FINITE SYSTEM SIZE CORRECTIONS TO SCATTERING IN ϕ^4 THEORY

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INTRODUCTION

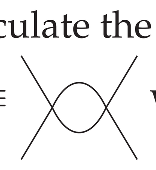
There are many questions surrounding the recent relativistic hydrodynamic calculations on the apparent formation of QGP in heavy-ion collisions at the LHC and RHIC. This calculation uses lattice QCD methods to extrapolate the calculation of the equation of state to infinite volumes. Crucially it is unknown whether the finite system size induced by the size of collided nuclei in heavy-ion collisions induces non-neglible contributions to the trace anomaly, and therefore equation of state, of QCD above its phase transition. The work shown here is a crucual first step in the direction of calculating these corrections. We need to first develop and understand the relevant mathematical tools and techniques in a simpler case, such as the scalar massive ϕ^4 model considered here. The quantization of available momentum modes, induced by the periodic boundary conditions considered here, offers a non-trivial challenge to some standard techniques. Most notably the usage of dimensional regularization is found to introduce too many mathematical, conceptual and practical problems to be of much use. A new regularization technique, dubbed denominator regularization, is shown to adapt wonderfully to finite system sizes, and is expected to be extremely useful in future calculations.

FINITE SIZED ϕ^4 THEORY

Starting from the ϕ^4 Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{1}$$

we want to calculate the finite system size corrections to $2\to 2$ scattering up to NLO (λ^2). For this we introduce the function

$(-i\lambda)^2 iV(p^2) \equiv$  which gives the following neat expression for the scattering amplitude:

$$\mathcal{M} = \lambda\left[1 + \lambda\left(\overline{V}(s) + \overline{V}(t) + \overline{V}(u)\right)\right]. \tag{2}$$

In order to calculate the renormalized $\overline{V}$ we need to regulate a divergent sum of the form

$$\sum_{\vec{k}\in\mathbb{Z}^3}\frac{1}{\left[\sum_{i=1}^3\left(\frac{k_i}{L_i}+x\,p^i\right)^2+\Delta^2\right]^{\frac{3}{2}}}$$

in order to isolate the divergences to be subtracted by counter terms according to $\overline{\textbf{MS}}$. This is unnecessarily complicated with the standard dimensional regularization when the length scales are asymmetric, so we can instead use denominator regularization as discussed below, together with a newly derived Equation 7. This gives the renormalized

$$\overline{V}(p^2,\{L_i\}) = -\frac{1}{2(4\pi)^2}\int_0^1 dx\left\{\log\left(\frac{\mu^2}{\Delta^2}\right) + 2\sum_{\vec{m}\in\mathbb{Z}^3}{}'\cos\left(2\pi x\sum m_i p^i L_i\right)K_0\left(2\pi\sqrt{\Delta^2\sum(m_iL_i)^2}\right)\right\}. \tag{3}$$

DENOMINATOR REGULARIZATION

If we want a regularization procedure that agrees with dimensional regularization in the infinite space cases where it works, we can notice the following:

$$\int_0^\infty dl\,\frac{2\pi^{\frac{d}{2}}l^{d-1}}{\Gamma(\frac{d}{2})(2\pi)^d}\mu^{2n-d}\left(l^2+\Delta^2\right)^{-n} = \frac{1}{\Gamma(n)(4\pi)^{\frac{d}{2}}}\left(\frac{\mu^2}{\Delta^2}\right)^{n-\frac{d}{2}}\Gamma\left(n-\frac{d}{2}\right). \tag{4}$$

We can therefore see there is the important quantity $n-\frac{d}{2}$, which if it is the same in both procedures, will give equal momentum dependent terms. Then instead of analytically continuing the measure of the integral by letting $d = 4-\epsilon$, we can instead analytically continue the exponent in the denominator, which is more amenable in the finite-system case. We can then see

$$\frac{1}{\Gamma(n)(4\pi)^{\frac{d}{2}}}\left(\frac{\mu^2}{\Delta^2}\right)^{n-\frac{d}{2}}\Gamma\left(n-\frac{d}{2}\right)\xrightarrow[d=4-\epsilon]{n=2}\frac{1}{8\pi^2\epsilon}+\frac{\log(4\pi)-\gamma}{16\pi^2}+\frac{1}{16\pi^2}\log\left(\frac{\mu^2}{\Delta^2}\right)+\mathcal{O}(\epsilon) \tag{5}$$

$$\frac{1}{\Gamma(n)(4\pi)^{\frac{d}{2}}}\left(\frac{\mu^2}{\Delta^2}\right)^{n-\frac{d}{2}}\Gamma\left(n-\frac{d}{2}\right)\xrightarrow[d=4]{n=2+\frac{\epsilon}{2}}\frac{1}{8\pi^2\epsilon}+\frac{-1}{16\pi^2}+\frac{1}{16\pi^2}\log\left(\frac{\mu^2}{\Delta^2}\right)+\mathcal{O}(\epsilon) \tag{6}$$

which shows how we can slightly modify the $\overline{\textbf{MS}}$ renormalization scheme, to get the correct result.

