Towards Colour Flow Evolution at $\mathcal{O}(\alpha_s^2)$

Ines Ruffa

Based on: Plätzer, S. & Ruffa, I. Towards Colour Flow Evolution at Two Loops. *JHEP* **06,** 007 (2021)

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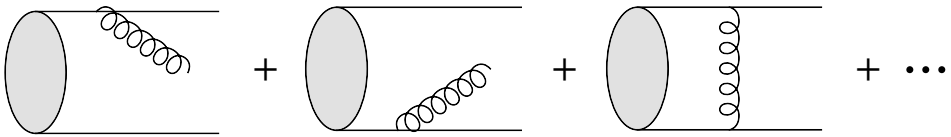
$\int dk \prod$ Doktoratskolleg Particles and Interactions

Soft gluon evolution

- Soft gluon evolution: Adding soft gluon corrections perturbatively - real emissions and virtual exchanges - to a hard, short-distance process.



- Formulate evolution equations which allow us to **resum large logarithms of soft origin**.

- The evolution equations are formulated in **colour space**. Treat the amplitude as a component of a vector in an abstract N-parton colour vector space

$$|\mathcal{M}_{\mathrm{N}}\rangle = \sum_{\sigma} \mathcal{M}_{\mathrm{N},\sigma} \, |\sigma\rangle \ .$$

[Catani, S. & Seymour, M. The Dipole formalism for the calculation of QCD jet cross-sections at next-to-leading order. *Phys. Lett. B* **378**, 287–301 (1996)]

Soft gluon evolution

General observable for n final state partons

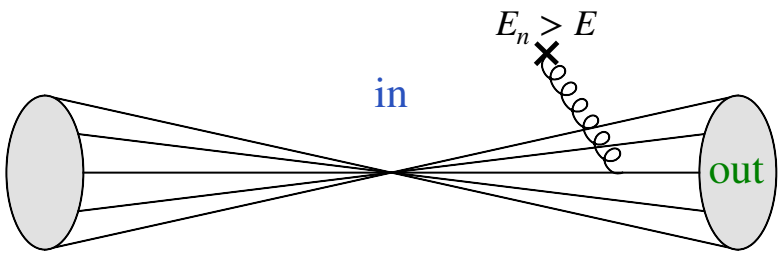
$$\sigma = \sum_{n\geq 0} \int \mathrm{Tr}\,[\mathbf{A}_n]\, \underbrace{u(q_1, ..., q_n)}_{\text{observable}}\, \overbrace{\mathrm{d}\Pi_n(q_1, ..q_n|Q)}^{\text{phase-space}}\,,$$

with the operator $\mathbf{A}_n$ ($\hat{=}\,|\mathcal{M}\rangle\langle\mathcal{M}|$) in colour space.

Measurement function for a **non-global observable** with resolution scale E:

$$u(q_1, ..., q_n) = u(q_1, ..., q_{n-1})\,[\Theta(q_n \in \Omega_{\text{in}})\theta(E - E_n) + \Theta(q_n \in \Omega_{\text{out}})\theta(E - E_n) + \Theta(q_n \in \Omega_{\text{out}})\theta(E_n - E)]\,.$$

Di-jet veto:



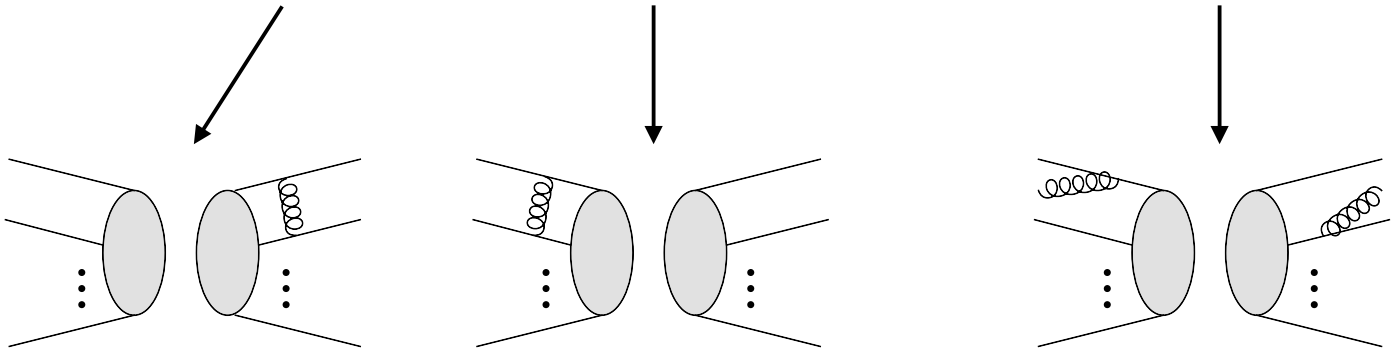
[Martínez, R. Á., De Angelis, M., Forshaw, J. R., Plätzer, S. & Seymour, M. H. Soft gluon evolution and non-global logarithms. *Journal of High Energy Physics* **2018** (2018)]

General observable for n final state partons

$$\sigma = \sum_{n\geq 0} \int \mathrm{Tr}\left[\mathbf{A}_n\right] \underbrace{u(q_1, ..., q_n)}_{\text{observable}} \overbrace{\mathrm{d}\Pi_n(q_1, ..q_n|Q)}^{\text{phase-space}} \ ,$$

Subtraction of IR divergences and renormalisation to absorb UV divergences lead to the **soft gluon evolution equations** in the resolution scale E **at leading order**:

$$E\frac{\partial}{\partial E}\mathbf{A}_n(E) = \mathbf{\Gamma}_n(E)\mathbf{A}_n(E) + \mathbf{A}_n(E)\mathbf{\Gamma}_n^\dagger(E) - \mathbf{D}_n\mathbf{A}_{n-1}(E)\mathbf{D}_n^\dagger E\delta(E - E_n) \ .$$



