Effective models of hadrons in Quantum Field Theory on the Light Front.

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Outline of the talk

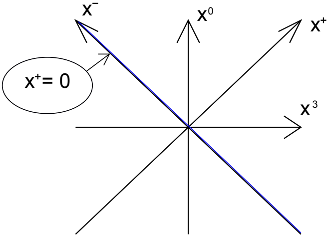
- Effective Light Front (LF) QCD Hamiltonian and spectral equation for meson mass squared in quark-antiquark approximation.
- Exact solution of spectral equation in the large quark mass limit.
- Modified form of previous effective LF Hamiltonian and the attempt to restore rotational symmetry.
- The model of effective Hamiltonian on the LF which can give the spectral equation coinciding with the equation obtained in AdS/QCD approach.
- The variant of effective LF Hamiltonian with the explicit rotational symmetry.

Relativistic physics of elementary particles is related to QFT. In QCD hadrons are to be described as bound states of quark and gluon fields. However this is difficult due to strong coupling. Nonperturbative methods like calculations on the space-time lattice are required. We use alternative approach based on the canonical formulation of field theory on the light front (LF) proposed by Dirac.

LF formulation of field theory allows to simplify the description of quantum vacuum state, identifying it with the state having zero LF longitudinal momentum

$$p_- = \frac{p_0 - p_3}{\sqrt{2}} \geq 0$$

(LF coordinates: $x^\pm = (x^0 \pm x^3)/\sqrt{2}$, $x^\perp = (x^1, x^2)$, x^+ plays the role of time x^0 and fields are quantized on the LF: $x^+ = 0$).



The component

$$P_+ = \frac{P_0 + P_3}{\sqrt{2}}$$

plays the role of Hamiltonian and depends on interaction, while P_- is kinematical quantity, independing on the coupling.
Moreover, on the LF not only free fields, but also interacting ones, can be represented in terms of creation and anihilation operators in LF Fock space:

$$\varphi(x) = \int\limits_0^\infty \frac{dp_- dp_\perp}{\sqrt{2p_-}} \left(a(p_-, x^\perp; x^+) e^{-ip_- x^-} + h.c. \right),$$

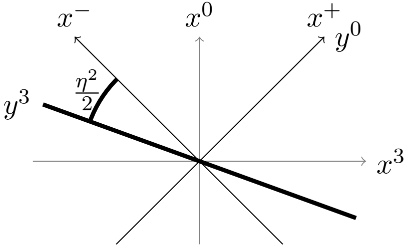
$$P_- = \int_0^\infty dp_- \int dp_\perp p_- a^+(p_-, x^\perp; x^+) a(p_-, x^\perp; x^+),$$

$$a(p_-, x^\perp; x^+)|0\rangle = 0.$$

Therefore LF formalism could be used as an alternative to other nonperturbative methods.

The difficulties of LF formulation are related to the singularity at $p_- \to 0$, so that zero ($p_- = 0$) mode of field is not defined. However this mode is responsible for vacuum effects (e.g. condensate description). It is not possible to ignore this mode in nonperturbative domain of QCD. The regularization $|x^-| \leq L$ and introduction of p. b. c. makes $p_- = \pi n/L$, $n \in \mathbb{Z}$, so that zero mode, $n = 0$, is present, but in canonical formalism it has no independent dynamics. It should be expressed through others, nonzero, modes via canonical constraints (note, that there the momentum $\Pi_{(0)}$ canonically conjugated to zero mode field $A_{(0)}$ is zero: $\Pi_{(0)} = 0$). The constraints are so complicated that they can not be practically solved for zero mode. To overcome this difficulty with zero mode we start to investigate how it arises when one goes from usual formulation of field theory on space-like hyperplanes to the LF.

With this aim we introduce approximating LF coordinates, $y^0 = x^+ + (\eta^2/2)x^-$, $y^3 = x^-$, $y^\perp = x^\perp$ with small parameter $\eta > 0$.



These coordinates are closely related to Lorentz coordinates in the Lorentz frame fastly moving w. r. to x^μ-frame: $x'^{\pm} = (\eta/\sqrt{2})^{\mp 1} x^{\pm}$, $x'^{\perp} = x^{\perp}$.