ANALYTIC CONTINUATION OF GENERALIZED EPSTEIN-ZETA FUNCTION

Using the Poisson summation formula, the essential analytic continutation of the generalized Epstein-Zeta function can be derived, giving

$$\sum_{\vec{n}\in\mathbb{Z}^p}\left(a_i^2n_i^2+b_in_i+c-i\varepsilon\right)^{-s} = \frac{1}{a_1\cdots a_p}\frac{1}{\Gamma(s)}\left[\pi^{p/2}\Gamma\left(s-\frac{p}{2}\right)\left(c-\sum\frac{b_i^2}{4a_i^2}-i\varepsilon\right)^{\frac{p}{2}-s}\right.$$
$$\left.+2\pi^s\sum_{\vec{m}\in\mathbb{Z}^p}{}'e^{-2\pi i\sum\frac{m_ib_i}{2a_i^2}}\left(\frac{c-\sum\frac{b_i^2}{4a_i^2}-i\varepsilon}{\sum\frac{m_i^2}{a_i^2}}\right)^{\frac{p}{4}-\frac{s}{2}}K_{s-\frac{p}{2}}\left(2\pi\sqrt{\left(c-\sum\frac{b_i^2}{4a_i^2}-i\varepsilon\right)\sum\frac{m_i^2}{a_i^2}}\right)\right], \tag{7}$$

which is crucial for the calculation of the scattering amplitude.

SUM OF SINC FUNCTIONS

In order to show that unitarity holds, we need an expression for the m-dimensional sum sum of sinc functions, defined as

$$\text{sinc}(x) = \begin{cases}\frac{\sin(x)}{x} & x\neq 0\\ 1 & x=0\end{cases},$$

which we can derive by using the d-dimensional Poisson-summation formula, giving:

$$\sum_{\vec{k}\in\mathbb{Z}^m}\text{sinc}(2\pi R\|\vec{k}\|) = \sum_{l=0}^\infty r_m(l)\,\text{sinc}(2\pi R\sqrt{l}) \tag{8}$$

$$= \frac{\pi^{\frac{1-m}{2}}}{2R\,\Gamma(\frac{3-m}{2})}\sum_{0\le l<R^2}{}^*\frac{r_m(l)}{\sqrt{R^2-l}^{\,m-1}}. \tag{9}$$

Testing suggests that this analytic continuation is valid for all real values of m, and is expected to hold for the entire complex plane. In the special case of $m=2$ we get a formula that is equivalent to one proposed by Ramanujan and proved by Hardy[1]:

$$\sum_{l=0}^\infty r_2(l)\,\text{sinc}(2\pi R\sqrt{l}) = \frac{1}{2R\pi}\sum_{0\le l<R^2}\frac{r_2(l)}{\sqrt{R^2-l}}. \tag{10}$$

OUTLOOK

There are many natural extension to this work currently being done. The ubuquity of the advantages of denominator regularization over dimensional regularization is being extenively investigated with encouraging preliminary results. The more involved numerics, with the main focus being on the calculation of the effective coupling (coming from a formal resummation of the bubble diagrams), becomes numerically misbehaved exactly where the effects are expected to be most relevant, neccessatating further analytical work. Finally the thermal field theory extension of the work shown here is a neccessary step in the direction of connecting to experiment.

UNITARITY

In order to verify the integrity of unitarity, we can equavalently verify that the optical theorem (i.e. $2\,\text{Im}[\mathcal{M}] = \sigma_{\text{tot}}$) holds. Its straightforward to show that the total cross-section

$$\sigma_{tot} = \frac{\lambda^2}{16\pi}\frac{\pi^{\frac{1-m}{2}}}{\Gamma\left(\frac{3-m}{2}\right)}\frac{1}{L\sqrt{s}}\sum_{0\le l\le R^2}{}^*\frac{r_m(l)}{\sqrt{R^2-l}^{\,m-1}}. \tag{11}$$

It can also be shown that the imaginary part of the scattering amplitude at NLO takes the form

$$2\,\text{Im}[\mathcal{M}] = \frac{\lambda^2}{16\pi}\frac{2R}{L\sqrt{s}}\sum_{\vec{k}\in\mathbb{Z}^m}\text{sinc}\left(2\pi R\|\vec{k}\|\right), \tag{12}$$

which we can use Equation 9 on to see that the optical theorem, and therefore also unitarity, holds. This provides a succesful non-trivial self-consistency check.

