Towards Colour Flow Evolution at $\mathcal{O}(\alpha_s^2)$

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Soft gluon evolution


\[ \cdots \]

- Formulate evolution equations which allow us to **resum large logarithms of soft origin**.

- The evolution equations are formulated in **colour space**. Treat the amplitude as a component of a vector in an abstract N-parton colour vector space

\[ |M_N \rangle = \sum_\sigma M_{N, \sigma} |\sigma \rangle . \]

Soft gluon evolution

General observable for \( n \) final state partons

\[
\sigma = \sum_{n \geq 0} \int \text{Tr} [A_n] u(q_1, ..., q_n) \left( \int \text{observable phase-space} \right) d\Pi_n(q_1, ..., q_n|Q),
\]

with the operator \( A_n (\hat{=} |\mathcal{M}\rangle \langle \mathcal{M}|) \) in colour space.

Measurement function for a non-global observable with resolution scale \( E \):

\[
u(q_1, ..., q_n) = u(q_1, ..., q_{n-1}) \left[ \Theta(q_n \in \Omega_{in}) \theta(E - E_n) + \Theta(q_n \in \Omega_{out}) \theta(E - E_n) + \Theta(q_n \in \Omega_{out}) \theta(E_n - E) \right].
\]

Di-jet veto:

General observable for $n$ final state partons

$$
\sigma = \sum_{n \geq 0} \int \text{Tr} [A_n] \frac{u(q_1, \ldots, q_n)}{\text{phase-space}} \frac{d\Pi_n(q_1, \ldots q_n | Q)}{\text{observable}},
$$

Subtraction of IR divergences and renormalisation to absorb UV divergences lead to the **soft gluon evolution equations** in the resolution scale $E$ at leading order:

$$
E \frac{\partial}{\partial E} A_n(E) = \Gamma_n(E) A_n(E) + A_n(E) \Gamma_n^\dagger(E) - D_n A_{n-1}(E) D_n^\dagger E \delta(E - E_n).
$$

Ingredients for soft evolution at $\mathcal{O}(\alpha_s^2)$

Focus on virtual corrections at $\mathcal{O}(\alpha_s^2)$ in order to determine the soft anomalous dimension:

- Analyse **colour structure** and **kinematic dependence separately**, cast results into a form that can also be handled by a numerical code.

- Use the **Feynman Tree theorem** to express loop-integrals as phase-space type integrals.

- Basis of colour vector space: **Colour-flow basis**
  Leads to a physical interpretation of flow of colour in a process, which can be depicted by colour-line diagrams, **basis states** labelled by permutations of colour indices.
Colour structure at one-loop level

In a **basis-independent** form write the **one-loop anomalous dimension** as

\[ N \Gamma^{(1)} = \frac{1}{2} \sum_{i \neq j} (T_i \cdot T_j) \Omega_{ij}^{(1)}. \]

The colour-flow basis allows us to keep track of contributions suppressed in \( N \) by writing the soft anomalous dimension as an **expansion in 't Hooft coupling**:

\[
[\tau|\Gamma|\sigma] = (\alpha_s N) [\tau|\Gamma^{(1)}|\sigma] + (\alpha_s N)^2 [\tau|\Gamma^{(2)}|\sigma] + \ldots.
\]

\[
(\alpha_s N)[\tau|\Gamma^{(1)}|\sigma] = (\alpha_s N) \left[ \left( \Gamma^{(1)}_{\sigma} + \frac{1}{N^2} \rho^{(1)} \right) \delta_{\sigma\tau} + \frac{1}{N} \Sigma^{(1)}_{\sigma\tau} \right].
\]

\( \Gamma^{(1)}_{\sigma} \): diagonal terms, enhanced in \( N \)

\( \rho^{(1)} \): diagonal terms, suppressed in \( N \)

\( \Sigma^{(1)}_{\sigma\tau} \): off-diagonal elements, swaps of colour labels between the permutations \( \sigma \) and \( \tau \)

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[Plätzer, S. Summing large-N towers in colour flow evolution. *The European Physical Journal C* 74 (2014)]