It was noticed that in the limit $\eta \to 0$ for a fixed L, the canonical formalism on the LF is restored. In this case, the canonical momentum $\Pi_{\perp(0)}$ conjugated to the zero mode $A_{\perp(0)}$ of the gluon field disappears, and the zero mode $A_{\perp(0)}$ again becomes a dependent variable (on the LF this is the result of solving the canonical constraints). However, if we consider the other limit, $L \to \infty$, $\eta \to 0$ at fixed $L_0 = \eta L$, for zero mode part of the Hamiltonian, then it is possible to preserve canonical momentum $\Pi_{\perp(0)}$ and obtain non-zero contribution from it to the mass squared and introduce the effective LF Hamiltonian:

$$P_+^{eff} = \frac{1}{2P_-}\left(\frac{1}{4L_0}\int d^2x^\perp \pi_k^a \pi_k^a\right)^2 + P_+^{can}.$$

where $\pi_k^a = 2L\Pi_{\perp(0)}$ and P_+^{can} is the canonical momentum P_+ on the LF.

The canonical QCD Hamiltonian on the LF:

$$H_{LF} = \int dx^- \int d^2x^\perp \Bigg[\frac{1}{2} \left(\partial_-^{-1} \left((D_k \Pi_k)^a + g\sqrt{2}\, \psi_+^+ \frac{\lambda^a}{2} \psi_+ \right) \right)^2 +$$

$$+ \frac{1}{2} (F_{12}^a)^2 + \frac{i}{\sqrt{2}} \psi_+^+ (D_\perp + M) \partial_-^{-1} (D_\perp - M) \psi_+ \Bigg].$$

where $\lambda^a = (\lambda^a)^+$, $a = 1 \ldots N^2 - 1$ Gell-Mann-like matrices,

$$A_\mu = \frac{\lambda^a}{2} A_\mu^a, \quad Tr(\lambda^a \lambda^b) = 2\delta^{ab}, \quad \left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2} \right] = i f^{abc} \frac{\lambda^c}{2},$$

$$\sum_a \left(\frac{\lambda^a}{2} \frac{\lambda^a}{2} \right) = \frac{1}{2} \left(N - \frac{1}{N} \right), \quad F_{\mu\nu} = \frac{\lambda^a}{2} F_{\mu\nu}^a.$$

Let us consider periodic boundary condition in x^- for $A_k(x)$ and antiperiodic boundary conditions for $\psi_+(x)$

$$A_k(x) \equiv A_{k(0)}(x^\perp) + \frac{1}{\sqrt{2L}} \sum_{n>0} \left(\frac{a_{nk}(x^\perp)}{\sqrt{2p_n}} e^{-ip_n x^-} + \text{h.c.} \right), \; p_n = \pi n / L,$$

$$\psi_+(x) = \frac{2^{-1/4}}{\sqrt{2L}} \sum_{m>0} (b_m(x^\perp) e^{-ip_m x^-} + d_m^+(x^\perp) e^{ip_m x^-}), \; m \in \mathbb{Z} + 1/2$$

where b_n^+ and d_n^+ correspond to quarks and antiquarks creation operators respectively.
So, the momentum operator is:
$P_- = \int d^2x^\perp \sum_{n>0} p_n \big(a_{nk}^+(x^\perp) a_{nk}(x^\perp) + b_n^+(x^\perp) b_n(x^\perp) + d_n^+(x^\perp) d_n(x^\perp)\big) \geqslant 0$

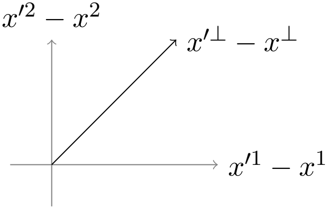
This model can be applied to the description of "constituent" quarks and gluons in hadrons. Zero mode of gluon field is used for the construction of the state which is taken to be invariant w. r. to the gauge symmetry transformations remaining after fixing the gauge $A^a_- = 0$.
We use gluon zero mode for the "string" connecting "constituent" particles separated in transverse coordinates. We take this "string" in the form of "path ordered" exponent

$$U_{x^\perp, x'^\perp} = Pexp\left(-ig\int_{x^\perp}^{x'^\perp}\sum_{k=1,2} dz^k A_{k(0)}(z^\perp)\right),$$