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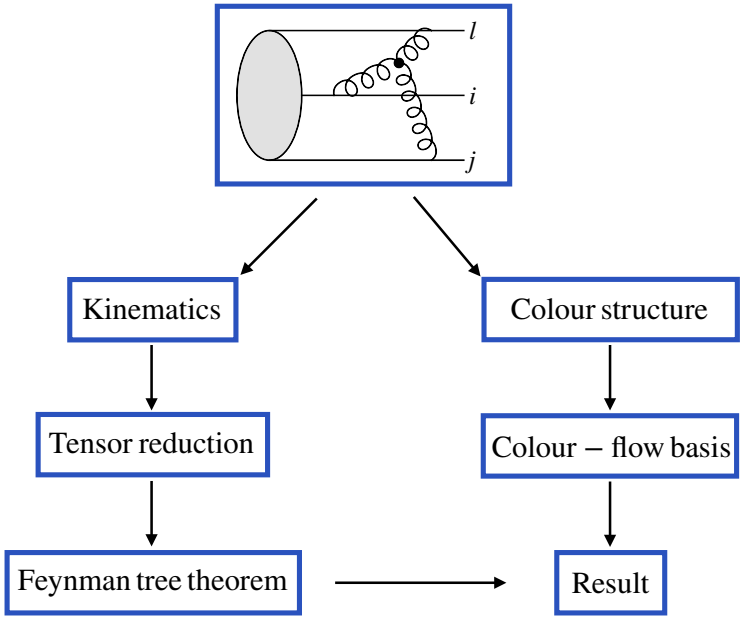
Ingredients for soft evolution at $\mathcal{O}(\alpha_s^2)$

Focus on virtual corrections at $\mathcal{O}(\alpha_s^2)$ in order to determine the soft anomalous dimension:

- Analyse colour structure and kinematic dependence separately, cast results into a form that can also be handled by a numerical code.
- Use the Feynman Tree theorem to express loop-integrals as phase-space type integrals.
- Basis of colour vector space: Colour-flow basis

  Leads to a physical interpretation of flow of colour in a process, which can be depicted by colour-line diagrams, basis states labelled by permutations of colour indices.



Colour structure at one-loop level

In a **basis-independent** form write the **one-loop anomalous dimension** as

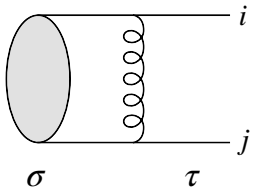
$$N\mathbf{\Gamma}^{(1)} = \frac{1}{2}\sum_{i\neq j}(\mathbf{T}_i\cdot\mathbf{T}_j)\,\Omega_{ij}^{(1)}\ .$$

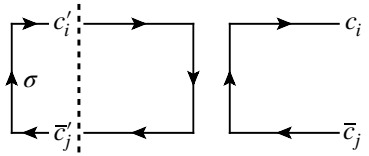
The colour-flow basis allows us to keep track of contributions suppressed in N by writing the soft anomalous dimension as an **expansion in 't Hooft coupling**:

$$[\tau|\mathbf{\Gamma}|\sigma\rangle = (\alpha_s N)\,[\tau|\mathbf{\Gamma}^{(1)}|\sigma\rangle + (\alpha_s N)^2\,[\tau|\mathbf{\Gamma}^{(2)}|\sigma\rangle + \ldots\ .$$

$$(\alpha_s N)[\tau|\mathbf{\Gamma}^{(1)}|\sigma\rangle = (\alpha_s N)\Bigg[\underbrace{\left(\Gamma_\sigma^{(1)} + \frac{1}{N^2}\rho^{(1)}\right)}_{\textit{diagonal elements}}\delta_{\sigma\tau} + \overbrace{\frac{1}{N}\Sigma_{\sigma\tau}^{(1)}}^{\textit{off-diagonal}}\Bigg]\ .$$

$\Gamma_\sigma^{(1)}$: diagonal terms, enhanced in N
$\rho^{(1)}$: diagonal terms, suppressed in N
$\Sigma_{\sigma\tau}^{(1)}$: off-diagonal elements, swaps of colour labels between the permutations σ and τ





[Plätzer, S. Summing large-N towers in colour flow evolution. *The European Physical Journal C* **74** (2014)]
[Martínez, R. Á., De Angelis, M., Forshaw, J. R., Plätzer, S. & Seymour, M. H. Soft gluon evolution and non-global logarithms. *Journal of High Energy Physics* **2018** (2018)]

Colour structure at two-loop level

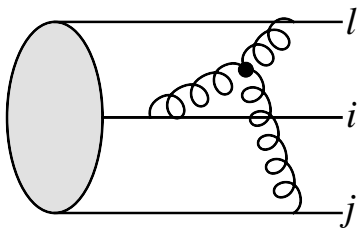
$$[\tau|\mathbf{\Gamma}|\sigma\rangle = (\alpha_s N)\,[\tau|\mathbf{\Gamma}^{(1)}|\sigma\rangle + (\alpha_s N)^2\,[\tau|\mathbf{\Gamma}^{(2)}|\sigma\rangle + \dots\,.$$

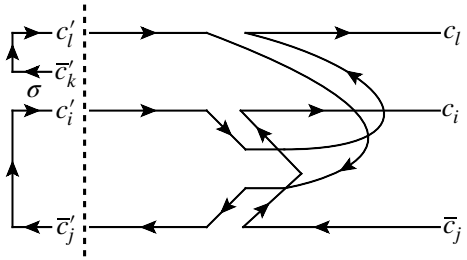
General structure of the **two-loop** soft anomalous dimension in the colour-flow basis:

$$(\alpha_s N)^2[\tau|\mathbf{\Gamma}^{(2)}|\sigma\rangle = (\alpha_s N)^2\Bigg[\left(\Gamma_\sigma^{(2)} + \frac{1}{N^2}\,(\rho_\sigma + \tilde{\rho}) + \frac{1}{N^4}\rho^{(2)}\right)\delta_{\sigma\tau}$$
$$+\,\text{off-diagonal contributions, “swap-terms”}\Bigg]\,.$$

first line: diagonal terms in a matrix representation of $\mathbf{\Gamma}^{(2)}$
second line: off-diagonal terms; “swap-terms”

Example of a contribution to the ρ_σ coefficient:





Kinematic dependence - Feynman tree theorem (FTT)

The FTT allows us to efficiently rewrite loop-integrals in terms of phase-space type integrals.