DISCUSSION

The results obtained so far are encouraging for the the feasibility of the calculation of finite system-size corrections to the trace anomaly in QCD. Many of the techniques developed appears to be readily generalizable and employable to more complicated situations. The derivations of the generalization to a formula by Ramanujan and Hardy, as well as the analytic continuation of the generalized Epstein-Zeta function, has potential mathematical interest outside of their applications which neccessatated their derivation.

SIZE OF CORRECTIONS

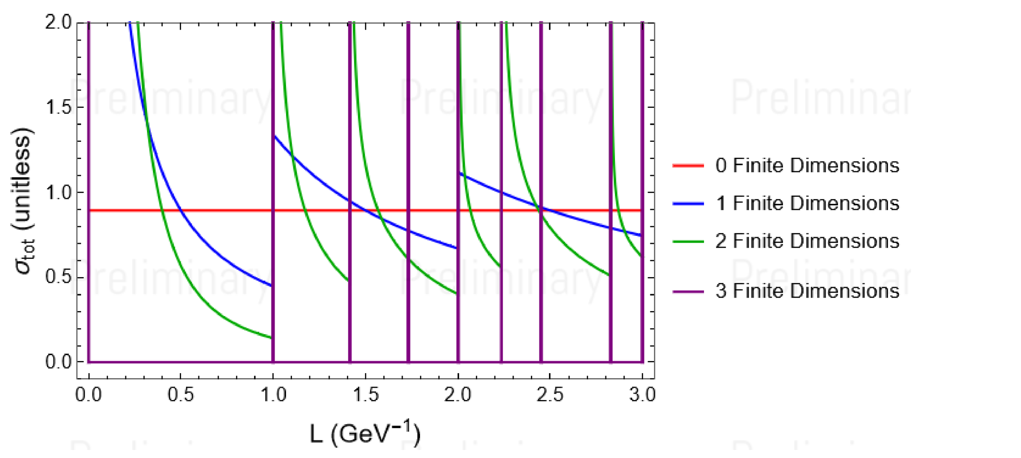


Figure 1: The total cross section (neglecting the scaling factor of $\frac{\lambda^2}{16\pi}$) as a function of the length-scale of the finite dimensions L. Here used are incoming particles each with mass 0.5GeV and spatial momentum of magnitude 1GeV.

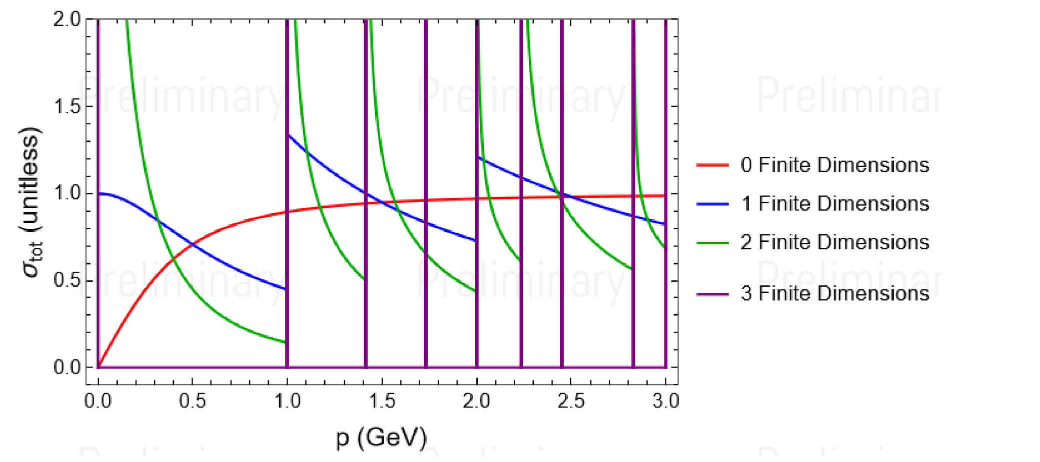


Figure 2: The total cross section (neglecting the scaling factor of $\frac{\lambda^2}{16\pi}$) as a function of the incoming spatial momentum p. Here used are incoming particles each with mass 0.5GeV and a length scale of 1GeV^{-1}.