$A_{k(0)}(z^\perp) = A^a_{k(0)}(z^\perp)(\lambda^a/2)$, where $\lambda^a/2$ are Gell-Mann like matrices for SU(N), g is coupling constant, $U_{x^\perp, x'^\perp} = U^{-1}_{x'^\perp, x^\perp}$. We choose the path as the straight line in transverse plane, from $x^\perp$ to $x'^\perp$ and introduce ultraviolet regularisation in a form of lattice along the path with parameter a.

Let us consider the action of new term in the P_+^{eff} on the state containing only quark and antiquark connected by such a string. Let us describe the path of this string between $x^\perp$ and $x'^\perp$ as follows: $x'^1 - x^1 = \rho\cos\varphi, \quad x'^2 - x^2 = \rho\sin\varphi,$ then

$$U_{x^\perp, x'^\perp} = \lim_{a\to 0} \prod_{i=1}^{i=\rho/a} \exp\{-iga\left(A_{1(0)}(z_i)\cos\varphi + A_{2(0)}(z_i)\sin\varphi\right)\}.$$



Introducing the ortonormal basis:

$$\frac{1}{\sqrt{N}} b^+_m(x^\perp) U_{x^\perp, x'^\perp} d^+_{m'}(x'^\perp)|0\rangle \equiv |mx^\perp, m'x'^\perp\rangle,$$

we project our effective Hamiltonian onto this basis. Let us denote the new term in the P_+^{eff} as $P_{+(0)}^{eff}$, then

$$P_{+(0)}^{eff}|mx^\perp, m'x'^\perp\rangle = \lim_{a\to 0} \frac{g^4 \left(N - \frac{1}{N}\right)^2 (x^\perp - x'^\perp)^2}{4(4L_0 a)^2 2 p_{m+m'}} |mx^\perp, m'x'^\perp\rangle.$$

Let us consider the action of the other terms in our effective Hamiltonian on the chosen basis states. Actually we consider the projection of our Hamiltonian on this basis and then remain with that part of the Hamiltonian which contains fermion modes and gluon zero modes only. This part of the Hamiltonian has the following form:

$$\int_{-L}^{L} dx^- \int d^2x^\perp \left[\frac{i}{\sqrt{2}} \psi_+^+ (D_\perp + M) \partial_-^{-1} (D_\perp - M) \psi_+ + \right.$$

$$\left. + g^2 \partial_-^{-1} \left(\psi_+^+ \frac{\lambda^a}{2} \psi_+ \right) \partial_-^{-1} \left(\psi_+^+ \frac{\lambda^a}{2} \psi_+ \right) \right].$$

It contains the four-fermion term, which can be rewritten as follows:

$$\int_{-L}^{L} dx^- \int_{-L}^{L} dx'^- \int d^2x^\perp \left[\left(\psi_+^+ \frac{\lambda^a}{2} \psi_+ \right)_{x^-} |x^- - x'^-| \left(\psi_+^+ \frac{\lambda^a}{2} \psi_+ \right)_{x'^-} \right]$$

Due to locality of 4-fermion term in $x^{\perp}$ it gives zero (after the projection on our basis states) except for the case when quark and antiquark are not separated in $x^{\perp}$.
If we want to restore the interaction between separated quark-antiquark we can introduce nonlocal modification of this 4-fermion term in gauge invariant way:

$$\frac{g^2}{\pi R^2}\int_{-L}^{L} dx^- \int d^2x^{\perp} \int\limits_{|x'^{\perp}-x^{\perp}|\leq R} d^2x'^{\perp}\partial_-^{-1}\left(\psi_+^+\frac{\lambda^a}{2}\psi_+\right)_{x^{\perp}}\times$$

$$\times\partial_-^{-1}\left(\psi_+^+(x^-,x'^{\perp})U_{x'^{\perp},x^{\perp}}\frac{\lambda^a}{2}U_{x^{\perp},x'^{\perp}}\psi_+(x^-,x'^{\perp})\right),$$

where $1/(\pi R^2)$ appears due to the averaging over the circle in transverse plane of finite radius R, when we integrate over $(x'^{\perp}-x^{\perp})$ at $|x'^{\perp}-x^{\perp}|\leq R$. To obtain spectral equation we consider the action of this term on the basis states.

