On the Difference between the FOPT and CIPT Approach for Hadronic τ Decays

Christoph Regner
University of Vienna

Based on arXiv:2008.00578 and arXiv:2105.11222
with André H. Hoang

Joint Annual Meeting of the APS and SPS, August 31 2021

Motivation - Hadronic τ Decays

Precise determination of the strong coupling α_s provided by investigations of τ hadronic spectral function moments:

$$A_W(s_0) = \frac{N_c}{2}|V_{ud}|^2\left[\delta_W^{\rm tree} + \delta_W^{(0)}(s_0) + \sum_{d\geq 4}\delta_W^{(d)}(s_0)\right]$$

Dominant theoretical uncertainty related to different expansion approaches to evaluate the perturbative series in $\delta_W^{(0)}$:

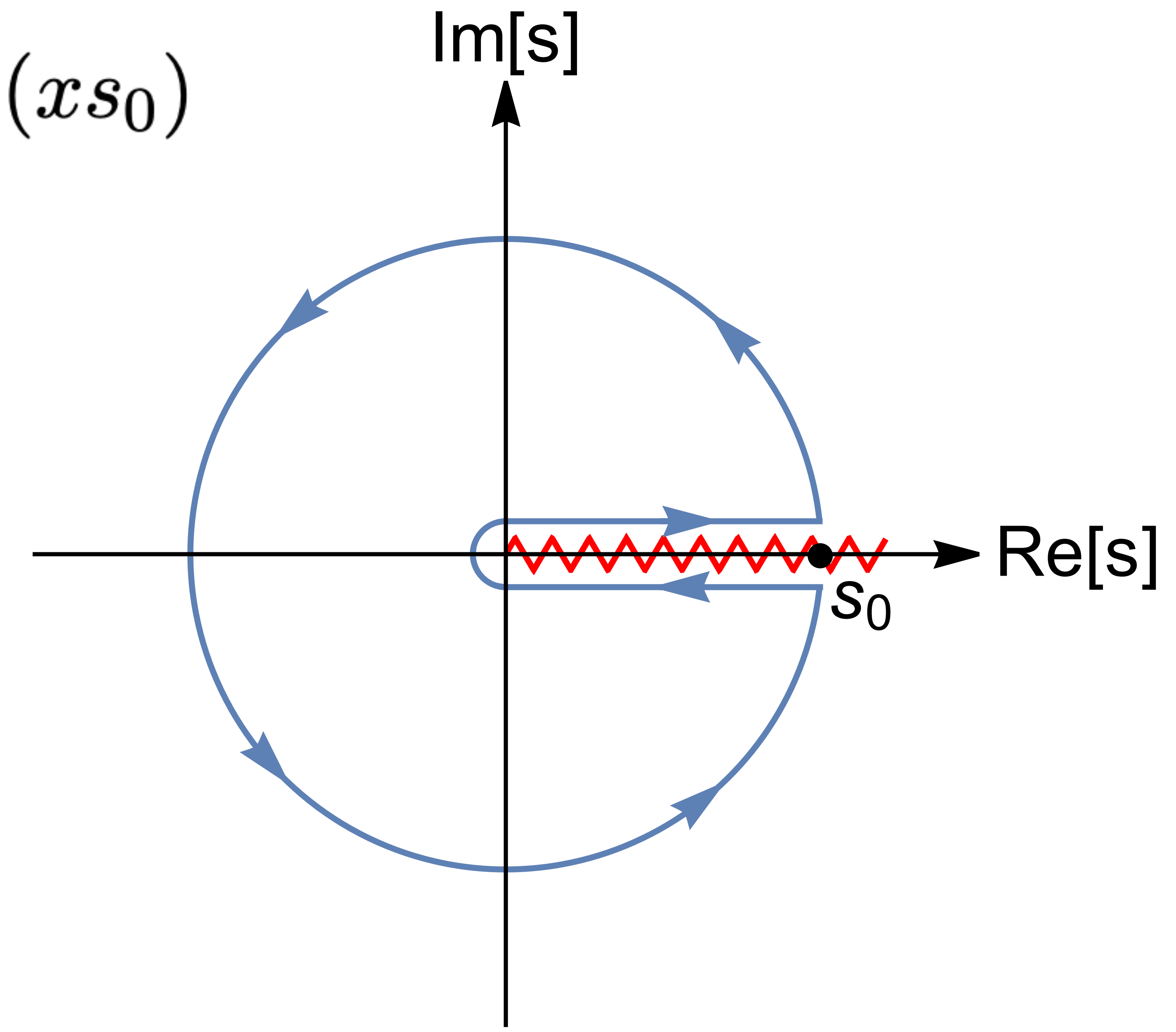
Fixed-order PT (FOPT) vs. Contour-improved PT (CIPT)

The perturbative QCD corrections $\delta_W^{(0)}$ are obtained from the Adler function $\hat{D}$: ($x \equiv s/s_0$)

$$\delta_W^{(0)}(s_0) = \frac{1}{2\pi i}\oint_{|x|=1}\frac{\mathrm{d}x}{x}W(x)\,\hat{D}(xs_0)$$

- CIPT expansion: $$\hat{D}(s) = \sum_{n=1}^{\infty}\bar{c}_n\, a^n(-x)$$
- FOPT expansion: $$\hat{D}(s) = \sum_{n=1}^{\infty}a_0^n\sum_{k=1}^{n}k\,\bar{c}_{n,k}\ln^{k-1}(-x)$$

$$a(-x) \equiv \frac{\beta_0\,\alpha_s(-s)}{4\pi} = \frac{\beta_0\,\alpha_s(-xs_0)}{4\pi} \qquad a_0 \equiv \frac{\beta_0\,\alpha_s(s_0)}{4\pi}$$