Colour structure at two-loop level

\[
[\tau | \Gamma | \sigma \rangle = (\alpha_s N) [\tau | \Gamma^{(1)} | \sigma \rangle + (\alpha_s N)^2 [\tau | \Gamma^{(2)} | \sigma \rangle + ... .
\]

General structure of the two-loop soft anomalous dimension in the colour-flow basis:

\[
(\alpha_s N)^2 [\tau | \Gamma^{(2)} | \sigma \rangle = (\alpha_s N)^2 \left[ \left( \Gamma^{(2)}_{\sigma} + \frac{1}{N^2} (\rho_\sigma + \tilde{\rho}) + \frac{1}{N^4} \rho^{(2)} \right) \delta_{\sigma \tau} + \text{off-diagonal contributions, \textquotedblleft swap-terms	extquotedblright} \right].
\]

first line: diagonal terms in a matrix representation of \( \Gamma^{(2)} \)
second line: off-diagonal terms; \textquotedblleft swap-terms\textquotedblright

Example of a contribution to the \( \rho_\sigma \) coefficient:

Kinematic dependence - Feynman tree theorem (FTT)

The FTT allows us to efficiently rewrite loop-integrals in terms of phase-space type integrals.

It instructs us to determine all single- and multiple-cut contributions and their permutations:

\[
L^{(n)}(p_1, ..., p_n) = - \left[ L^{(n)}_{1-\text{cut}}(p_1, ..., p_n) + ... + L^{(n)}_{n-\text{cut}}(p_1, ..., p_n) \right],
\]

where

\[
L^{(n)}_{r-\text{cut}} = -i \int \frac{d^d k}{i \pi^{d/2}} \left[ (2\pi i)^r \tilde{\delta}(k_1)...\tilde{\delta}(k_r) G(k_{r+1})...G(k_n) + \text{unequal permutations} \right].
\]

In particular, it turns out that we need to **reformulate the FTT** for:

- **eikonal propagators** → if the pole lies in upper half-plane the propagator is not cut, e.g. for soft momentum $k$

$$p_i + k \quad i$$

$$p_j - k \quad j$$

$$p_i + k \quad i$$

$$p_j - k \quad j$$

$$p_i + k \quad i$$

$$p_j - k \quad j$$

$$p_i + k \quad i$$

$$p_j - k \quad j$$
In particular, it turns out that we need to **reformulate the FTT** for:

- **eikonal propagators** → if the pole lies in upper half-plane the propagator is not cut, e.g. for soft momentum \( k \)

  ![Diagram of eikonal propagators]

- **double propagators** → formulate a “derivative” identity in order to cut:

  \[
  \frac{1}{[k^2 - i0(T \cdot k)|T \cdot k|]^2} - \frac{1}{[k^2 + i0(T \cdot k)^2]^2} = -2\pi i \delta'(k^2) \theta(T \cdot k).
  \]

- **two-loop level integrals** → apply to one one-loop subintegral and then shift momentum.
Summary

- This analysis provides **building blocks** to:
  - building precise MC event generators since we are using methods of parton showers to address complex resummation procedures,
  - amplitude evolution and resummation of non-global logarithms at next-to-leading logarithmic order (tightly linked to previous point).

- **Impact:**
  - Conceptual: can now determine imaginary part of soft anomalous dimension at two-loop order; insight on **3-parton correlations**; uncovered the **cancellation of IR divergences** differentially in phase space.
  - Technical: gained expressions for the **colour structures**, which can now be included in an **evolution algorithm**.
Thank you!
Back-up: Colour-space formalism

Process with \( n_q \) quarks and anti-quarks and \( n_g \) gluons; decompose the amplitude into kinematic part and colour structure:

\[
\mathcal{M}_{i_1 \ldots i_{nq} \bar{i}_1 \ldots \bar{i}_{nq}}^{a_1 \ldots a_{ng}} (...) = \sum_{\sigma} \mathcal{M}_{\sigma} (...) C_{\sigma}^{i_1 \ldots i_{nq} a_1 \ldots a_{ng}} \bar{i}_1 \ldots \bar{i}_{nq}
\]

\[
= \sum_{\sigma} \mathcal{M}_{\sigma} (...) \langle c_1 \ldots c_{2n_q+n_g} \mid \sigma \rangle
\]

\[
= \langle c_1 \ldots c_{2n_q+n_g} \mid \mathcal{M}_{2n_q+n_g} (...) \rangle .
\]

We sum over the different orderings of generators which are absorbed into the colour factor \( C \). In the last line use

\[
|\mathcal{M}_N\rangle = \sum_{\sigma} \mathcal{M}_{N, \sigma} \mid \sigma \rangle ,
\]

where \( |\mathcal{M}_N\rangle \) is a vector in a \( N \)-parton colour space.