The spectral equation has the following form:
$2P_-P_+^{eff}|f\rangle = m^2|f\rangle$ at $P_\perp|f\rangle = 0$, where $|f\rangle$ is the superposition of basis states:

$$|f\rangle = \sum_{m,m'>0} \delta_{n,m+m'} \int d^2x^\perp d^2x'^\perp f_{m,m'}(x'^\perp - x^\perp)|mx^\perp, m'x'^\perp\rangle,$$

$$2P_-|f\rangle = 2p_n|f\rangle, \quad \langle mx^\perp, m'x'^\perp|2P_-P_+^{eff}|f\rangle = m^2\langle mx^\perp, m'x'^\perp|f\rangle.$$

Using ortonormality property of basis states we obtain the eigenvalue equation for wave functions $f_{m,m'}(x'^\perp - x^\perp)$:

$$m^2 f_{m,m'}(x^{\perp}) =$$

$$= \left(\frac{g^4 \left(N - \frac{1}{N}\right)^2 \rho^2}{4(L_0 \Delta)^2} + p_n \left(\frac{1}{p_m} + \frac{1}{p'_m} \right) \left(M^2 - \nabla_{\perp}^2 \right) \right) f_{m,m'}(x^{\perp}) -$$

$$- \frac{g^2 \left(N - \frac{1}{N}\right)}{2L\pi R^2} \sum_{m_1, m_2 > 0} \delta_{n, m_1 + m_2} \frac{p_n}{(p_m - p_{m_1})^2} f_{m_1, m_2}(x^{\perp}) -$$

$$- \frac{g^2 \left(N - \frac{1}{N}\right)}{4L\pi R^2} \sum_{m_1 > 0} p_n \left(\frac{1}{(p_m + p_{m_1})^2} + \frac{1}{(p_{m'} + p_{m_1})^2} \right) f_{m,m'}(x^{\perp}),$$

$$\nabla_{\perp}^2 = \partial_1^2 + \partial_2^2, \quad \rho^2 = (x^1)^2 + (x^2)^2, \quad p_n = p_m + p_{m'}.$$

To write this equation in $L \to \infty$ limit let us introduce new variables:

$$\xi = \frac{p_m}{p_n}, \quad 1-\xi = \frac{p_{m'}}{p_n}, \quad \xi' = \frac{p_{m_1}}{p_n}, \quad dp_{m_1} \simeq \frac{\pi}{L}, \quad d\xi' = \frac{dp_{m_1}}{p_n} \simeq \frac{\pi}{Lp_n},$$

$$\frac{\pi}{Lp_n} \sum_{m_1>0} p_n^2 \left(\frac{1}{(p_m + p_{m_1})^2} + \frac{1}{(p_{m'} + p_{m_1})^2} \right) \simeq$$

$$\simeq \int_0^\infty d\xi' \left(\frac{1}{(\xi + \xi')^2} + \frac{1}{(1 - \xi + \xi')^2} \right) = -\frac{1}{\xi} - \frac{1}{1-\xi},$$

$$\frac{\pi}{Lp_n} \sum_{m_1>0} p_n^2 \left(\frac{1}{(p_m - p_{m_1})^2} \right) \simeq \int_0^1 \frac{d\xi'}{(\xi' - \xi)^2}.$$