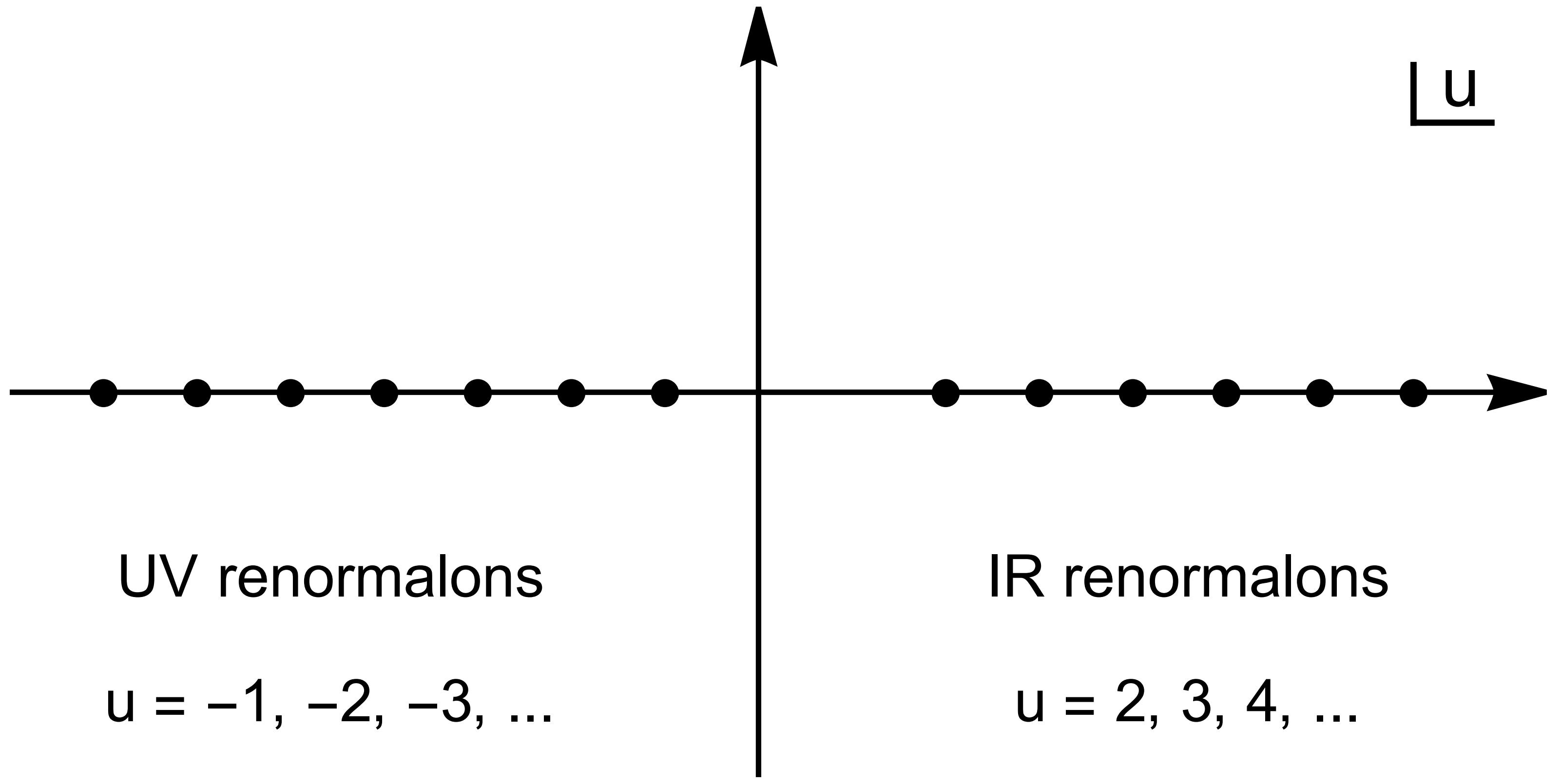
Renormalons and Borel Summation

Perturbative series in QCD in general not convergent, but asymptotic.

$$\hat{D}(s) = \sum_n \bar{c}_n \, a^n(-s) \overset{n\to\infty}{\sim} \sum_n n! \, a^n(-s) \longrightarrow \text{Borel summation:} \;\; B[\hat{D}](u) = \sum_{n=1}^{\infty} \frac{\bar{c}_n}{\Gamma(n)} \, u^{n-1}$$

The Borel transform $B[\hat{D}](u)$ of $\hat{D}(s)$ is defined by the Borel integral:

$$\hat{D}(-s_0) = \int_0^\infty \mathrm{d}u \, B[\hat{D}](u) \, \mathrm{e}^{-\frac{u}{a_0}}$$



- Borel transfrom in large-β_0 approximation:

$$B[\hat{D}](u) = \frac{128}{3\beta_0} \frac{\mathrm{e}^{\frac{5}{3}u}}{2-u} \sum_{k=2}^{\infty} \frac{(-1)^k \, k}{(k^2-(1-u)^2)^2}$$

IR renormalons lead to ambiguities in the definition of the Borel integral.

Important connection: IR renormalons $\Longleftrightarrow$ non-pert. OPE corrections

$$\hat{D}^{\mathrm{OPE}}(s) = \frac{1}{(-s)^2} \langle G^2 \rangle + \sum_{p=3}^{\infty} \frac{1}{(-s)^p} \left[\langle \bar{O}_{2p,0} \rangle + (a(-x))^{-1} \langle \bar{O}_{2p,-1} \rangle \right]$$

Most importantly: $$B(u) = \frac{1}{(2-u)} \quad \Longleftrightarrow \quad \langle G^2 \rangle$$