Back-up: Colour-space formalism - Example

\[ \mathcal{M}^{i_1a_1a_2}_i(...) = \mathcal{M}_1(...) (t^{a_1} t^{a_2})^{i_i} + \mathcal{M}_2(...) (t^{a_2} t^{a_1})^{i_i} \]

\[ = \sum_\sigma \mathcal{M}_\sigma(...) C^{i_1a_1a_2}_{\sigma_i} = \langle i_i a_1 a_2 | \mathcal{M}_4(...) \rangle \]

where the sum over \( \sigma \) is a sum over the different orderings of the generators. The amplitude vector for a N-parton colour space is defined by

\[ |\mathcal{M}_N\rangle = \sum_\sigma \mathcal{M}_{N,\sigma} |\sigma\rangle . \]

The property of colour conservation can be written as

\[ e^{i \sum_i \theta^a T^a_i} |\mathcal{M}\rangle = |\mathcal{M}\rangle \rightarrow \sum_{i=1}^n T_i |\mathcal{M}(p_1, ..., p_n)\rangle = 0 . \]

The colour charge operators are defined by

\[ \langle c_1...c_i...c_n | T^a_i | d_1...d_i...d_n \rangle = \delta^{c_1}_{d_1}...(T^a)^{c_i}_{d_i}...\delta^{c_n}_{d_n} , \]

where for a quark \( (T^a)^{c_i}_{d_i} = (t^a)^{c_i}_{d_i} \), an antiquark \( (T^a)^{c_i}_{d_i} = (\bar{t}^a)^{c_i}_{d_i} = -(t^a)^{c_i}_{d_i} \) or a gluon \( (T^a)^{c_i}_{d_i} = if^{aci}_{d_i} \) holds.

Back-up: Colour-flow basis I

Colour-flow basis states are labelled by permutations of colour indices

\[|\sigma\rangle = \left| \begin{array}{ccc} 1 & \cdots & n \\ \sigma(1) & \cdots & \sigma(n) \end{array} \right|.\]

We obtain the colour-flow Kronecker deltas by acting with a basis (with a certain ordering of the labels) onto the colour-flow basis

\[\langle c|\sigma\rangle = \left\langle \begin{array}{ccc} 1 & \cdots & n \\ 1 & \cdots & n \end{array} \left| \begin{array}{ccc} 1 & \cdots & n \\ \sigma(1) & \cdots & \sigma(n) \end{array} \right\rangle = \delta_{\alpha_1}^{\beta_\sigma(1)} \cdots \delta_{\alpha_n}^{\beta_\sigma(n)}.\]

The colour-flow basis is not orthogonal

\[S_{\sigma\tau} = \langle \sigma|\tau\rangle = \delta_{\beta_{\tau(1)}}^{\alpha_1} \cdots \delta_{\beta_{\tau(n)}}^{\alpha_n} \delta_{\alpha_1}^{\beta_\sigma(1)} \cdots \delta_{\alpha_n}^{\beta_\sigma(n)} = N^n - \#\text{transpositions}.\]

Therefore we can introduce “square bracket notation”, such that we can define

\[\sum_{\sigma} |\sigma\rangle [\sigma] = \sum_{\sigma} |\sigma\rangle \langle \sigma| = 1 \quad \langle \sigma|\tau\rangle = [\tau|\sigma\rangle = \delta_{\sigma\tau}.\]

[Plätzer, S. Summing large-N towers in colour flow evolution. *The European Physical Journal C* 74 (2014)]