Thus we obtain the spectral equation in the following form:

$$\left(\frac{M^2-\nabla_\perp^2}{\xi(1-\xi)}+\frac{g^4\left(N-\frac{1}{N}\right)^2\rho^2}{4(4L_0\Delta)^2}-\right.$$

$$\left.-\frac{g^2\left(N-\frac{1}{N}\right)}{4\pi^2R^2}\left(\frac{1}{\xi(1-\xi)}\right)-C\right)f(\xi,x^\perp)-$$

$$-\frac{g^2\left(N-\frac{1}{N}\right)}{2\pi^2R^2}P\int_0^1\frac{d\xi'\ f(\xi',x^\perp)}{(\xi'-\xi)^2}=m^2f(\xi,x^\perp),$$

where we renormalize the singularity at $\xi'=\xi$ in the integral by introducing into equation the principal value symbol,

$$P\frac{1}{x^2}=\frac{1}{2}\left(\frac{1}{(x+i\varepsilon)^2}+\frac{1}{(x-i\varepsilon)^2}\right),$$

and arbitrary renormalization constant C as the new parameter.

However we can solve this equation in the large quark mass limit. Let us rewrite the spectral equation in terms of dimensionless variables: $\overline{x} = x^{\perp}/R$, $\overline{m} = mR$, $\overline{M} = MR$,

$$\beta = \frac{g^2\left(N - \frac{1}{N}\right)}{2L_0 a} R^2, \quad \gamma = \frac{g^2\left(N - \frac{1}{N}\right)}{2\pi^2}, \quad \overline{\nabla}_{\perp}^2 = \nabla_{\perp}^2 R^2$$

and change the variable ξ to $\omega = 2\xi - 1$, $-1 \leq \omega \leq 1$,

$$\left(\frac{\overline{M}^2 - \overline{\nabla}_{\perp}^2 - \frac{\gamma}{2}}{1 - \omega^2} + \frac{\beta^2}{4}\overline{x}^2\right)\overline{f}(\omega, \overline{x}) - \frac{\gamma}{2} P \int_{-1}^{1} \frac{d\omega' \; \overline{f}(\omega', \overline{x})}{(\omega' - \omega)^2} =$$

$$= \frac{\overline{m}^2 + C}{4}\, \overline{f}(\omega', \overline{x}).$$

Large quark mass limit corresponds to $\omega^2 \ll 1$. It is convinient to introduce new variable $s = \overline{M}\omega$, $-\overline{M} \leq s \leq \overline{M}$, and expand the equation in $\overline{M}^{-1}$. Assume also that $\gamma\overline{M} \equiv \gamma_1$ is finite in this limit then we get the following result:

$$\left(-\overline{\nabla}_{\perp}^{2}+s^{2}+\frac{\beta^{2}}{4}\overline{x}^{2}\right)\widetilde{f}(s,\overline{x})-\frac{\gamma_{1}}{2}P\int_{-\infty}^{\infty}\frac{ds'\ \widetilde{f}(s',\overline{x})}{(s'-s)^{2}}=\frac{\overline{m}^{2}+C}{4}\ \widetilde{f}(s',\overline{x})$$

where we fix the arbitrary parameter C:

$$C=\overline{M}^{2}+\frac{\gamma}{2}.$$

Let us make Fourier transformation in s:

$$\varphi(z,\overline{x})=\int ds\exp\left(isz\right)\widetilde{f}(s,\overline{x}),$$

$$P\int\frac{ds\,ds'\exp\left\{i(s-s')z\right\}}{(s-s')^{2}}\exp\left(is'z\right)\ \widetilde{f}(s',\overline{x})=-\pi|z|\varphi(z,\overline{x}).$$

Thus we obtain the following equation:

$$\left(-\overline{\nabla}_\perp^2 - \partial_z^2 + \frac{\beta^2}{4}\overline{x}^2 + \frac{\pi}{2}\gamma_1|z|\right)\varphi(z,\overline{x}) = \frac{\overline{m}^2}{4}\,\varphi(z,\overline{x}).$$