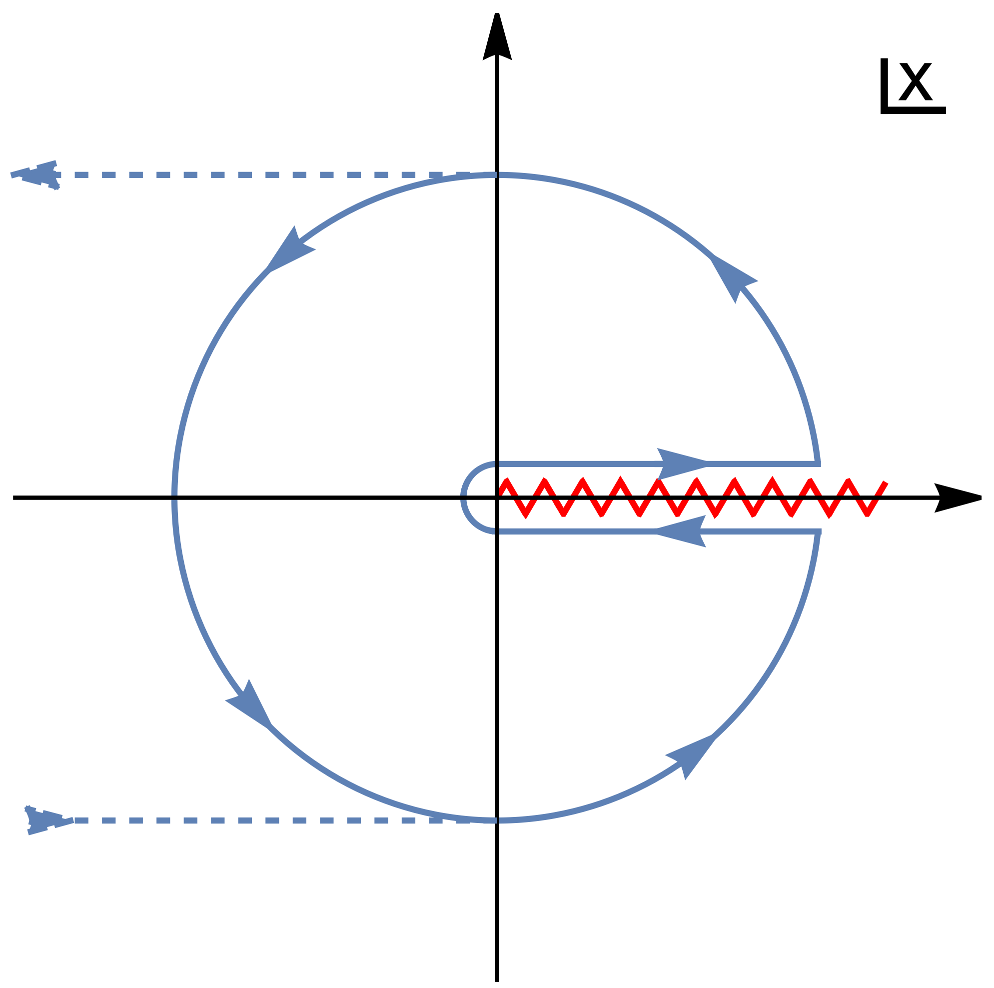
FOPT vs. CIPT Borel Representations

- FOPT spectral function moment Borel representation:
  (= Previously accepted Borel representation for **FOPT and CIPT** approach)

$$\delta^{(0),\mathrm{FOPT}}_{W,\mathrm{Borel}}(s_0) = \mathrm{PV}\int_0^\infty \mathrm{d}u\,\frac{1}{2\pi i}\oint_{|x|=1}\frac{\mathrm{d}x}{x}\,W(x)\,B[\hat{D}](u)\,\mathrm{e}^{-\frac{u}{a(-x)}}$$

- CIPT spectral function moment Borel representation: [New!]

$$\delta^{(0),\mathrm{CIPT}}_{W,\mathrm{Borel}}(s_0) = \int_0^\infty \mathrm{d}\bar{u}\,\frac{1}{2\pi i}\oint_{\mathcal{C}_x}\frac{\mathrm{d}x}{x}\,W(x)\,\left(\frac{a(-x)}{a_0}\right)B[\hat{D}]\left(\frac{a(-x)}{a_0}\bar{u}\right)\mathrm{e}^{-\frac{\bar{u}}{a_0}}$$



Importance of path $\mathcal{C}_x$: CIPT Borel representation contains additional poles on the negative real x-axis ⟶ Contour must be deformed away form $|x| = 1$.

Borel representations related via complex-valued change of variables: $u = \dfrac{a(-x)}{a_0}\,\bar{u}$

