Large Charge, Semiclassics and Superfluids

mostly based on
G. Badel, GC, A. Monin and R. Rattazzi 1909.01269

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Weak coupling

QFT observables are calculable when there is an expansion around leading trajectory γ in the PI:

$$\langle \mathcal{O} \rangle = e^{-[S_0+S_1+\ldots]} = \mathcal{O}(\gamma) + \delta_q \mathcal{O}$$

- Semiclassical observables: $\mathcal{O}(\gamma) \gg \delta_q \mathcal{O}$. E.g.: everyday life.

- Quantum observables: $\delta_q \mathcal{O} \gg \mathcal{O}(\gamma)$. E.g.: HE scattering of few EW particles.

Multilegged amplitudes, such as those associated with the production of $n \gg 1$ particles, *apparently* induce large quantum fluctuations - how to deal with them?

Background: $1 \to n$ in massive $\lambda\phi^4$

- Near threshold production of n particles from an external source violates unitarity at tree-level for $\lambda n \gg 1$:

$$\sigma_{1\to n} \propto n!\lambda^{n-1}\left(\frac{\epsilon}{m}\right)^{\frac{3n}{2}}, \qquad \epsilon = \frac{E - nm}{n}.$$

- Loop corrections uncontrolled for $\lambda n \gg 1$:

$$\sigma_{loop} = \sigma_{tree}\left[1 + \lambda\left(An^2 + Bn\right) + \lambda^2\left(\frac{1}{2}A^2n^4 + Cn^3 + \ldots\right)\right]$$

Rubakov 1995

Is this process strongly coupled?

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Rubakov 1995

Is this process strongly coupled? No!

Background: $1 \to n$ in massive $\lambda\phi^4$

- All large effects resum to a simple exponential form:

$$\sigma_{1\to n} \sim e^{\frac{1}{\lambda}F(\lambda n, \epsilon)}$$

Libanov, Rubakov, Son, Troitsky 1994

- Exponentiation arises from a non-trivial semiclassical trajectory γ in the PI

Son 1995

- Unfortunately, it is technically hard to explicitly find γ and compute $\sigma_{1\to n}$

Today we will solve an analogous but technically simpler problem.

Setup

Wilson-Fisher fixed point in $\lambda|\phi|^4$ in $d = 4 - \varepsilon$:

$$\mathcal{L} = \partial\bar{\phi}\partial\phi + \frac{\lambda}{4}\left(\bar{\phi}\phi\right)^2, \qquad \frac{\lambda_*}{(4\pi)^2} = \frac{\varepsilon}{5} + \frac{3}{25}\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right).$$

- Scale (and conformal) invariance:

$$\langle\bar{\mathcal{O}}(x)\mathcal{O}(0)\rangle \propto \frac{1}{x^{2\Delta_{\mathcal{O}}}}$$

- $U(1)$ symmetry:

$$\langle\bar{\phi}^n(x)\phi^m(0)\rangle \propto \delta_{n,m}$$

In the limit $\lambda_* \sim \varepsilon \to 1$ this theory is in the same universality class of the O(2) model

Setup

Goal: computing the scaling dimension Δ_{ϕ^n} for $\lambda_* \to 0$, $n \to \infty$ with $\lambda_* n =$ fixed

$$\langle \bar{\phi}^n(x_f)\phi^n(x_i)\rangle \propto \frac{1}{|x_f - x_i|^{2\Delta_{\phi^n}}}$$

- Simplest multi-legged amplitude
- Explicit realization of the superfluid phase for large charge operators in CFTs