We can write $\varphi(z,\overline{x}) = \varphi_1(\overline{x})\varphi_2(z)$, where $\varphi_1(\overline{x})$ and $\varphi_2(z)$ satisfy the following equations:

$$\left(-\overline{\nabla}_\perp^2 + \frac{\beta^2}{4}\overline{x}^2\right)\varphi_1(\overline{x}) = \frac{\overline{m}_1^2}{4}\,\varphi_1(\overline{x}),$$

$$\left(-\partial_z^2 + \frac{\pi}{2}\gamma_1|z|\right)\varphi_2(z) = \frac{\overline{m}_2^2}{4}\,\varphi_2(z), \quad \overline{m}^2 = \overline{m}_1^2 + \overline{m}_2^2.$$

The first equation is the equation of 2-dimensional quantum harmonic oscillator and the second one is the Airy equation. Therefore we know the spectrum of $\overline{m}_1^2$ and $\overline{m}_2^2$:

$$\overline{m}_1^2(n_1,n_2) = 4\beta\sum_{k=1,2}\left(n_k + \frac{1}{2}\right), \quad \overline{m}_2^2(n_3) = 4\left(\frac{\pi\gamma_1}{2}\right)^{\frac{2}{3}}|\zeta_{n_3}|,$$

where ζ_n are zeros of corresponding Airy eigenfunctions.

For this roots we have the following: $|\zeta_0| \approx 1$ and for $n > 0$

$$|\zeta_n| \approx \left(\frac{3\pi}{4}\left(n + \frac{1}{2}\right)\right)^{\frac{2}{3}}.$$

Thus $\overline{m}_2^2(n_3 = 0) \approx 4\left(\frac{\pi\gamma_1}{2}\right)^{\frac{2}{3}}, \quad \overline{m}_2^2(n_3 > 0) \approx 4\left(\frac{\pi^2\gamma_1}{8}\left(n + \frac{1}{2}\right)\right)^{\frac{2}{3}}$

The obtained result is interesting. However, it shows the absence of 3-D rotational symmetry. It makes difficult to classify the spectrum in total orbital momentum and spin

To restore the 3-D rotational symmetry we propose several modifications of our effective Hamiltonian. In these modifications we introduce additionally zero mode of A_- assuming that it doesn't depend on x^-, $x^\perp$ and Abelian. In a principle, the p.b.c. don't allow to throw out zero mode of this field because $trPexp\left(-ig\int_{-L}^{L} A_- dx^-\right)$ is gauge invariant, and the light front gauge $A_- = 0$ is not correct (one may take only $\partial_- A_- = 0$).

Formulation of the model:
$H^{eff} = H_G + H_\psi$, where
$H_\psi = \frac{i}{2}\int d^2x^\perp dx^- \, \chi^+(x)\,(D_\perp + M)\,D_-^{-1}\,(D_\perp - M)\,\chi(x)$, a
$H_G = \frac{1}{2P_-}\left[\int \frac{d^2x^\perp}{g^2 L_0}\left(\pi_k^a(x^\perp)\pi_k^a(x^\perp)\right)\right]^2 + \frac{\varkappa^2}{2P_-}\frac{\Pi_-^2 P_-^2}{M^2}$
Let us take the quark-antiquark states in the following form
$|f(P)\rangle = \int d^2x_1^\perp d^2x_2^\perp dx_1^- dx_2^- \, f_p(x_1 - x_2)\exp\left(-iP_\perp\frac{(x_1+x_2)^\perp}{2} - iP_-\frac{(x_1+x_2)^-}{2}\right)\times$
$\times\frac{1}{(2\pi)^2}\int\limits_0^{+\infty}\frac{dq_1 dq_2}{\sqrt{2q_1q_2}}\exp\left(iq_1x_1^- + iq_2x_2^-\right) b_{q_1}^+(x_1^\perp)U_{x_1,x_2}d_{q_2}^+(x_2^\perp)\,|0\rangle$
Here $U_{x_1,x_2}(A) = Pexp\left(-i\int_{x_1^\perp}^{x_2^\perp} dz^k A_{k(0)}(z^\perp)\right)e^{iA_-(x_1^- - x_2^-)} = U_{x_1^\perp,x_2^\perp}(A_\perp)e^{iA_-(x_1^- - x_2^-)}$, $P_\perp, P_-$–are momenta, and Π_-–is the canonical momentum conjugated to A_-.