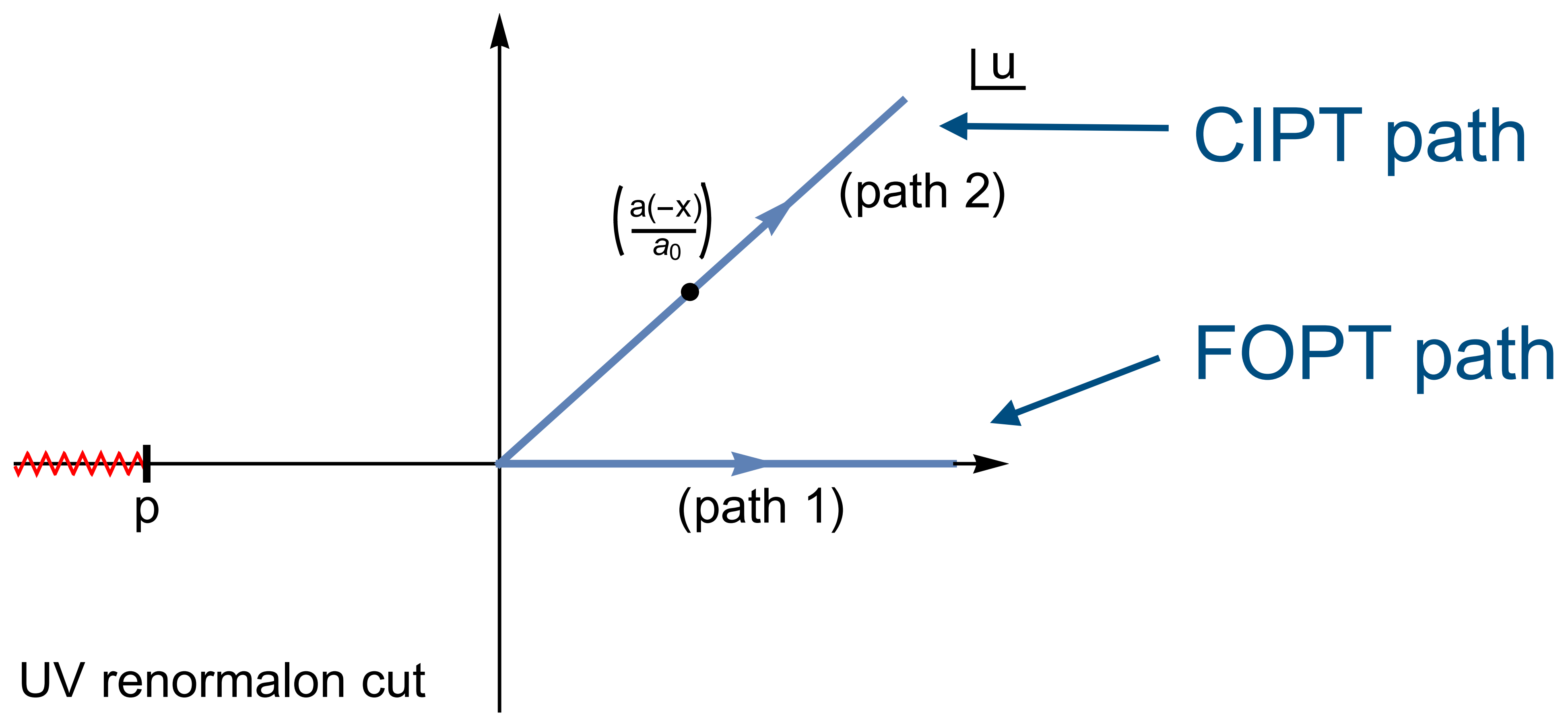
We want to address the following questions:

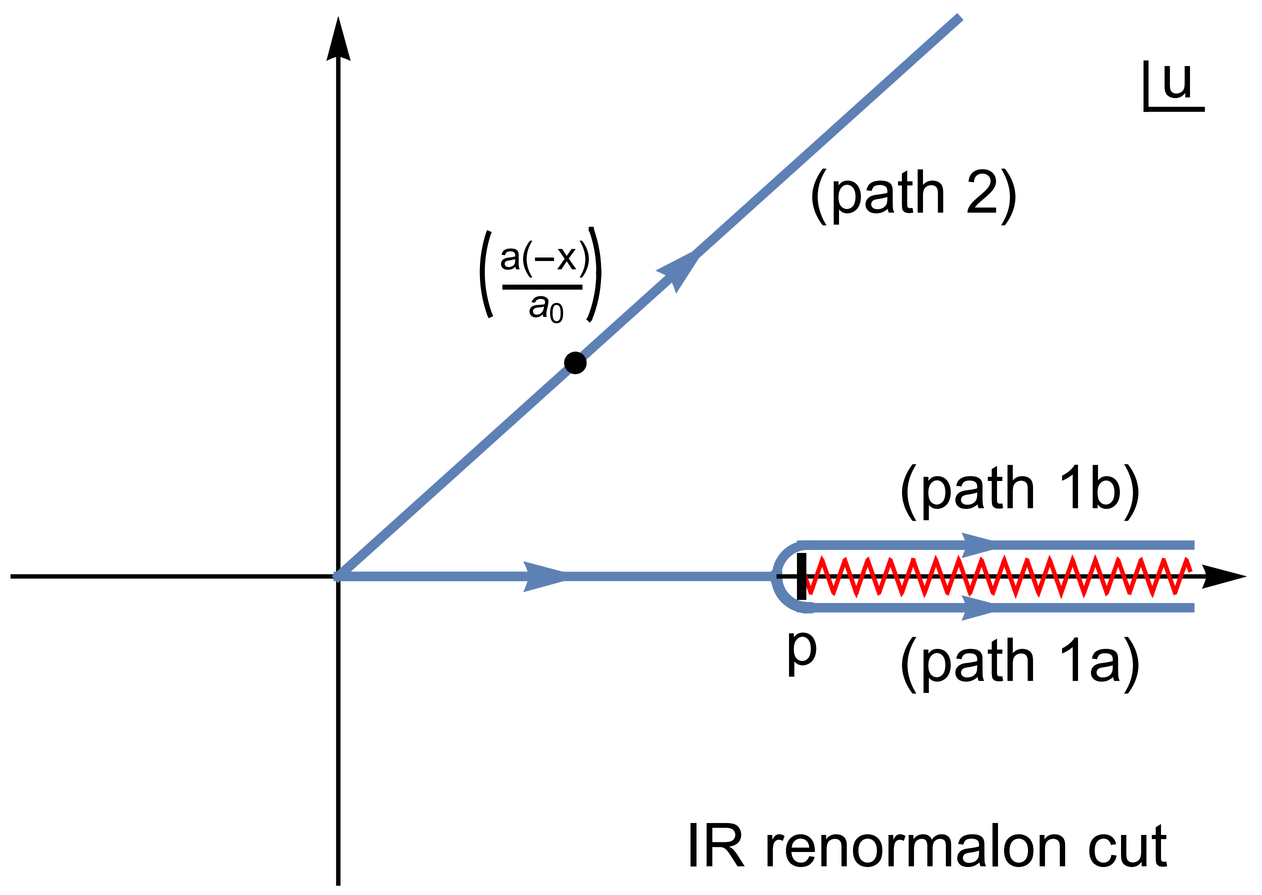
- How can it happen that CIPT and FOPT "converge" to different values (Borel sums)?
- Can one predict the FOPT-CIPT discrepancy for a given Borel model?
- Is it possible to construct moments with a small (vanishing) discrepancy between the FOPT and CIPT Borel sums?
- Implications for α_s determinations?