A basis-independent object/an operator can be written as

\[ A = \sum_{\sigma, \tau} A_{\sigma \tau} |\tau\rangle \langle \sigma| = \sum_{\sigma, \tau} [\tau |A|\sigma] |\tau\rangle \langle \sigma| , \]

and a trace of an operator is given by

\[ \text{Tr}[A] = \sum_{\tau} [\tau |A|\tau] = \sum_{\sigma, \tau} [\tau |A|\sigma] \langle \sigma|\tau\rangle = \text{Tr}[AS] = \sum_{\sigma, \tau} A_{\tau \sigma} S_{\sigma \tau} . \]

Therefore, for the amplitude squared we have

\[ |M|^2 = \langle M| M \rangle = \text{Tr}[|M\rangle \langle M|] = \sum_{\sigma, \tau} M_{\sigma} M_{\tau}^{*} \langle \sigma|\tau\rangle = \text{Tr}[MS] . \]

In the definitions above we have used the “square bracket” vectors, one may as well write

\[ \sum_{\sigma, \tau} |\tau\rangle \langle \sigma| = S_{\tau \sigma} \rightarrow \sum_{\sigma, \tau} S_{\tau \sigma}^{-1} |\tau\rangle \langle \sigma| = 1 , \]

and thus,

\[ A |\sigma\rangle = \sum_{\rho, \lambda} S_{\rho \lambda}^{-1} |\rho\rangle \langle \lambda|A|\sigma\rangle = \sum_{\rho, \lambda, \tau} S_{\rho \lambda}^{-1} S_{\lambda \tau} A_{\tau \sigma} |\rho\rangle = \sum_{\tau} A_{\tau \sigma} |\tau\rangle . \]

Consider the virtual exchange for a quark line $i$ and an anti-quark line $j$:

\[
\rightarrow - (t^a)^{c_i'}_{c_i} (t^a)^{\bar{c}_j'}_{\bar{c}_j} = - T_R \left[ \delta^{c_i'}_{\bar{c}_j} \delta^{\bar{c}_j'}_{c_i} - \frac{1}{N} \delta^{c_i}_{\bar{c}_j} \delta^{\bar{c}_j'}_{c_i} \right]
\]

This leads to the following colour flows:

\[
\sigma_{c_i', \bar{c}_j'} \rightarrow \Gamma^{(1)}_{\sigma}
\]

\[
\sigma_{c_i', \bar{c}_m, \bar{c}_j, \bar{c}_n} \rightarrow \Sigma^{(1)}_{\sigma \tau}
\]

\[
\frac{1}{N} \delta^{c_i'}_{\bar{c}_m} \delta^{\bar{c}_m'}_{\bar{c}_j} \delta^{\bar{c}_n}_{c_i} \delta^{c_n'}_{c_j} \rightarrow \rho^{(1)}
\]

\[
(\alpha_s N)[\tau | \Gamma^{(1)}_{\sigma}| \sigma] = (\alpha_s N) \left[ \left( \Gamma^{(1)}_{\sigma} + \frac{1}{N^2} \rho^{(1)} \right) \delta_{\sigma \tau} + \frac{1}{N} \Sigma^{(1)}_{\sigma \tau} \right]
\]

[Plätzer, S. Summing large-N towers in colour flow evolution. *The European Physical Journal C* 74 (2014)]
Back-up: Colour structure at two-loop level

Two-loop soft anomalous dimension in a basis-independent form:

\[
\Gamma^{(2)} = \frac{1}{2} \sum_{i \neq j} \left[ (T_i \cdot T_j)(T_i \cdot T_j) \Omega^{(2)}_{ij} + T_i^a T_i^b T_j^a \tilde{\Omega}^{(2)}_{ij} \right] \\
+ \sum_{i \neq j \neq l} \left[ (T_i \cdot T_j)(T_i \cdot T_l) \Omega^{(2)}_{ijl} + \frac{1}{2} f^{abc} T_i^a T_j^b T_l^c \hat{\Omega}^{(2)}_{ijl} \right] \\
+ \sum_{i \neq j} T_R(T_i \cdot T_j) \left[ \frac{1}{2} \Omega^{(2)}_{ij,\text{self-en.}} + \Omega^{(2)}_{ij,\text{vertex-corr.}} \right].
\]
General structure of the two-loop soft anomalous dimension in the colour-flow basis:

\[
(\alpha_s N)^2 [\tau | \Gamma^{(2)} | \sigma] = (\alpha_s N)^2 \left[ \left( \Gamma^{(2)}_\sigma + \frac{1}{N^2} (\rho_\sigma + \tilde{\rho}) + \frac{1}{N^4} \rho^{(2)} \right) \delta_{\sigma\tau} \right.
\]

\[
+ \frac{1}{N} \left( \Sigma^{(2)}_{\sigma\tau} + \tilde{\Sigma}^{(2)}_{\sigma\tau} \right) + \frac{1}{N^3} \tilde{\Sigma}^{(2)}_{\sigma\tau} + \frac{1}{N^2} \left( \Sigma^{(2)}_{\sigma\tau} + \Sigma^{''(2)}_{\sigma\tau} \right) \right].
\]

first line: diagonal terms in a matrix representation of \( \Gamma^{(2)} \)
second line: off-diagonal terms; “swap-terms”

Example of a contribution to the \( \Gamma^{(2)}_\sigma \) coefficient:

General structure of the two-loop soft anomalous dimension in the colour-flow basis:

\[
(\alpha_s N)^2 \langle \tau | \Gamma^{(2)} | \sigma \rangle = (\alpha_s N)^2 \left[ \left( \Gamma^{(2)}_\sigma + \frac{1}{N^2} (\rho_\sigma + \tilde{\rho}) + \frac{1}{N^4} \rho^{(2)} \right) \delta_{\sigma \tau} 
\right.
\]
\[
+ \frac{1}{N} \left( \Delta\Sigma^{(2)}_{\sigma \tau} + \tilde{\Delta}\Sigma^{(2)}_{\sigma \tau} \right) + \frac{1}{N^3} \tilde{\Delta}\Sigma^{(2)}_{\sigma \tau} + \frac{1}{N^2} \left( \Delta\Sigma^{(2)}_{\sigma \tau} + \tilde{\Delta}\Sigma^{(2)}_{\sigma \tau} \right) \right] .
\]

first line: diagonal terms in a matrix representation of \( \Gamma^{(2)} \)
second line: off-diagonal terms; “swap-terms”

Example of a contribution to the \( \Sigma^{(2)}_{\sigma \tau} \) coefficient:
Example for two-loop matrix element in colour-flow basis:

\[
|\tau|T_gT_iT_j|\sigma\rangle = \sqrt{T_R}N^2\delta_{\sigma\tau}\left[\frac{1}{N^2}\left(\lambda_i\tilde{\lambda}_j\delta_{c_i\sigma-1}(\tau_i) - \lambda_j\tilde{\lambda}_j\delta_{c_j\sigma-1}(\tau_j)\right)
\right.
\]
\[
+ (\lambda_j - \tilde{\lambda}_j) \left(\lambda_i\tilde{\lambda}_j\delta_{c_i\sigma-1}(\tau_i) - \lambda_i\tilde{\lambda}_i\delta_{c_i\sigma-1}(\tau_i)\right)
\]
\[
+ (\lambda_i - \tilde{\lambda}_i) \left(\lambda_j\tilde{\lambda}_j\delta_{c_j\sigma-1}(\tau_j) - \lambda_j\tilde{\lambda}_j\delta_{c_j\sigma-1}(\tau_j)\right)\right]
\]
\[
+ \sum_{(a,b)} \delta_{\sigma\tau(a,b)}N\sqrt{T_R}\]
\]
\[
\left[\lambda_i\lambda_j\lambda_l\delta_{(a,b)(c_i,c_j)}\left(\delta_{c_l\sigma-1}(\tau_l) - \delta_{c_i\sigma-1}(\tau_i)\right)
\right.
\]
\[
+ \lambda_i\lambda_j\lambda_l\delta_{c_i\sigma-1}(\tau_i) - \delta_{c_i\sigma-1}(\tau_i)
\]
\[
+ \lambda_i\lambda_j\lambda_l\delta_{c_j\sigma-1}(\tau_j) - \delta_{c_j\sigma-1}(\tau_j)
\]
\[
+ \lambda_i\lambda_j\lambda_l\delta_{c_l\sigma-1}(\tau_l) - \delta_{c_l\sigma-1}(\tau_l)\right]
\]
\[
+ \sum_{(a,b)} \sum_{(b,c)} \delta_{\sigma\tau(a,b)(b,c)}\sqrt{T_R}\]
\]
\[
\left[\lambda_i\lambda_j\lambda_l\delta_{c_i\sigma-1}(\tau_i) - \delta_{c_i\sigma-1}(\tau_i)\right)
\]
\[
+ \lambda_i\lambda_j\lambda_l\delta_{c_j\sigma-1}(\tau_j) - \delta_{c_j\sigma-1}(\tau_j)
\]
\[
+ \lambda_i\lambda_j\lambda_l\delta_{c_l\sigma-1}(\tau_l) - \delta_{c_l\sigma-1}(\tau_l)\right]
\]
Back-up: Feynman tree theorem