Further, we define the parameter $\varkappa^2$ so that to restore rotational symmetry in H_G: $\varkappa^2 = \lim_{a \to 0} \frac{g^4\left(N - \frac{1}{N}\right)^2}{4(L_0)^2 a^2}$.
Furthermore, we introduce the boost-invariant coordinate $x^3 \equiv (P_- x^- / M)$. Then we get the following spectral equation:

$$\left(\varkappa^2 (x_\perp^2 + (x^3)^2) + 4 \frac{\nabla_\perp^2 - M^2}{1 + (4\partial_3^2)/M^2} - m^2 \right) f_p(x^\perp, x^3) = 0$$

In the limit $M^2 \gg \nabla_\perp^2$, $M^2 \gg \partial_3$ we obtain the equation which has the expected the rotational invariance:

$$\left(4\Delta - \varkappa^2 \vec{r}^2 - 4M^2 + m^2 \right) f_p(\vec{r}) = 0$$

where $(\vec{r})^2 = x_\perp^2 + (x^3)^2$, $\Delta = \nabla_\perp^2 + \partial_3^2$, $\partial_3 = (M/P_-)\partial_-$.

The model of effective Hamiltonian on the LF which can give the spectral equation coinciding with the equation obtained in AdS/QCD approach.

$H^{eff}=H_G+H_\psi$, where

$H_\psi=\frac{i}{2}\int d^2x^\perp dx^-\,\chi^+(x)\,(D_\perp+M)\,D_-^{-1}\,(D_\perp-M)\,\chi(x)$, and

$H_G=\frac{\varkappa^2}{4P_-}\left\{\Pi_3^2+\Pi_\perp^2,\,\frac{\tilde{P}_-}{P_-}\right\}=\frac{\varkappa^2}{4P_-}\left\{\frac{\Pi_-^2 P_-^2}{M^2}+\Pi_\perp^2,\,\frac{\tilde{P}_-}{P_-}\right\}$

Here we introduce the new operator

$\tilde{P}_-=\left(\int[d^3y]\,\chi^+(y)(-iD_-^{-1})\chi(y)\right)^{-1}$ and use the same quark-antiquark states as in a model before.

Then we get the following spectral equation

$$\left(4\frac{-\nabla_\perp^2+M^2}{1-z^2}+\frac{\varkappa^2}{4}\left(1-z^2\right)\left(x^\perp\right)^2-\right.$$

$$\left.-\frac{\varkappa^2}{M^2}\left[\left(1-z^2\right)\partial_{zz}^2-2z\partial_z-1\right]-m^2\right)\hat{f}_p(x^\perp,\tilde{q})=0,\ \text{where}$$

$z=2\tilde{q}/P_-$, $\tilde{q}=q+A_-$

Let us introduce dementionless variables
$Mx^{\perp}\sqrt{1-z^2}=\tilde{x}$, $\nabla^2_{\perp}/(1-z^2)=M^2\nabla^2_{\tilde{x}}$ and redenote
$\tilde{x}^2\equiv|\vec{x}_{\perp}|^2 \qquad \nabla^2_{\tilde{x}}\equiv\vec{\nabla}^2_{\perp} \qquad \hat{f}_p(x^{\perp},\tilde{q})\equiv$
$\hat{\varphi}_p(x^{\perp}M\sqrt{1-z^2},2\tilde{q}/P_-)=\hat{\varphi}_p(\vec{x}_{\perp},z)$.
Then we can rewrite the spectral equation as follows:

$$\left(M^2\vec{\nabla}^2_{\perp}-\frac{\varkappa^2}{16M^2}|\vec{x}_{\perp}|^2+\frac{\varkappa^2}{4M^2}\left[\left(1-z^2\right)\partial^2_{zz}-2z\partial_z-1\right]-\right.$$
$$\left.-\frac{M^2}{1-z^2}+\frac{m^2}{4}\right)\hat{\varphi}_p(\vec{x},z)=0$$

We get the spectral equation with separated variables $\vec{x}$, z and have two equations: one is 2-D harmonic oscillator and the other is Jacobi equation. They can be solved exactly.