Analytic Properties of the Borel Representations

- FOPT Borel representation: $\delta^{(0),\mathrm{FOPT}}_{W,\mathrm{Borel}}(s_0) = \mathrm{PV}\int_0^\infty \mathrm{d}u\, \frac{1}{2\pi i}\oint_{|x|=1}\frac{\mathrm{d}x}{x}\, W(x)\, B[\hat{D}](u)\, \mathrm{e}^{-\frac{u}{a(-x)}}$

- CIPT Borel representation: $\delta^{(0),\mathrm{CIPT}}_{W,\mathrm{Borel}}(s_0) = \int_0^\infty \mathrm{d}\bar{u}\, \frac{1}{2\pi i}\oint_{\mathcal{C}_x}\frac{\mathrm{d}x}{x}\, W(x)\left(\frac{a(-x)}{a_0}\right) B[\hat{D}]\left(\frac{a(-x)}{a_0}\bar{u}\right) \mathrm{e}^{-\frac{\bar{u}}{a_0}}$





UV Renormalons: $B^{\mathrm{UV}}_{\hat{D},p,\gamma} = \frac{1}{(p+u)^\gamma}$

- FOPT and CIPT Borel representations equivalent.
- Closing paths 1 and 2 does not encircle cuts.

$$u = \frac{a(-x)}{a_0}\,\bar{u}$$

IR Renormalons: $B^{\mathrm{IR}}_{\hat{D},p,\gamma} = \frac{1}{(p-u)^\gamma}$

- FOPT and CIPT Borel representations inequivalent.
- Closing paths 1a (1b) and 2 contains cuts.
- FOPT: Regularization prescription (e.g. PV) must be imposed.
- CIPT: Automatically well-defined due to complex-valued $a(-x)$.

Analytic Properties of the Borel Representations

- FOPT Borel representation: $\delta^{(0),\mathrm{FOPT}}_{W,\mathrm{Borel}}(s_0) = \mathrm{PV}\int_0^\infty \mathrm{d}u\,\frac{1}{2\pi i}\oint_{|x|=1}\frac{\mathrm{d}x}{x}\,W(x)\,B[\hat{D}](u)\,\mathrm{e}^{-\frac{u}{a(-x)}}$

- CIPT Borel representation: $\delta^{(0),\mathrm{CIPT}}_{W,\mathrm{Borel}}(s_0) = \int_0^\infty \mathrm{d}\bar{u}\,\frac{1}{2\pi i}\oint_{\mathcal{C}_x}\frac{\mathrm{d}x}{x}\,W(x)\,\left(\frac{a(-x)}{a_0}\right)B[\hat{D}]\left(\frac{a(-x)}{a_0}\bar{u}\right)\mathrm{e}^{-\frac{\bar{u}}{a_0}}$

Performing first the u-integration for a generic IR renormalon contribution $B^{\mathrm{IR}}_{\hat{D},p,\gamma} = \frac{1}{(p-u)^\gamma}$ yields

- FOPT: $\hat{D}^{\mathrm{FOPT}}_{p,\gamma}(xs_0) = -(-a(-x))^{1-\gamma}\,e^{-\frac{p}{a(-x)}}\,\Gamma\left(1-\gamma,-\frac{p}{a(-x)}\right) - \mathrm{sig}[\mathrm{Im}[x]]\,(i\pi)\,\frac{(a(-x))^{1-\gamma}}{\Gamma(\gamma)}\,e^{-\frac{p}{a(-x)}}$
- CIPT: $\hat{D}^{\mathrm{CIPT}}_{p,\gamma}(xs_0) = -(-a(-x))^{1-\gamma}\,e^{-\frac{p}{a(-x)}}\,\Gamma\left(1-\gamma,-\frac{p}{a(-x)}\right)$

FOPT expression $\hat{D}^{\mathrm{FOPT}}_{p,\gamma}(xs_0)$

- Analytic in the entire x-plane except for the cut along the positive real x-axis.
- Coincides with the analytic properties one expects form the physical (hadron level) Adler function.
- FOPT expansion compatible with standard OPE corrections assumed in the literature.

CIPT expression $\hat{D}^{\mathrm{CIPT}}_{p,\gamma}(xs_0)$

- Also contains the cut along the positive real x-axis.
- Additional **unphysical cut** along the negative real x-axis.
  → OPE corrections in general not of standard form.
- OPE corrections differ for CIPT and FOPT approach.

Asymptotic Separation

Difference between the FOPT and CIPT Borel representations can be calculated analytically.

"Asymptotic separation" for an IR renormalon $B^{\rm IR}_{\hat{D},p,\gamma} = \frac{1}{(p-u)^\gamma}$ and a generic weight function $W(x) = (-x)^m$ given by:

$$\Delta(m,p,\gamma,s_0) \equiv \delta^{(0),\rm CIPT}_{\{(-x)^m,p,\gamma\},\rm Borel}(s_0) - \delta^{(0),\rm FOPT}_{\{(-x)^m,p,\gamma\},\rm Borel}(s_0)$$

$$= \frac{1}{2\Gamma(\gamma)} \oint_{\mathcal{C}_x} \frac{{\rm d}x}{x} (-x)^m \,{\rm sig}[{\rm Im}[x]]\, (a(-x))^{1-\gamma}\, e^{-\frac{p}{a(-x)}}$$