Hellerman, Orlando, Reffert, Watanabe 2015

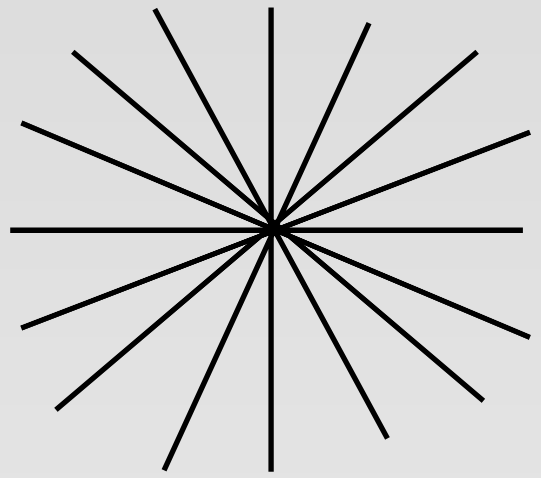
Overview

1. From Feynman diagrams to semiclassics in the epsilon expansion
2. The double-scaling limit from the state-operator map
3. Large charge operators in CFTs
4. Summary and outlook

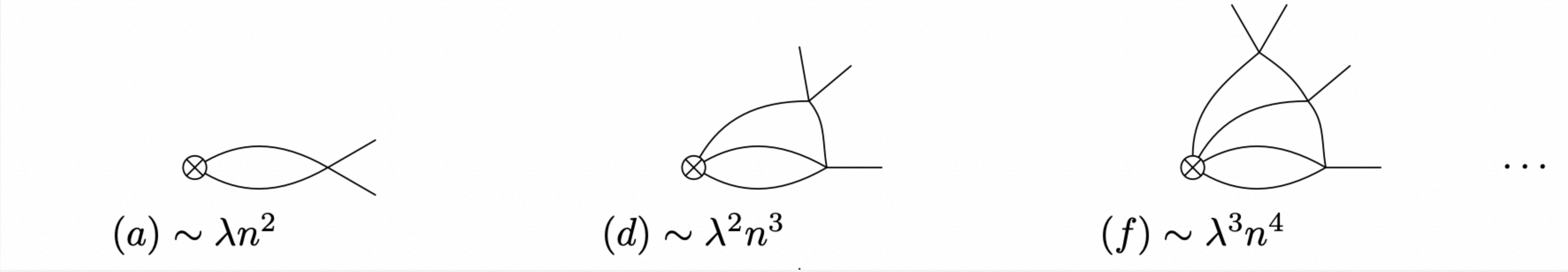
From Feynman diagrams to semiclassics in the epsilon expansion

Diagrammatics

- At tree-level ϕ^n corresponds to n out coming legs

$\otimes \equiv$ 

- Loop corrections may attach to any leg and lead to large combinatoric factors



$$\Delta_{\phi^n} = n + \lambda(\# n^2 + \dots) + \lambda^2(\# n^3 + \dots) + \lambda^3(\# n^4 + \dots) + \dots$$

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- Perturbation theory breaks down for $\lambda n \gg (4\pi)^2$

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- Series can be reorganized as a double expansion:

$$\begin{aligned}\Delta_{\phi^n} &= n \sum_\ell c_{\ell,\ell} (\lambda n)^\ell + n^0 \sum_\ell c_{\ell,\ell-1} (\lambda n)^\ell + \ldots \\ &= \frac{1}{\lambda} F_0(\lambda n) + F_1(\lambda n) + \ldots\end{aligned}$$

Can we systematically resume the λn series and compute and compute $F_0(\lambda n)$ exactly?

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Can we systematically resume the λn series and compute and compute $F_0(\lambda n)$ exactly?

Yes! By expanding the PI around a suitable trajectory.

Sketch of the idea

$$\int \mathcal{D}\phi\, \bar{\phi}^n(x_f)\phi^n(x_i)e^{-\frac{S}{\lambda}} = \int \mathcal{D}\phi\, e^{-\frac{1}{\lambda}\left(S+\lambda n \log \bar{\phi}_f+\lambda n \log \phi_i\right)}$$

Operator insertion source non trivial saddle-point trajectory $\phi_{cl} \equiv \phi_{cl}(\lambda n, x_f - x_i)$

$$\langle \bar{\phi}^n(x_f)\phi^n(x_i)\rangle = e^{-\frac{1}{\lambda}S_{cl}(\lambda n, x_f - x_i) - S_1 - \lambda S_2 + \ldots}$$

$$\Delta_{\phi^n} = \frac{1}{\lambda}F_0(\lambda n) + F_1(\lambda n) + \ldots$$

Semiclassical expansion holds for $\lambda \ll 1$ but any λn: it resums infinitely many loop diagrams!