It instructs us to determine all single- and multiple-cut contributions and their permutations:

$$L^{(n)}(p_1, ..., p_n) = -\left[L^{(n)}_{1-\mathrm{cut}}(p_1, ..., p_n) + ... + L^{(n)}_{n-\mathrm{cut}}(p_1, ..., p_n)\right] \ ,$$

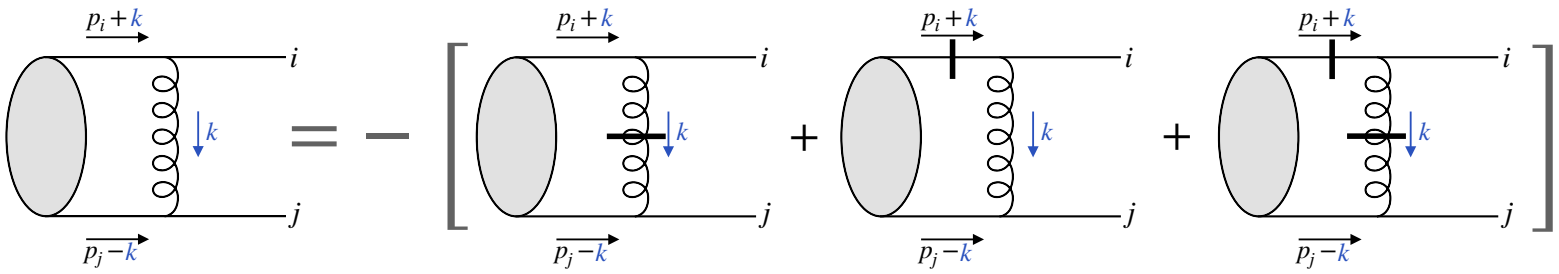
where

$$L^{(n)}_{r-\mathrm{cut}} = -i \int \frac{\bar{\mathrm{d}}^d k}{i\pi^{d/2}} \left[(2\pi i)^r \tilde{\delta}(k_1)...\tilde{\delta}(k_r) G(k_{r+1})...G(k_n) \right.$$
$$\left. + \text{unequal permutations}\right] \ .$$

[Catani, S., Gleisberg, T., Krauss, F., Rodrigo, G. & Winter, J.-C. From loops to trees by-passing Feynman's theorem. *Journal of High Energy Physics* **2008,** 065 (2008)]

In particular, it turns out that we need to **reformulate the FTT** for:

- **eikonal propagators** $\rightarrow$ if the pole lies in upper half-plane the propagator is not cut, e.g. for soft momentum k



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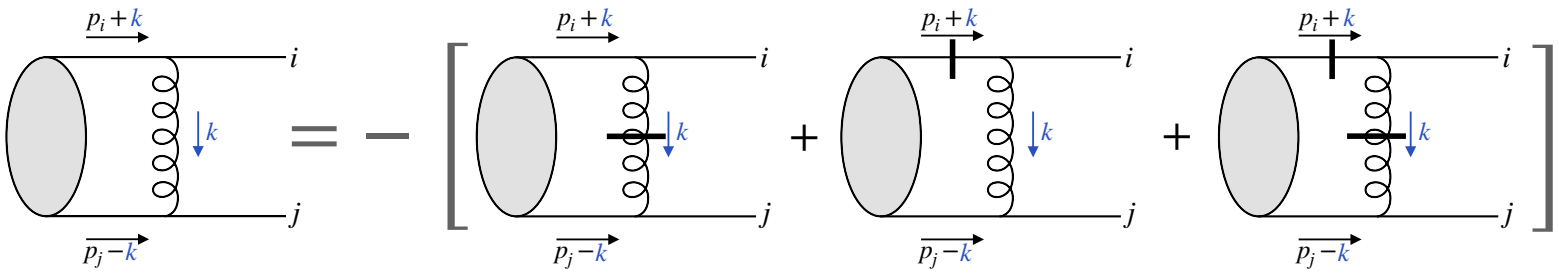
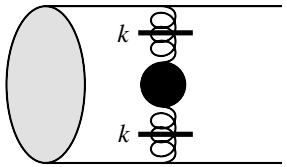


- **double propagators** $\rightarrow$ formulate a "derivative" identity in order to cut:



$$\frac{1}{[k^2 - i0(T\cdot k)|T\cdot k|]^2} - \frac{1}{[k^2 + i0(T\cdot k)^2]^2} = -2\pi i\delta'(k^2)\theta(T\cdot k)\ .$$

- **two-loop level integrals** $\rightarrow$ apply to one one-loop subintegral and then shift momentum.

Summary

- This analysis provides **building blocks** to:
  - building precise MC event generators since we are using methods of parton showers to address complex resummation procedures,
  - amplitude evolution and resummation of non-global logarithms at next-to-leading logarithmic order (tightly linked to previous point).
- Impact:
  - Conceptual: can now **determine imaginary part** of soft anomalous dimension at two-loop order; insight on **3-parton correlations**; uncovered the **cancellation of IR divergences** differentially in phase space.
  - Technical: gained expressions for the **colour structures**, which can now be included in an **evolution algorithm**.

Thank you!