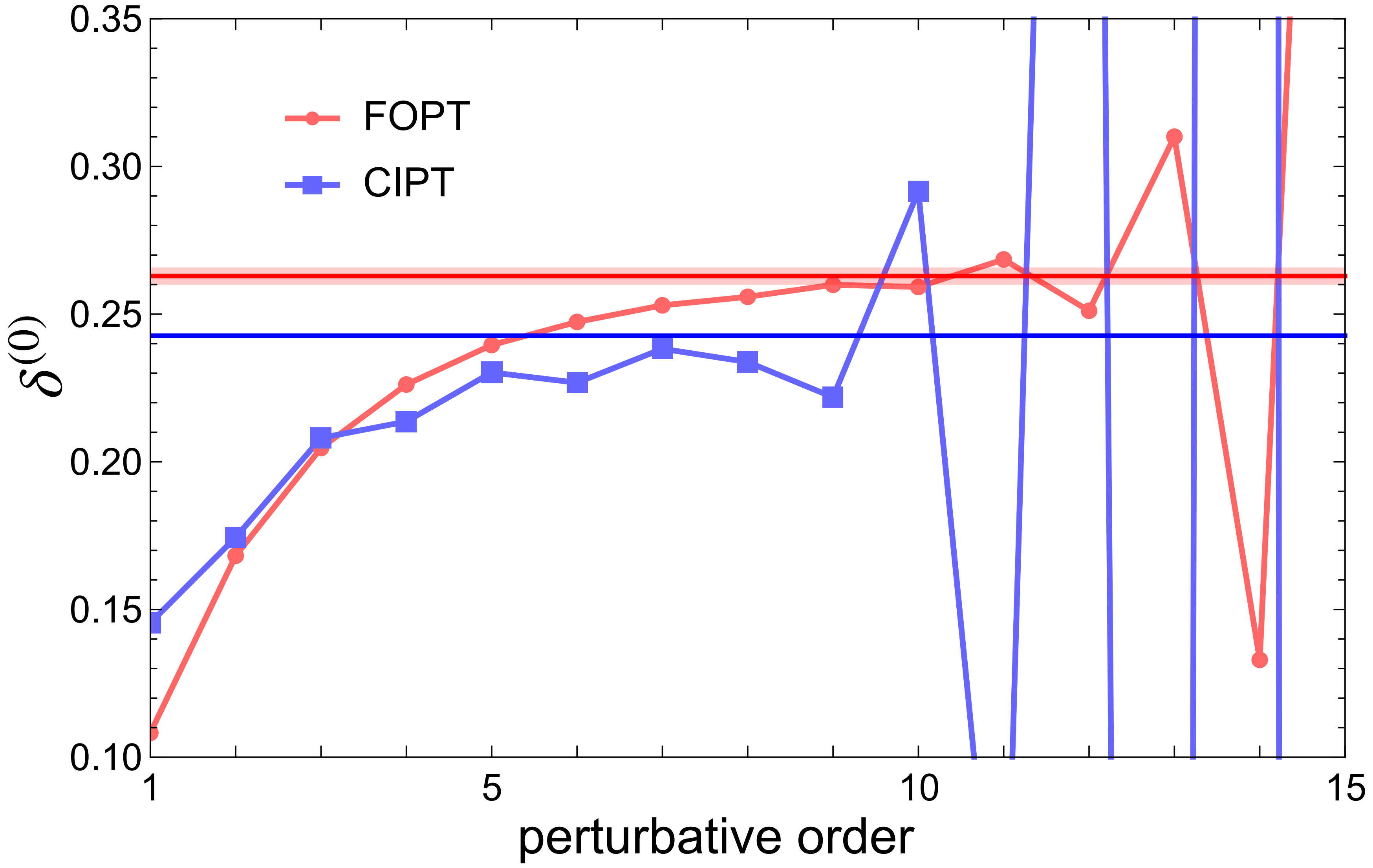
Properties of the asymptotic separation:

- Fully analytic result.
- Renormalization scheme invariant.
- Exhibits same kind of power-suppression as FOPT Borel sum ambiguity: $e^{-\frac{p}{a(-x)}} \sim \left(\frac{\Lambda^2_{\rm QCD}}{s_0}\right)^p$
- Much larger than FOPT Borel sum ambiguity for Borel models with a sizable gluon condensate cut.
- Asymptotic separation provides quantitative description of the CIPT-FOPT discrepancy for any given Borel model.