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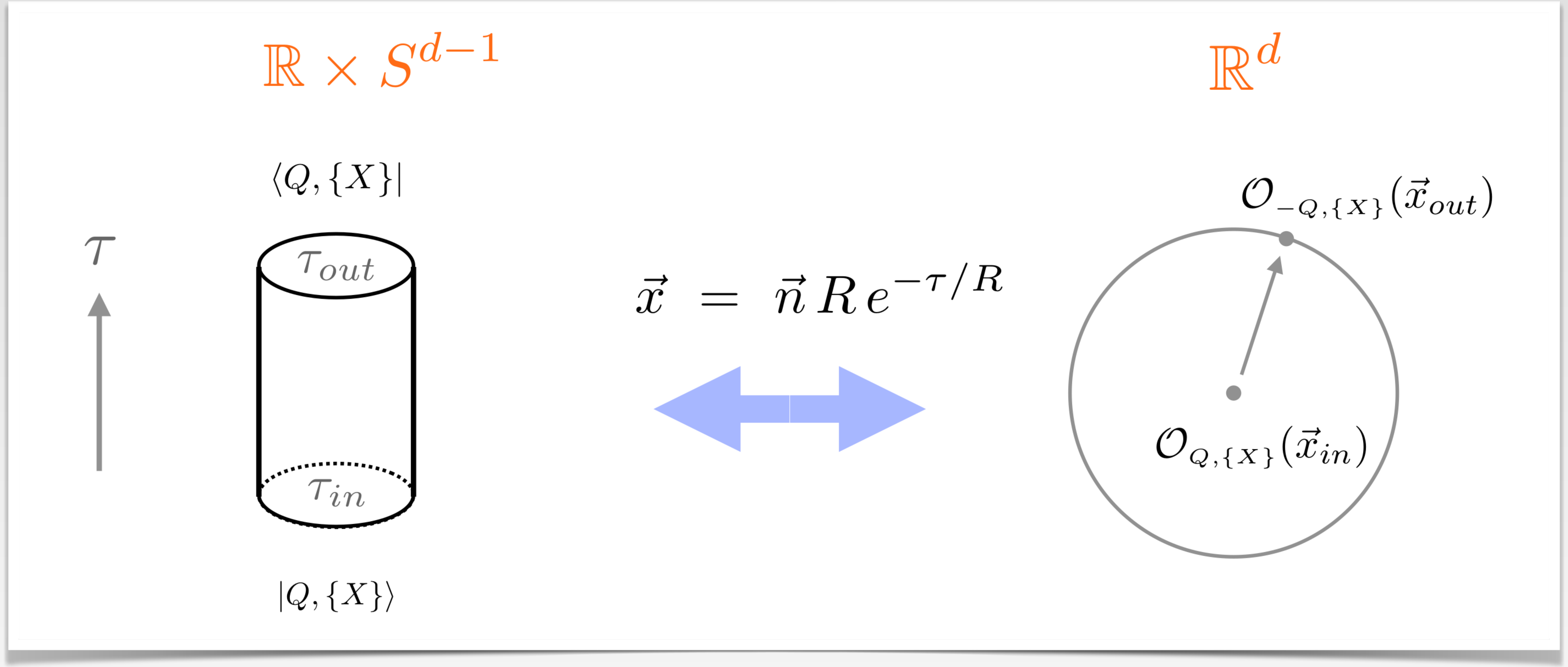
Semiclassical expansion holds for $\lambda \ll 1$ but any λn: it resums infinitely many loop diagrams! In practice it is hard to solve the problem directly in flat space…

The double-scaling limit from the state-operator map

The cylinder and the state-operator correspondence



$$\langle \mathcal{O}^\dagger(x_f)\mathcal{O}(x_i)\rangle_{\text{cyl}} \overset{\tau_i \to -\infty}{=} e^{-E_{\mathcal{O}}(\tau_f - \tau_i)}, \qquad E_{\mathcal{O}} = \Delta_{\mathcal{O}}/R$$

