_k = (e^{\hat{G}}\Psi_1)(e^{\hat{G}}\Psi_2) \ldots (e^{\hat{G}}\Psi_k)$$

$R_{\hat{G}} = e^{\hat{G}}$ – it is R - operator

Other R - operators:

Complex conjugation $(\bullet)^*$, transposition $(\bullet)^T$,

Hermitian conjugation $(\bullet)^+ = (\bullet)^T(\bullet)^*$

Operators T_1 and T of time inversion

$$T_1 = R_{-x^0} R_{\hat{\gamma}^1 \hat{\gamma}^3} (\bullet)^*, \qquad \Psi_V \to \Psi_V$$

"Rewinding the film", annihilation operator must become creation one, and vice versa

$$R = R_{\hat{\gamma}^{05}} R_{\hat{\gamma}^{26}} (\bullet)^T - \text{reverse } R\Psi_1 \Psi_2 \ldots \Psi_k = \Psi_k \ldots \Psi_2 \Psi_1$$

$$R\Psi_{\mathrm{V}} = \Psi_{\mathrm{alt}}, R\Psi = \Psi, R\overline{\Psi} = \overline{\Psi}$$

$$T = RT_1 = R_{-x^0} R_{\hat{\gamma}^7} (\bullet)^+, \quad \Psi_V \to \Psi_{alt}$$

Charge conjugation C

The CPT operator must be antiunitary. Operator P is unitary, T is antiunitary and converts the vacuum into an alternative one. Consequently, the charge conjugation operator C must be unitary and convert the vacuum into an alternative one.

$$C_1 = R_{-q} R_{i\hat{\gamma}^{56}}, \qquad \Psi_V \to \Psi_V$$

$$C = RC_1 = R_{-q} R_{-i\hat{\gamma}^{02}} (\bullet)^T, \qquad \Psi_V \to \Psi_{alt}$$

Invariance of the mass term of the spinor Lagrangian

$$L_m = m\overline{\Psi}\Psi = m\,(M\Psi,\Psi)$$

Invariant form $(M\Phi,\Psi)$.

Infinitesimal Lorentz transformation $\Psi' = S\Psi$,

$S = \exp(\gamma^{\mu\nu}\mathrm{d}\omega_{\mu\nu}) = 1 + \gamma^{\mu\nu}\mathrm{d}\omega_{\mu\nu}$,

Invariance of L_m =>

$$M = k_+\gamma_+^1\gamma_+^2\ldots\gamma_+^p + k_-\gamma_-^1\gamma_-^2\ldots\gamma_-^q = M_+ + M_-$$

– a similar result was obtained by us in the theory of superalgebraic spinors.

Reality of the mass term of the spinor Lagrangian

Reality of the mass term =>

$$\Psi^{+} M \Psi = \Psi^{+} M^{+} \Psi ;$$

M must be diagonal => $M^{+} = M$;

The same norm for $\overline{\Psi}$ as for Ψ =>

$$M^{+} M = 1 \Rightarrow (M)^{2} = 1,$$

$$\left[\begin{array}{l} M = M_{+} = \pm i^{[p/2]} \gamma_{+}^{1} \gamma_{+}^{2} \ldots \gamma_{+}^{p} . \\ M = M_{-} = \pm i^{q-[q/2]} \gamma_{-}^{1} \gamma_{-}^{2} \ldots \gamma_{-}^{q} . \end{array}\right.$$

Decomposition of the spinor transformation in the Clifford algebra

$$\Psi' = \mathrm{e}^{1+dG}\Psi = (1+dG)\Psi;$$

$$dG = dk_0 + \gamma^{a_1} dk_{a_1} + \gamma^{a_1 a_2} dk_{a_1 a_2} + \ldots + \gamma^{a_1 \ldots a_n} dk_{a_1 \ldots a_n};$$

$$dk_{a_1 \ldots a_l} \sim F^{\mu}_{a_1 \ldots a_l} dx_{\mu} =>$$

$$m\gamma^{\mu} dx_{\mu} \Psi = i\partial^{\mu} dx_{\mu} \Psi| = \hat{p}^{\mu} dx_{\mu} \Psi = \hat{p}_{\mu} dx^{\mu} \Psi =>$$

$$\Psi' = (1 - i(\hat{p}_{\mu} dx^{\mu} + i\gamma^{\lambda\nu} A_{\lambda\nu\mu} dx^{\mu} + \ldots))\Psi,$$

$$\nabla_{\mu} = \partial_{\mu} + \gamma^{\lambda\nu} A_{\lambda\nu\mu} + \ldots, \text{ where } \Gamma_{\mu} = \gamma^{\lambda\nu} A_{\lambda\nu\mu} \text{ is spin bundle.}$$

Multidimension Dirac equations. Time and space axes

$$\gamma^{\mu}, \text{signature}\,(\text{p}, \text{q})$$

$$\gamma^{\mu} i \nabla_{\mu} \Psi = m\, \Psi \qquad (1)$$

or $\gamma'^{\mu} = i\gamma^{\mu}, \text{signature}\,(\text{p}, \text{q})$

$$\gamma'^{\mu} \nabla_{\mu} \Psi = m\, \Psi. \qquad (2)$$

Which axes correspond to time and which to space?

Decomposition in momenta determines which axes correspond to time and space

If $n=p+q$ is even => M is either M_+ or M_-.

If $n=p+q$ is odd => $M = M_+ = M_-$.

Existence of the momentum decomposition determines which axes correspond to time and which to space.

- odd p and M_+ is used, or even q and M_- is used => p time axes with a positive signature and q spatial axes with a negative signature. Dirac equation (1).
- even p and M_+ is used, or odd q and M_- is used => q time axes with a negative signature and p spatial axes with a positive signature. Dirac equation (2).

Conclusions

- The vacuum of the Universe is not invariant under the action of the operators T and C, and is invariant under the action of the operators P, TC and CPT.
- The properties of the mass term of the spinor Lagrangian imply the properties of the generalized Dirac conjugation matrix: $\boldsymbol{M}^{+} = \boldsymbol{M}$; $\boldsymbol{M}^{2} = \mathbf{1}$.
- If $n=p+q$ is even => $\boldsymbol{M}$ is either $\boldsymbol{M}_{+}$ or $\boldsymbol{M}_{-}$.
- If $n=p+q$ is odd => $\boldsymbol{M} = \boldsymbol{M}_{+} = \boldsymbol{M}_{-}$.
- Existence of the momentum decomposition determines which axes correspond to time and which to space.

Thank you for attention!