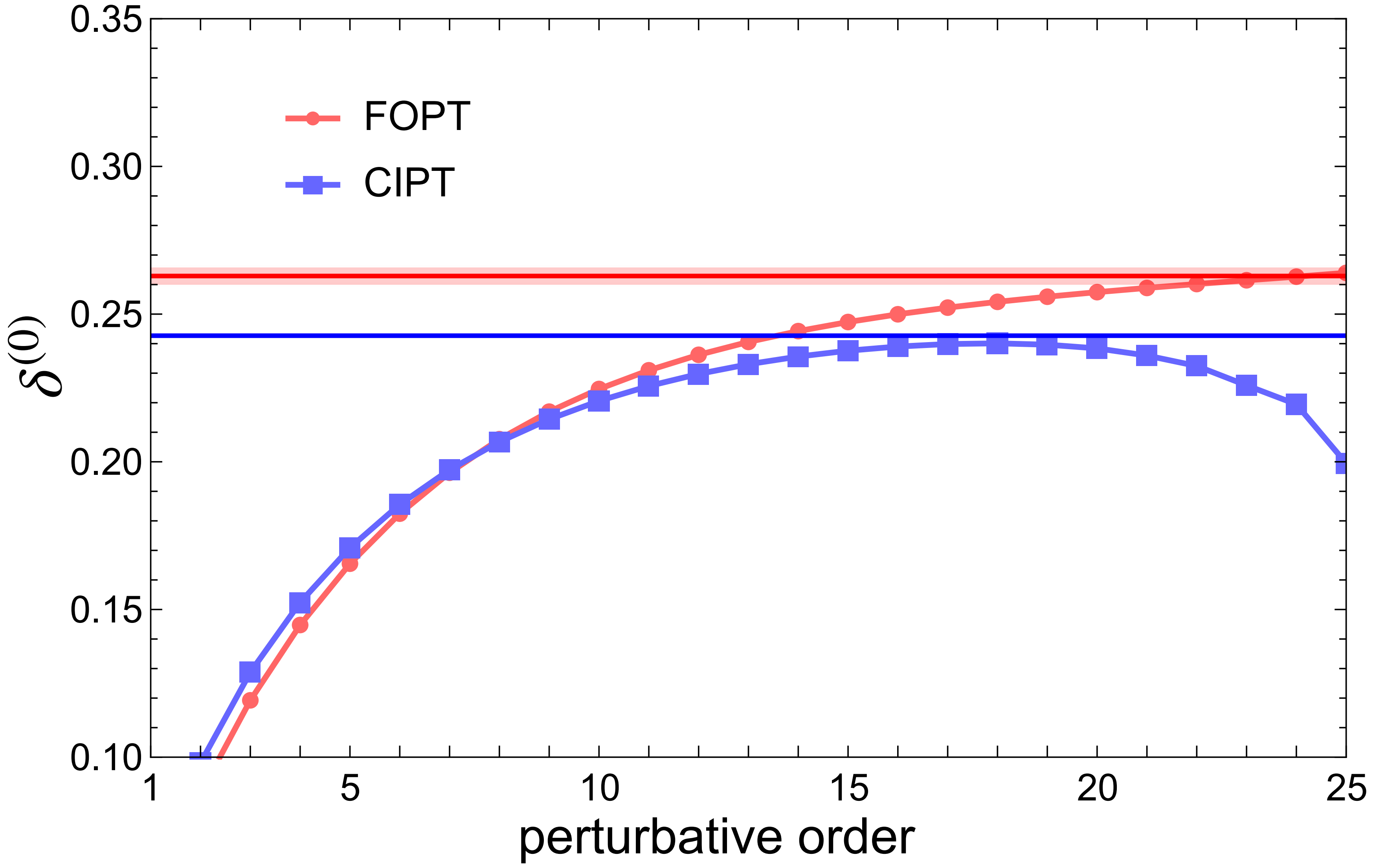
Numerical Tests — Hadronic τ Decay Rate

Hadronic τ decay rate R_τ related to the weight function $W_\tau = (1-x)^3(1+x) = 1 - 2x + 2x^3 - x^4$.

Large-β_0 Borel function: $B[\hat{D}](u) = \frac{128}{3\beta_0}\frac{e^{\lambda u}}{2-u}\sum_{k=2}^{\infty}\frac{(-1)^k\,k}{(k^2-(1-u)^2)^2}$



$(a)\quad \lambda = \frac{5}{3}$ ($\overline{\text{MS}}$ scheme)



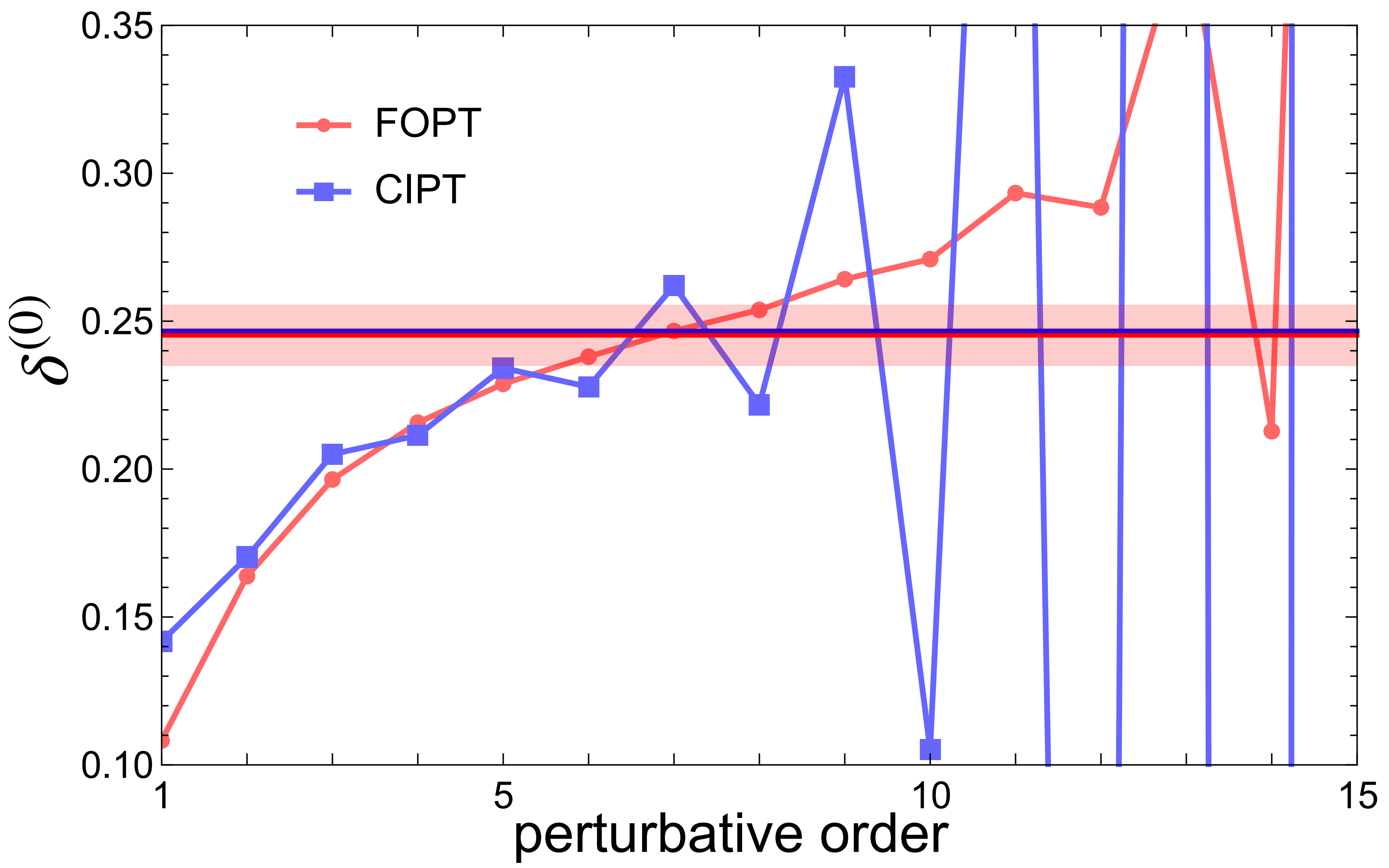
$(b)\quad \lambda = \frac{20}{3}$

- Agreement of CIPT series behavior with CIPT Borel sum (blue horizontal line) depends on the scheme.
- Better agreement in schemes with small values for $\alpha_s(m_\tau)$.