Finite time path-integral

- ϕ^n: lowest dimensional operator of charge n

- Consider arbitrary charge n state $|\psi_n\rangle$ on $\mathbb{R}\times S^{d-1}$:

$$\langle\psi_n|e^{-HT}|\psi_n\rangle \underset{T\to\infty}{=} \mathcal{N}e^{-E_{\phi^n}T}, \qquad E_{\phi^n} = \Delta_{\phi^n}/R$$

- Generalize 2d particle angular momentum m state: $\langle\phi, r|m, n\rangle \propto e^{im\phi}$

$$\phi = \frac{\rho}{\sqrt{2}}e^{i\chi}, \qquad \bar{\phi} = \frac{\rho}{\sqrt{2}}e^{-i\chi},$$

$$\langle\psi_n|e^{-HT}|\psi_n\rangle = \frac{\int_{\rho=f}^{\rho=f}\mathcal{D}\rho\mathcal{D}\chi e^{-S_{cyl}-i\frac{n}{\Omega_{d-1}}[\int d\Omega_{d-1}(\chi_f-\chi_i)]}}{\int\mathcal{D}\phi\mathcal{D}\bar{\phi}\,e^{-S_{cyl}}}$$

Saddle-point

- Saddle-point solution:

$$\chi = -i\mu\tau + \text{const.}\,, \qquad \rho = f$$

- Superfluid symmetry breaking pattern:

$$\{\, \not{D}\,,\, \not{Q}\,\} \longrightarrow \bar{D} = D + \mu\, Q$$

- To leading order, the scaling dimension follows from the classical action

$$\langle \psi_n | e^{-HT} | \psi_n \rangle \simeq e^{-S_{cl}} \propto \exp\left[-\Delta_{\phi^n} T/R\right]$$

The result

$$\Delta_{\phi^n} = nF_0(\lambda_* n) + F_1(\lambda_* n) + \dots$$

$$F_0(16\pi^2 x) = \frac{3\left[9x - \sqrt{81x^2-3}\right]^{1/3} + 3^{2/3}\left[9x - \sqrt{81x^2-3}\right]}{\left[\left(9x - \sqrt{81x^2-3}\right)^{2/3} + 3^{1/3}\right]^2}$$
$$+ \frac{9 \times 3^{1/3} x \left[9x - \sqrt{81x^2-3}\right]^{2/3}}{2\left[\left(9x - \sqrt{81x^2-3}\right)^{2/3} + 3^{1/3}\right]^2}$$

Badel GC Monin Rattazzi 2019

Analogous but more complicated expression for the 1-loop correction $F_1(\lambda_* n)$

One-loop result: small λn

For $\lambda_* n \ll (4\pi)^2$ (recall $\lambda_* \sim \varepsilon = 4 - d$):

$$\Delta_{\phi^n} = n\left(\frac{d}{2} - 1\right) + \frac{\varepsilon}{10}\, n(n-1) - \frac{\varepsilon^2}{50} n(n^2 - 4n) + \mathcal{O}\left(\varepsilon^2 n, \varepsilon^3 n^4\right).$$

- Perfect agreement with diagrammatic calculation:

$$\Delta_{\phi^n} = n\left[\left(\frac{d}{2} - 1\right) + \frac{\varepsilon}{10}\,(n-1) - \frac{\varepsilon^2}{100}(2n^2 - 8n + 5) + \mathcal{O}\left(\varepsilon^3 n^4\right)\right].$$

One-loop result: large $\lambda_* n$

For $\lambda_* n \gg (4\pi)^2$ (recall $\lambda_* \sim \varepsilon = 4 - d$)::

$$\Delta_{\phi^n} = \frac{1}{\lambda_*}\left[\alpha_1 \left(\lambda_* n\right)^{\frac{d}{d-1}} + \alpha_2 \left(\lambda_* n\right)^{\frac{d-2}{d-1}} + \ldots\right]$$