Back-up: Colour-space formalism

Process with n_q quarks and anti-quarks and n_g gluons; decompose the amplitude into kinematic part and colour structure:

$$\begin{aligned}\mathcal{M}^{i_1\cdots i_{n_q}\ a_1\ldots a_{n_g}}_{\bar{i}_1\cdots\bar{i}_{n_q}}(\ldots) &= \sum_\sigma \mathcal{M}_\sigma(\ldots)\ C^{i_1\ldots i_{n_q}\ a_1\ldots a_{n_g}}_{\sigma\ \bar{i}_1\ldots\bar{i}_{n_q}} \\ &= \sum_\sigma \mathcal{M}_\sigma(\ldots)\,\langle c_1\ldots c_{2n_q+n_g}|\sigma\rangle \\ &= \langle c_1\ldots c_{2n_q+n_g}|\mathcal{M}_{2n_q+n_g}(\ldots)\rangle\ .\end{aligned}$$

We sum over the different orderings of generators which are absorbed into the colour factor C. In the last line use

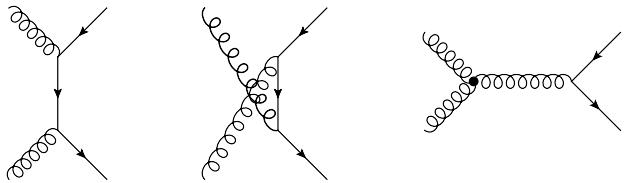
$$|\mathcal{M}_{\mathrm{N}}\rangle = \sum_\sigma \mathcal{M}_{\mathrm{N},\,\sigma}\,|\sigma\rangle\ ,$$

where $|\mathcal{M}_{\mathrm{N}}\rangle$ is a vector in a N-parton colour space.

[Catani, S. & Seymour, M. The Dipole formalism for the calculation of QCD jet cross-sections at next-to-leading order. *Phys. Lett. B* **378,** 287–301 (1996)]

Back-up: Colour-space formalism - Example

$$\mathcal{M}^{i}{}_{\bar{i}}{}^{a_1 a_2}(\ldots) = \mathcal{M}_1(\ldots)(t^{a_1} t^{a_2})^{i}{}_{\bar{i}} + \mathcal{M}_2(\ldots)(t^{a_2} t^{a_1})^{i}{}_{\bar{i}}$$
$$= \sum_{\sigma} \mathcal{M}_{\sigma}(\ldots) C_{\sigma}{}^{i}{}_{\bar{i}}{}^{a_1 a_2} = \langle i\bar{i}\, a_1 a_2 | \mathcal{M}_4(\ldots)\rangle$$



where the sum over σ is a sum over the **different orderings of the generators**. The amplitude vector for a N-parton colour space is defined by

$$|\mathcal{M}_{\mathrm{N}}\rangle = \sum_{\sigma} \mathcal{M}_{\mathrm{N},\,\sigma}\, |\sigma\rangle \ .$$

The property of colour conservation can be written as

$$e^{i\sum_i \theta_a T_i^a}\, |\mathcal{M}\rangle = |\mathcal{M}\rangle \rightarrow \sum_{i=1}^{n} \mathbf{T}_i\, |\mathcal{M}(p_1, \ldots, p_n)\rangle = 0 \ .$$

The colour charge operators are defined by

$$\langle c_1 \ldots c_i \ldots c_n |\, \mathbf{T}_i^a | d_1 \ldots d_i \ldots d_n \rangle = \delta_{d_1}^{c_1} \ldots (T^a)^{c_i}{}_{d_i} \ldots \delta_{d_n}^{c_n} \ ,$$

where for a quark $(T^a)^{c_i}{}_{d_i} = (t^a)^{c_i}{}_{d_i}$, an antiquark $(T^a)^{c_i}{}_{d_i} = (\overline{t^a})^{c_i}{}_{d_i} = -(t^a)^{c_i}{}_{d_i}$ or a gluon $(T^a)^{c_i}{}_{d_i} = i f^{a c_i}{}_{d_i}$ holds.

[Catani, S. & Grazzini, M. The soft-gluon current at one-loop order. *Nuclear Physics B* **591** (Dec. 2000)]

Back-up: Colour-flow basis I

Colour-flow basis states are labelled by **permutations of colour indices**

$$|\sigma\rangle = \left| \begin{array}{ccc} 1 & \dots & n \\ \sigma(1) & \dots & \sigma(n) \end{array} \right\rangle .$$

We obtain the colour-flow Kronecker deltas by acting with a basis (with a certain ordering of the labels) onto the colour-flow basis

$$\langle c|\sigma\rangle = \left\langle \begin{array}{ccc} 1 & \dots & n \\ 1 & \dots & n \end{array} \middle| \begin{array}{ccc} 1 & \dots & n \\ \sigma(1) & \dots & \sigma(n) \end{array} \right\rangle = \delta^{\alpha_1}_{\beta_{\sigma(1)}} \dots \delta^{\alpha_n}_{\beta_{\sigma(n)}} .$$

The colour-flow basis is **not orthogonal**

$$S_{\sigma\tau} = \langle\sigma|\tau\rangle = \delta^{\alpha_1}_{\beta_{\tau(1)}} \dots \delta^{\alpha_n}_{\beta_{\tau(n)}} \delta^{\beta_{\sigma(1)}}_{\alpha_1} \dots \delta^{\beta_{\sigma(n)}}_{\alpha_n} = N^{n-\#\text{transpositions}} .$$

Therefore we can introduce "square bracket notation", such that we can define

$$\sum_\sigma |\sigma\rangle\,[\sigma| = \sum_\sigma |\sigma]\,\langle\sigma| = \mathbb{1} \qquad \langle\sigma|\tau] = [\tau|\sigma\rangle = \delta_{\sigma\tau} .$$