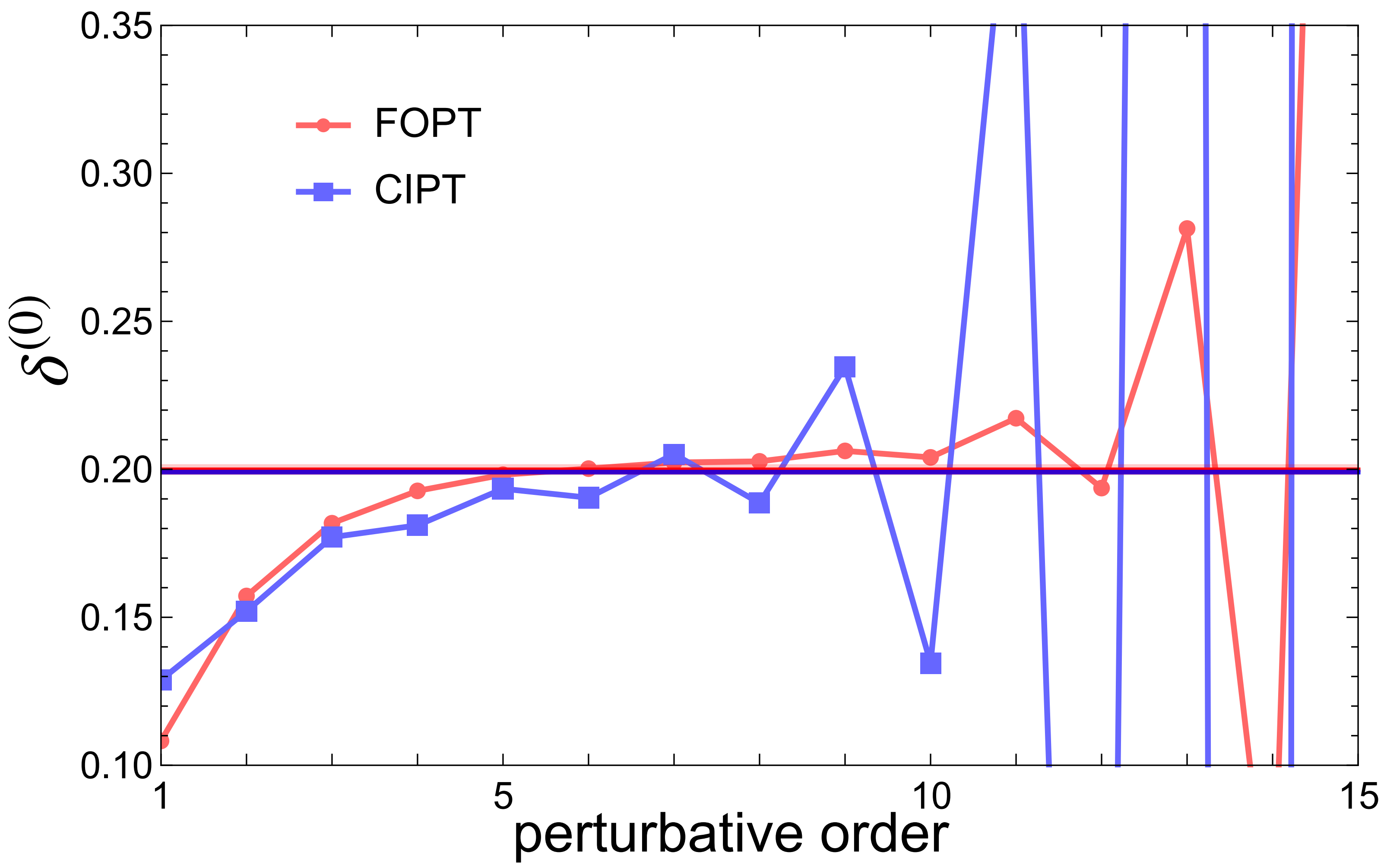
Numerical Tests

Moments with a small asymptotic separation for the weight $W(x) = (1-x)^2\,(1 + cx + x^2)$.

- large-β_0 Borel transform of the Adler function $(\overline{\mathrm{MS}}\ \text{scheme})$: $B[\hat{D}](u) = \frac{128}{3\beta_0}\frac{e^{\frac{5}{3}u}}{2-u}\sum_{k=2}^{\infty}\frac{(-1)^k\,k}{(k^2-(1-u)^2)^2}$



(a) $c = 0$



(b) $c = 0.75$

- Vanishing contribution to asymptotic separation from gluon condensate renormalon in large-β_0 approximation.
- Moments with small CIPT-FOPT discrepancy can be designed.
- Might be useful for α_s determinations.

Summary

- CIPT and FOPT Borel representations and their corresponding Borel sums differ.
- CIPT expansion not compatible with the standard OPE approach.
- Difference between the CIPT and FOPT Borel sum (= asymptotic separation) can be calculated analytically.
- Asymptotic separation can provide an explanation for the CIPT-FOPT discrepancy if the Adler function's Borel function has a large gluon condensate cut.

Outlook:

- Important prediction: Discrepancy between CIPT and FOPT series vanishes in infrared-subtracted perturbation theory.

  ⟶ Ongoing work for high precision α_s determination.