- Coefficients explicitly computed as series in the coupling $\alpha_i = \# + \lambda_* \# + \ldots$

One-loop result: large $\lambda_* n$

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Interpretation:

- Heavy radial mode: $m_\rho \sim \mu \sim (\lambda_* n)^{\frac{1}{d-1}}$ → **superfluid EFT**

Hellerman Orlando Reffert Watanabe 2015

Large charge operators in CFTs

General setup

$$\langle Q|e^{-HT}|Q\rangle_{\mathbb{R}\times S^{d-1}} = \int_{\widehat{Q}=Q} \mathcal{D}\Phi e^{-S} = e^{-\Delta_Q T/R}$$

- $Q \gg 1$: expect classical trajectory Φ_{class} dominates
- Φ_{class} non-linearly realizes $SO(d+1,1) \times U(1)_Q$
- *Universal* features describable in terms of Goldstone modes
- Simplest (but not only) option: superfluid

Monin Pirtskhalava Rattazzi Seibold 2016

The conformal superfluid

Summary: $SO(d+1,1) \times U(1) \longrightarrow SO(d) \times \bar{D}$

- 1 Goldstone mode $\chi(x) = -i\mu\tau + \pi(x)$

- "radial" modes: generically gapped at $\mu \sim \frac{Q^{\frac{1}{d-1}}}{R}$

(But moduli in SCFTs: Hellerman Maeda Orlando Reffert Watanabe 2017-2021)

⬇

Can write $\mathscr{L}\left(\chi\right)$ systematically in a derivative expansion $\partial/\mu \propto E/Q^{\frac{1}{d-1}}$

$$
\begin{aligned}
\mathcal{L} = \ & c(\partial\chi)^d && \Big\} = Q^{\frac{d}{d-1}} \\
& +c_1(\partial\chi)^{d-2}\left\{\mathcal{R} + (d-2)(d-1)\frac{[\partial_\mu(\partial\chi)]^2}{(\partial\chi)^2}\right\} && \multirow{2}{*}{$\Bigg\} = Q^{\frac{d-2}{d-1}}$} \\
& +c_2(\partial\chi)^{d-2}\mathcal{R}_{\mu\nu}\frac{\partial^\mu\chi\partial^\nu\chi}{(\partial\chi)^2} && \\
& +\dots && \Big\} = Q^{\frac{d-4}{d-1}}
\end{aligned}
$$

$$
\left.\mathcal{L} = c(\partial\chi)^d \right\} = Q^{\frac{d}{d-1}}
$$

$$
\left.\begin{aligned} &+c_1(\partial\chi)^{d-2}\left\{\mathcal{R} + (d-2)(d-1)\frac{[\partial_\mu(\partial\chi)]^2}{(\partial\chi)^2}\right\} \\ &+c_2(\partial\chi)^{d-2}\mathcal{R}_{\mu\nu}\frac{\partial^\mu\chi\partial^\nu\chi}{(\partial\chi)^2}\end{aligned}\right\} = Q^{\frac{d-2}{d-1}}
$$

$$
\left. + \dots \right\} = Q^{\frac{d-4}{d-1}}
$$

Simple prediction for the ground state energy - matches the result in the ε-expansion!

$$
\Delta_0(Q) = Q^{\frac{d}{d-1}}\left[\alpha_1 + \alpha_2 Q^{-\frac{2}{d-1}} + \dots\right] + \text{1-loop}
$$

$$
\text{1-loop} = \begin{cases} -0.0937\dots & d=3 \\ -\frac{1}{48\sqrt{3}}\log Q & d=4 \end{cases}
$$

Hellerman Orlando Reffert Watanabe 2015, Monin 2016, GC 2020

Summary

Summary

- Multi-legged amplitudes are related to semiclassical physics
- In the WF fixed-point correlation functions of ϕ^n can be explicitly computed semiclassically in the double scaling limit $\lambda_* \to 0$, $\lambda_* n =$ fixed
- A similar semiclassical behavior underlies a general *effective* description of large charge operators in generic CFTs.

Any lessons for the multi-particle production problem?

THANK YOU!