[Plätzer, S. Summing large-N towers in colour flow evolution. *The European Physical Journal C* **74** (2014)]
[Martínez, R. Á., De Angelis, M., Forshaw, J. R., Plätzer, S. & Seymour, M. H. Soft gluon evolution and non-global logarithms. *Journal of High Energy Physics* **2018** (2018)]

Back-up: Colour-flow basis II

A basis-independent object/an operator can be written as

$$\mathbf{A} = \sum_{\sigma,\tau} \mathcal{A}_{\sigma\tau} \left|\tau\right\rangle \left\langle\sigma\right| = \sum_{\sigma,\tau} \left[\tau|\mathbf{A}|\sigma\right] \left|\tau\right\rangle \left\langle\sigma\right| \ ,$$

and a **trace of an operator** is given by

$$\mathrm{Tr}[\mathbf{A}] = \sum_{\tau} \left[\tau|\mathbf{A}|\tau\right\rangle = \sum_{\sigma,\tau} \left[\tau|\mathbf{A}|\sigma\right] \left\langle\sigma|\tau\right\rangle = \mathrm{Tr}[\mathcal{A}S] = \sum_{\sigma,\tau} \mathcal{A}_{\tau\sigma} S_{\sigma\tau} \ .$$

Therefore, for the **amplitude squared** we have

$$|\mathcal{M}|^2 = \left\langle\mathcal{M}|\mathcal{M}\right\rangle = \mathrm{Tr}\left[\left|\mathcal{M}\right\rangle \left\langle\mathcal{M}\right|\right] = \sum_{\sigma,\tau} \mathcal{M}_\sigma \mathcal{M}^*_\tau \left\langle\sigma|\tau\right\rangle = \mathrm{Tr}[\mathcal{M}S] \ .$$

In the definitions above we have used the "square bracket" vectors, one may as well write

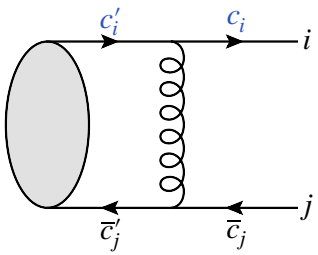
$$\sum_{\sigma,\tau} \left|\tau\right\rangle \left\langle\sigma\right| = S_{\tau\sigma} \rightarrow \sum_{\sigma,\tau} S^{-1}_{\tau\sigma} \left|\tau\right\rangle \left\langle\sigma\right| = \mathbb{1} \ ,$$

and thus,

$$\mathbf{A} \left|\sigma\right\rangle = \sum_{\rho,\lambda} S^{-1}_{\rho\lambda} \left|\rho\right\rangle \left\langle\lambda|\mathbf{A}|\sigma\right\rangle = \sum_{\rho,\lambda,\tau} S^{-1}_{\rho\lambda} S_{\lambda\tau} \mathcal{A}_{\tau\sigma} \left|\rho\right\rangle = \sum_{\tau} \mathcal{A}_{\tau\sigma} \left|\tau\right\rangle \ .$$

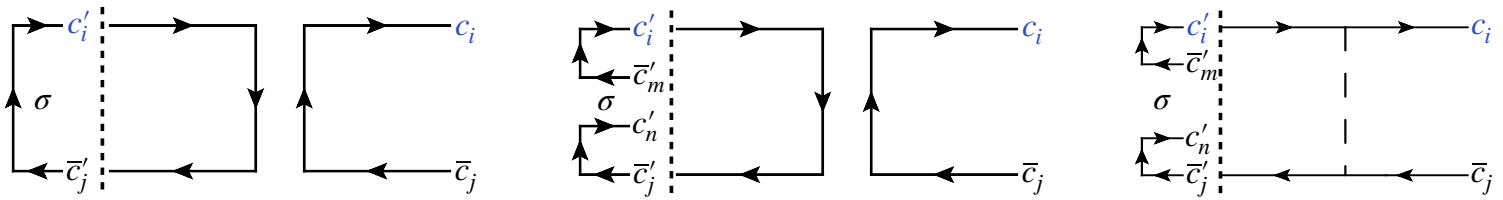
[Plätzer, S. Summing large-N towers in colour flow evolution. *The European Physical Journal C* **74** (2014)]

Consider the virtual exchange for a **quark line** i and and **anti-quark line** j:



$$\rightarrow -(t^a)^{c_i}{}_{c_i'}(t^a)^{\bar{c}_j'}{}_{\bar{c}_j} = -T_R\left[\delta^{c_i}_{\bar{c}_j}\delta^{\bar{c}_j'}_{c_i'} - \frac{1}{N}\delta^{c_i}_{c_i'}\delta^{\bar{c}_j'}_{\bar{c}_j}\right]$$

This leads to the following colour flows:



$$\underbrace{\delta^{c_i'}_{\bar{c}_j'}\quad \delta^{\bar{c}_j'}_{c_i'}\delta^{c_i}_{\bar{c}_j}}_{\rightarrow\Gamma^{(1)}_{\sigma}} \qquad \underbrace{\delta^{c_i'}_{\bar{c}_m'}\delta^{c_n'}_{\bar{c}_j'}\quad \delta^{c_i}_{\bar{c}_j}\delta^{\bar{c}_j'}_{c_i'}}_{\rightarrow\Sigma^{(1)}_{\sigma\tau}} \qquad \frac{1}{N}\underbrace{\delta^{c_i'}_{\bar{c}_m'}\delta^{c_n'}_{\bar{c}_j'}\quad \delta^{c_i}_{c_i'}\delta^{\bar{c}_j'}_{\bar{c}_j}}_{\rightarrow\rho^{(1)}}$$