The normalised solutions have the following form

$$\hat{f}_p^{nl_\phi r}(x^\perp,\tilde{q}) = N_{nl_\phi r}\frac{\sqrt{2}M}{(2\pi)^{3/2}\sqrt{P_-}}e^{il_\phi\phi}(Mx^\perp)^{|l_\phi|}\times$$

$$\times\left(1-\left(\frac{2\tilde{q}}{P_-}\right)^2\right)^{\left(|l_\phi|+\mu_2+1\right)/2}\exp\left[-\frac{\varkappa}{M^2}(8Mx^\perp)^2\left(1-\left(\frac{2\tilde{q}}{P_-}\right)^2\right)\right]\times$$

$$\times L_n^{|l_\phi|}\left[\frac{\varkappa}{4M^2}\left(Mx^\perp\right)^2\left(1-\left(\frac{2\tilde{q}}{P_-}\right)^2\right)\right]\times P_r^{(\mu_2,\mu_2)}\left(1-\left(\frac{2\tilde{q}}{P_-}\right)^2\right),$$

where $\mu_2 = M^2/\varkappa$, L_n^μ are Laguerre polynomials and $P_r^{\mu,\nu}$ are Jacobi polynomials.

And the spectrum is

$$m^2(n, l_\phi, r) =$$
$$8M^2Q\left[(2n + |l_\phi| + 1) + 2Q(1 + (r + \mu_2 + 1)(r + \mu_2))\right] =$$
$$= 2\varkappa\left[2n + |l_\phi| + \frac{\varkappa}{2M^2}\left(r + \frac{M^2}{\varkappa} + 1\right)\left(r + \frac{M^2}{\varkappa}\right)\right] + const$$
$$n, r \in \mathbb{N} \cup 0, \qquad l_\phi \in \mathbb{Z}$$

Let us note that in the limit $M^2 \gg \varkappa$ the spectral equation again coincides with the 3-D harmonic oscillator one, and rotational symmetry again restores.

The same equation was recently proposed by Guy de Teramond and Stanley Brodsky using the LF AdS/QCD approach.

The model formally supporting rotational symmetry.

Consider effective LF Hamiltonian of the following simple form:

$$P_+^{eff} = \frac{1}{2P_-}\Bigg(\varkappa^2\left(\Pi_\perp^2 + \Pi_3^2\right) +$$
$$+2\int d^2y^\perp \int_0^\infty \Bigg[b_k^+(y^\perp)\left(M^2 - D_\perp^2 + (k_3 + A_3)^2\right) b_k(y^\perp) +$$
$$+d_k^+(y^\perp)\left(M^2 - (D_\perp^*)^2 + (k_3 - A_3)^2\right) d_k(y^\perp)\Bigg]\Bigg),$$

$$\Pi_3 \equiv P_-\Pi_-/M, \qquad D_i = \partial_i - iA_i,$$
$$D_i^* = \partial_i + iA_i, \qquad y^3 \equiv P_- y^-/M,$$
$$A_3 \equiv MA_-/P_-, \qquad k_3 \equiv Mq/P_-,$$
$$k_3 x^3 = kx^-, \qquad \vec{x} = (x^1, x^2, x^3).$$

This equation has the previous 3-D rotation invariant form but now at any M.

$$\left(\varkappa^2 \vec{x}^2 + 4\left(M^2 - \vec{\nabla}^2\right) - m^2\right) f_p(x) = 0$$

Conclusion

• Using the intuition related to the investigation of limit transition to the LF from Hamiltonians on the space-like hyperplanes approaching to the LF we have found the way how to formulate LF QCD Hamiltonian with dynamical zero mode of gluon field. It have led to the introduction of additional term with this zero mode that produces quadratic relative quark-antiquark transverse coordinate potential in spectral equation.

• We solved this equation exactly in the large quark mass limit. However, this solution supports only 2-D rotational symmetry.

• We have proposed the modified forms of the effective LF Hamiltonian which support 3-D rotational symmetry of spectral equation at least in large quark mass limit.