The FTT is based on the identity

\[ G_A(k) = G(k) + 2\pi i \delta(k^2) \theta(T \cdot k) , \]

which equals to

\[ \frac{1}{k^2 - i0|T \cdot k|(T \cdot k)} = \frac{1}{k^2 + i0(T \cdot k)^2} + 2\pi i \delta(k^2) \theta(T \cdot k) , \]

where we have introduced the timelike vector \((T^\mu) = (\sqrt{2}, \vec{0})\). For the loop integral with advanced propagators only we can write

\[ L_A^{(n)}(p_1, \ldots, p_n) = -i \int \frac{d^d k}{i\pi^{d/2}} \prod_{m=1}^{n} G_A(k_m) = 0 . \]

Plugging in, we find

\[ L_A^{(n)}(p_1, \ldots, p_n) = -i \int \frac{d^d k}{i\pi^{d/2}} \prod_{m=1}^{n} \left[ G(k_m) + 2\pi i \delta(k^2) \right] \]

\[ = L^{(n)}(p_1, \ldots, p_n) + L_{1-\text{cut}}^{(n)}(p_1, \ldots, p_n) + \ldots + L_{n-\text{cut}}^{(n)}(p_1, \ldots, p_n) . \]

Example for one-loop diagram with soft momentum $k$

\[
0 = \mu^{2\varepsilon} \int \frac{d^d k}{i\pi^{d/2}} G_A(k) \frac{1}{[2p_i \cdot k - 2i0(p_i^0)^2]} \frac{1}{[-2p_j \cdot k + 2i0(p_j^0)^2]} ,
\]

which leads to

\[
0 = \mu^{2\varepsilon} \int \frac{d^d k}{i\pi^{d/2}} \left( \frac{1}{[k^2 + 2i0(k^0)^2]} + (2\pi i)\delta(k^2)\theta(k^0) \right) \frac{1}{[-2p_j \cdot k + 2i0(p_j^0)^2]} \left( \frac{1}{[2p_i \cdot k + 2i0(p_i^0)^2]} + (2\pi i)\delta(2p_i \cdot k) \right) .
\]
Start with the identity

\[
\frac{1}{k^2 - i0(T \cdot k)^2} = \frac{1}{k^2 + i0(T \cdot k)^2} + 2\pi i \delta(k^2),
\]

where \((T^\mu) = (\sqrt{2}, \vec{0})\) and take the derivative with respect to component of loop momentum

\[
\frac{k^\mu - i0 \ T^\mu(T \cdot k)}{[k^2 - i0(T \cdot k)]^2} - \frac{k^\mu + i0 \ T^\mu(T \cdot k)}{[k^2 + i0(T \cdot k)^2]^2} = -i\pi \frac{\partial}{\partial k^\mu} \delta(k^2).
\]

Using the relation \(\frac{\partial f(k^2)}{\partial k^\mu} = 2k^\mu \frac{\partial f(k^2)}{\partial k^2}\) and some space-like vector \((S \cdot T = 0)\) to contract in order to get rid of the \(T\)-dependent numerator structure one finds

\[
\frac{1}{[k^2 - i0(T \cdot k)^2]^2} - \frac{1}{[k^2 + i0(T \cdot k)^2]^2} = -2\pi i \delta'(k^2).
\]
Back-up: FTT at two-loop level - An example