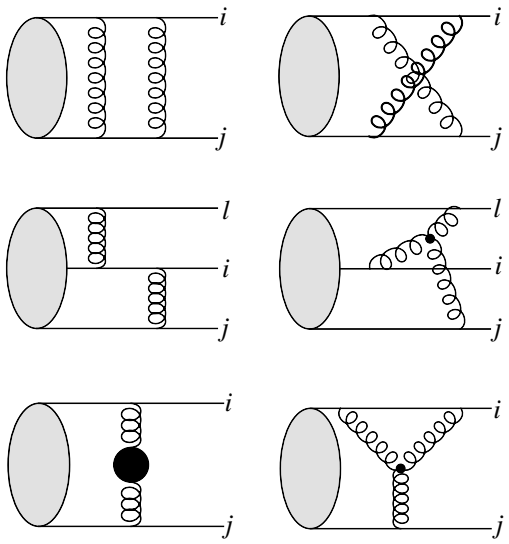
$$\boxed{(\alpha_s N)[\tau|\mathbf{\Gamma}^{(1)}|\sigma\rangle = (\alpha_s N)\left[\left(\Gamma^{(1)}_{\sigma} + \frac{1}{N^2}\rho^{(1)}\right)\delta_{\sigma\tau} + \frac{1}{N}\Sigma^{(1)}_{\sigma\tau}\right]}$$

[Plätzer, S. Summing large-N towers in colour flow evolution. *The European Physical Journal C* **74** (2014)]

Back-up: Colour structure at two-loop level

Two-loop soft anomalous dimension in a basis-independent form:

$$\begin{aligned}\boldsymbol{\Gamma}^{(2)} = \frac{1}{2}\sum_{i\neq j}\left[(\mathbf{T}_i\cdot\mathbf{T}_j)(\mathbf{T}_i\cdot\mathbf{T}_j)\,\Omega^{(2)}_{ij} + \mathbf{T}^a_i\mathbf{T}^b_i\mathbf{T}^b_j\mathbf{T}^a_j\,\tilde{\Omega}^{(2)}_{ij}\right]\\ + \sum_{i\neq j\neq l}\left[(\mathbf{T}_i\cdot\mathbf{T}_j)(\mathbf{T}_i\cdot\mathbf{T}_l)\,\Omega^{(2)}_{ijl} + \frac{1}{2}f^{abc}\mathbf{T}^a_i\mathbf{T}^b_j\mathbf{T}^c_l\,\hat{\Omega}^{(2)}_{ijl}\right]\\ + \sum_{i\neq j} T_R(\mathbf{T}_i\cdot\mathbf{T}_j)\left[\frac{1}{2}\Omega^{(2)}_{ij,\mathrm{self-en.}} + \Omega^{(2)}_{ij,\mathrm{vertex-corr.}}\right]\,.\end{aligned}$$



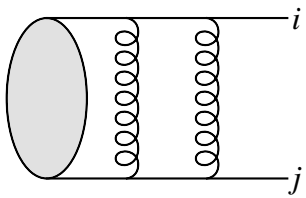
Back-up: Colour structure at two-loop level - Examples I

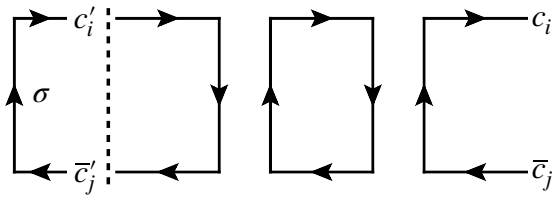
General structure of the **two-loop** soft anomalous dimension in the colour-flow basis:

$$(\alpha_s N)^2 [\tau|\mathbf{\Gamma}^{(2)}|\sigma\rangle = (\alpha_s N)^2 \left[\left(\Gamma_\sigma^{(2)} + \frac{1}{N^2}(\rho_\sigma + \tilde{\rho}) + \frac{1}{N^4}\rho^{(2)} \right) \delta_{\sigma\tau} \right.$$
$$\left. + \frac{1}{N}\left(\Sigma_{\sigma\tau}^{(2)} + \hat{\Sigma}_{\sigma\tau}^{(2)} \right) + \frac{1}{N^3}\tilde{\Sigma}_{\sigma\tau}^{(2)} + \frac{1}{N^2}\left(\Sigma'^{(2)}_{\sigma\tau} + \Sigma''^{(2)}_{\sigma\tau} \right) \right] .$$

first line: diagonal terms in a matrix representation of $\mathbf{\Gamma}^{(2)}$
second line: off-diagonal terms; "swap-terms"

Example of a contribution to the $\Gamma_\sigma^{(2)}$ coefficient:





[Plätzer, S. & Ruffa, I. Towards Colour Flow Evolution at Two Loops. *JHEP* **06**, 007 (2021)]

Back-up: Colour structure at two-loop level - Examples II

General structure of the **two-loop** soft anomalous dimension in the colour-flow basis:

$$(\alpha_s N)^2 [\tau|\mathbf{\Gamma}^{(2)}|\sigma\rangle = (\alpha_s N)^2 \left[\left(\Gamma_\sigma^{(2)} + \frac{1}{N^2}(\rho_\sigma + \tilde{\rho}) + \frac{1}{N^4}\rho^{(2)} \right) \delta_{\sigma\tau} \right.$$
$$\left. + \frac{1}{N}\left(\Sigma_{\sigma\tau}^{(2)} + \hat{\Sigma}_{\sigma\tau}^{(2)} \right) + \frac{1}{N^3}\tilde{\Sigma}_{\sigma\tau}^{(2)} + \frac{1}{N^2}\left(\Sigma'^{(2)}_{\sigma\tau} + \Sigma''^{(2)}_{\sigma\tau} \right) \right] .$$

first line: diagonal terms in a matrix representation of $\mathbf{\Gamma}^{(2)}$
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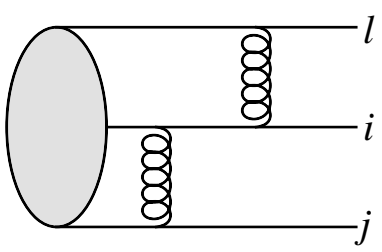
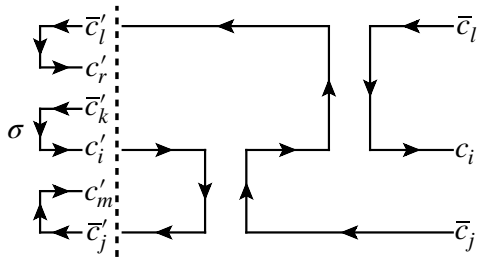
Example of a contribution to the $\Sigma'^{(2)}_{\sigma\tau}$ coefficient:





[Plätzer, S. & Ruffa, I. Towards Colour Flow Evolution at Two Loops. *JHEP* **06**, 007 (2021)]

Example for two-loop matrix element in colour-flow basis:

$$
\begin{aligned}
[\tau|\mathbf{T}_g\mathbf{T}_l\mathbf{T}_i\mathbf{T}_j|\sigma\rangle = \sqrt{T_R}N^2\delta_{\sigma\tau}\Bigg[&\frac{1}{N^2}\Big((\lambda_l-\bar{\lambda}_l)\Big(\lambda_i\bar{\lambda}_j\delta_{c_i\sigma^{-1}(\bar{c}_j)}-\lambda_j\bar{\lambda}_i\delta_{c_j\sigma^{-1}(\bar{c}_i)}\Big)\\
&+(\lambda_j-\bar{\lambda}_j)\left(\bar{\lambda}_i\lambda_l\delta_{c_l\sigma^{-1}(\bar{c}_i)}-\lambda_i\bar{\lambda}_l\delta_{c_i\sigma^{-1}(\bar{c}_l)}\right)\\
&+(\lambda_i-\bar{\lambda}_i)\Big(\bar{\lambda}_l\lambda_j\delta_{c_j\sigma^{-1}(\bar{c}_l)}-\bar{\lambda}_j\lambda_l\delta_{c_l\sigma^{-1}(\bar{c}_j)}\Big)\Big)\Bigg]\\
&+\sum_{(a,b)}\delta_{\sigma\tau_{(a,b)}}N\sqrt{T_R}\\
&\Big[\lambda_i\bar{\lambda}_j\lambda_l\delta_{(a,b)(c_i,c_l)}\Big(\delta_{c_l\sigma^{-1}(\bar{c}_j)}-\delta_{c_i\sigma^{-1}(\bar{c}_j)}\Big)\\
&+\bar{\lambda}_i\lambda_j\lambda_l\delta_{(a,b)(c_j,c_l)}\Big(\delta_{c_j\sigma^{-1}(\bar{c}_i)}-\delta_{c_l\sigma^{-1}(\bar{c}_i)}\Big)\\
&+\lambda_i\lambda_j\bar{\lambda}_l\delta_{(a,b)(c_i,c_j)}\Big(\delta_{c_i\sigma^{-1}(\bar{c}_l)}-\delta_{c_j\sigma^{-1}(\bar{c}_l)}\Big)\\
&+\lambda_i\bar{\lambda}_j\bar{\lambda}_l\Big(\delta_{(a,b)(c_i,\sigma^{-1}(\bar{c}_l))}\delta_{c_i\sigma^{-1}(\bar{c}_j)}-\delta_{(a,b)(c_i,\sigma^{-1}(\bar{c}_j))}\delta_{c_i\sigma^{-1}(\bar{c}_l)}\Big)\\
&+\bar{\lambda}_i\lambda_j\bar{\lambda}_l\Big(\delta_{(a,b)(\sigma^{-1}(\bar{c}_i),c_j)}\delta_{c_j\sigma^{-1}(\bar{c}_l)}-\delta_{(a,b)(c_j,\sigma^{-1}(\bar{c}_l))}\delta_{c_j\sigma^{-1}(\bar{c}_i)}\Big)\\
&+\bar{\lambda}_i\bar{\lambda}_j\lambda_l\Big(\delta_{(a,b)(\sigma^{-1}(\bar{c}_j),c_l)}\delta_{c_l\sigma^{-1}(\bar{c}_i)}-\delta_{(a,b)(\sigma^{-1}(\bar{c}_i),c_l)}\delta_{c_l\sigma^{-1}(\bar{c}_j)}\Big)\Big]\\
&+\sum_{(a,b)}\sum_{(b,c)}\delta_{\sigma\tau_{(a,b)(b,c)}}\sqrt{T_R}\\
&\Big[\lambda_i\lambda_j\delta_{(a,b)(c_i,c_j)}\Big(\lambda_l\delta_{(b,c)(c_j,c_l)}-\bar{\lambda}_l\delta_{(b,c)(c_j,\sigma^{-1}(\bar{c}_l))}\Big)\\
&-\lambda_i\lambda_l\delta_{(a,b)(c_i,c_l)}\Big(\lambda_j\delta_{(b,c)(c_l,c_j)}-\bar{\lambda}_j\delta_{(b,c)(c_l,\sigma^{-1}(\bar{c}_j))}\Big)\\
&-\lambda_i\bar{\lambda}_j\delta_{(a,b)(c_i,\sigma^{-1}(\bar{c}_j))}\Big(\lambda_l\delta_{(b,c)(\sigma^{-1}(\bar{c}_j),c_l)}-\bar{\lambda}_l\delta_{(b,c)(\sigma^{-1}(\bar{c}_j),\sigma^{-1}(\bar{c}_l))}\Big)\\
&+\bar{\lambda}_i\lambda_l\delta_{(a,b)(\sigma^{-1}(\bar{c}_i),c_l)}\Big(\lambda_j\delta_{(b,c)(c_l,c_j)}-\bar{\lambda}_j\delta_{(b,c)(c_l,\sigma^{-1}(\bar{c}_j))}\Big)\\
&-\bar{\lambda}_i\lambda_j\delta_{(a,b)(\sigma^{-1}(\bar{c}_i),c_j)}\Big(\lambda_l\delta_{(b,c)(c_j,c_l)}-\bar{\lambda}_l\delta_{(b,c)(c_j,\sigma^{-1}(\bar{c}_l))}\Big)\\
&-\bar{\lambda}_i\bar{\lambda}_l\delta_{(a,b)(\sigma^{-1}(\bar{c}_i),\sigma^{-1}(\bar{c}_l))}\Big(\lambda_j\delta_{(b,c)(\sigma^{-1}(\bar{c}_l),c_j)}-\bar{\lambda}_j\delta_{(b,c)(\sigma^{-1}(\bar{c}_l),\sigma^{-1}(\bar{c}_j))}\Big)\\
&+\lambda_i\bar{\lambda}_l\delta_{(a,b)(c_i,\sigma^{-1}(\bar{c}_l))}\Big(\lambda_j\delta_{(b,c)(\sigma^{-1}(\bar{c}_l),c_j)}-\bar{\lambda}_j\delta_{(b,c)(\sigma^{-1}(\bar{c}_l),\sigma^{-1}(\bar{c}_j))}\Big)\\
&+\bar{\lambda}_i\bar{\lambda}_j\delta_{(a,b)(\sigma^{-1}(\bar{c}_i),\sigma^{-1}(\bar{c}_j))}\Big(\lambda_l\delta_{(b,c)(\sigma^{-1}(\bar{c}_j),c_l)}-\bar{\lambda}_l\delta_{(b,c)(\sigma^{-1}(\bar{c}_j),\sigma^{-1}(\bar{c}_l))}\Big)\Big]
\end{aligned}
$$

Back-up: Feynman tree theorem

The FTT is based on the identity

$$G_A(k) = G(k) + 2\pi i \delta(k^2)\theta(T \cdot k) \ ,$$

which equals to

$$\frac{1}{k^2 - i0|T \cdot k|(T \cdot k)} = \frac{1}{k^2 + i0(T \cdot k)^2} + 2\pi i \, \delta(k^2)\theta(T \cdot k) \ ,$$

where we have introduced the timelike vector $(T^\mu) = (\sqrt{2}, \vec{0})$. For the loop integral with advanced propagators only we can write

$$L_A^{(n)}(p_1, ..., p_n) = -i \int \frac{đ^d k}{i\pi^{d/2}} \prod_{m=1}^{n} G_A(k_m) = 0 \ .$$

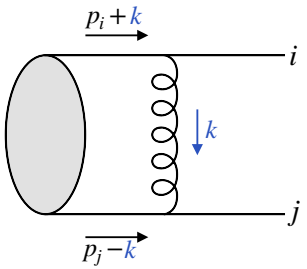
Plugging in, we find

$$\begin{aligned} L_A^{(n)}(p_1, ..., p_n) &= -i \int \frac{đ^d k}{i\pi^{d/2}} \prod_{m=1}^{n} \Big[G(k_m) + 2\pi i \, \tilde{\delta}(k) \Big] \\ &= L^{(n)}(p_1, ..., p_n) + L^{(n)}_{1-\mathrm{cut}}(p_1, ..., p_n) + ... + L^{(n)}_{\mathrm{n-cut}}(p_1, ..., p_n) \ . \end{aligned}$$

[Catani, S., Gleisberg, T., Krauss, F., Rodrigo, G. & Winter, J.-C. From loops to trees by-passing Feynman's theorem. *Journal of High Energy Physics* **2008,** 065 (2008)]

Back-up: Eikonal Propagators

Example for one-loop diagram with soft momentum k



$$0 = \mu^{2\varepsilon} \int \frac{đ^d k}{i\pi^{d/2}} G_A(k) \frac{1}{[2p_i \cdot k - 2i0(p_i^0)^2]} \frac{1}{[-2p_j \cdot k + 2i0(p_j^0)^2]} \ ,$$

which leads to

$$0 = \mu^{2\varepsilon} \int \frac{đ^d k}{i\pi^{d/2}} \left(\frac{1}{[k^2 + 2i0(k^0)^2]} + (2\pi i)\delta(k^2)\theta(k^0) \right) \frac{1}{[-2p_j \cdot k + 2i0(p_j^0)^2]}$$
$$\left(\frac{1}{[2p_i \cdot k + 2i0(p_i^0)^2]} + (2\pi i)\delta(2p_i \cdot k) \right) \ .$$

Back-up: Derivative Rule for double propagators

Start with the identity

$$\frac{1}{k^2 - i0(T\cdot k)^2} = \frac{1}{k^2 + i0(T\cdot k)^2} + 2\pi i\delta(k^2)\ ,$$

where $(T^\mu) = (\sqrt{2}, \vec{0})$ and take the derivative with respect to component of loop momentum

$$\frac{k^\mu - i0\ T^\mu(T\cdot k)}{[k^2 - i0(T\cdot k)]^2} - \frac{k^\mu + i0\ T^\mu(T\cdot k)}{[k^2 + i0(T\cdot k)^2]^2} = -i\pi\frac{\partial}{\partial k^\mu}\delta(k^2)\ .$$

Using the relation $\frac{\partial f(k^2)}{\partial k^\mu} = 2k^\mu\frac{\partial f(k^2)}{\partial k^2}$ and some space-like vector $(S\cdot T = 0)$ to contract in order to get rid of the T-dependent numerator structure one finds

$$\boxed{\frac{1}{[k^2 - i0(T\cdot k)^2]^2} - \frac{1}{[k^2 + i0(T\cdot k)^2]^2} = -2\pi i\,\delta'(k^2)\ .}$$

[Plätzer, S. & Ruffa, I. Towards Colour Flow Evolution at Two Loops. *JHEP* **06,** 007 (2021)]

Back-up: FTT at two-loop level - An example