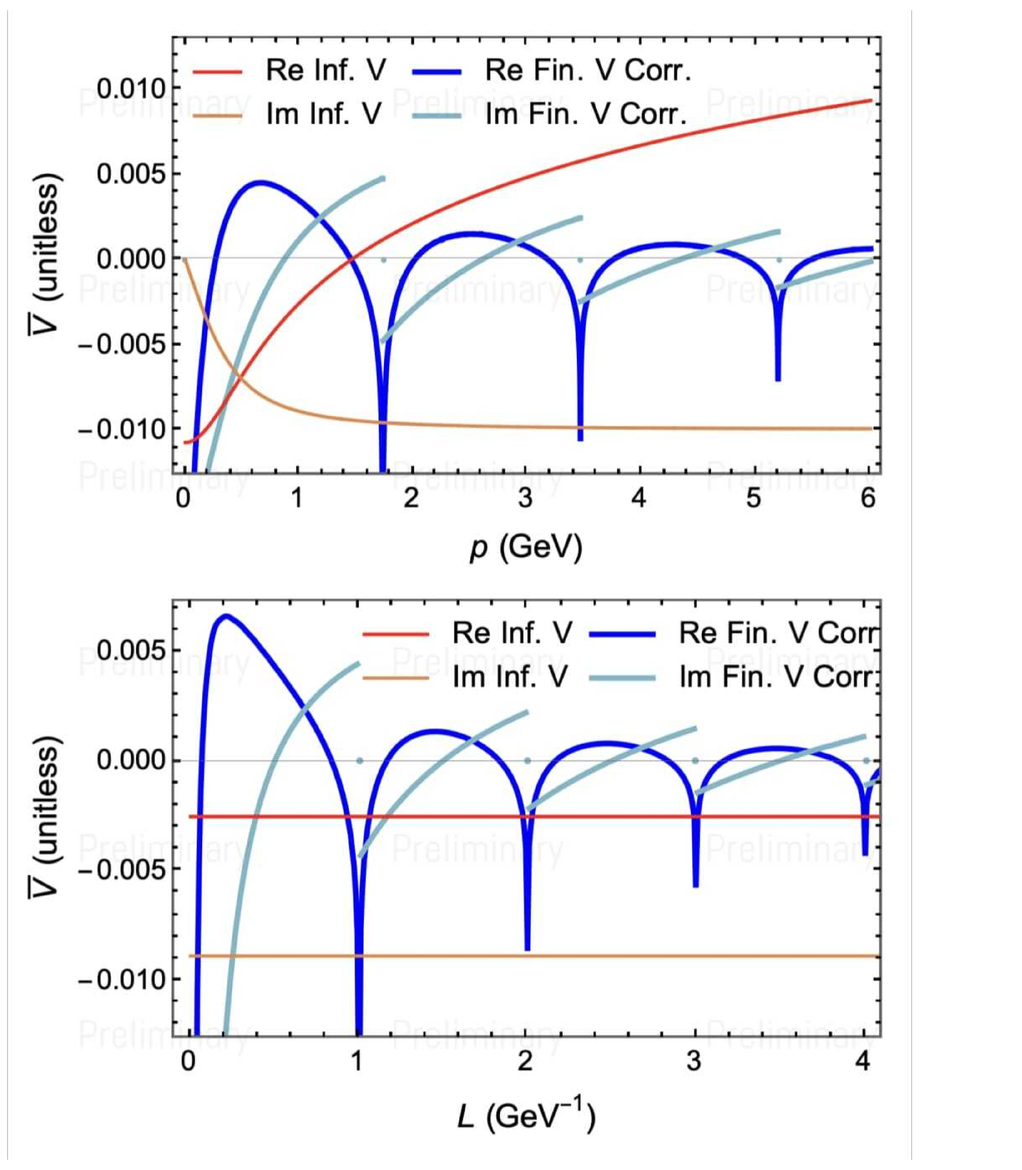


Figure 3: (Top) A plot of the contributions to $\overline{V}(p^2,\{L_i\})$ for the s channel as a function of $p\equiv\|\vec{p}\|$ for $n=1$ compact dimension. The real part of the infinite volume contribution is in red; the imaginary part of the infinite volume contribution is in orange. The real part of the finite length correction is in blue; the imaginary part of the finite length correction is in blue-gray. We take $\mu = 1$ GeV, $m = 0.5$ GeV, and $L = \frac{1}{\sqrt{3}}$ GeV^{-1}. (Bottom) The same comparison but as a function of L for fixed $\|\vec{p}\| = 1$ GeV.

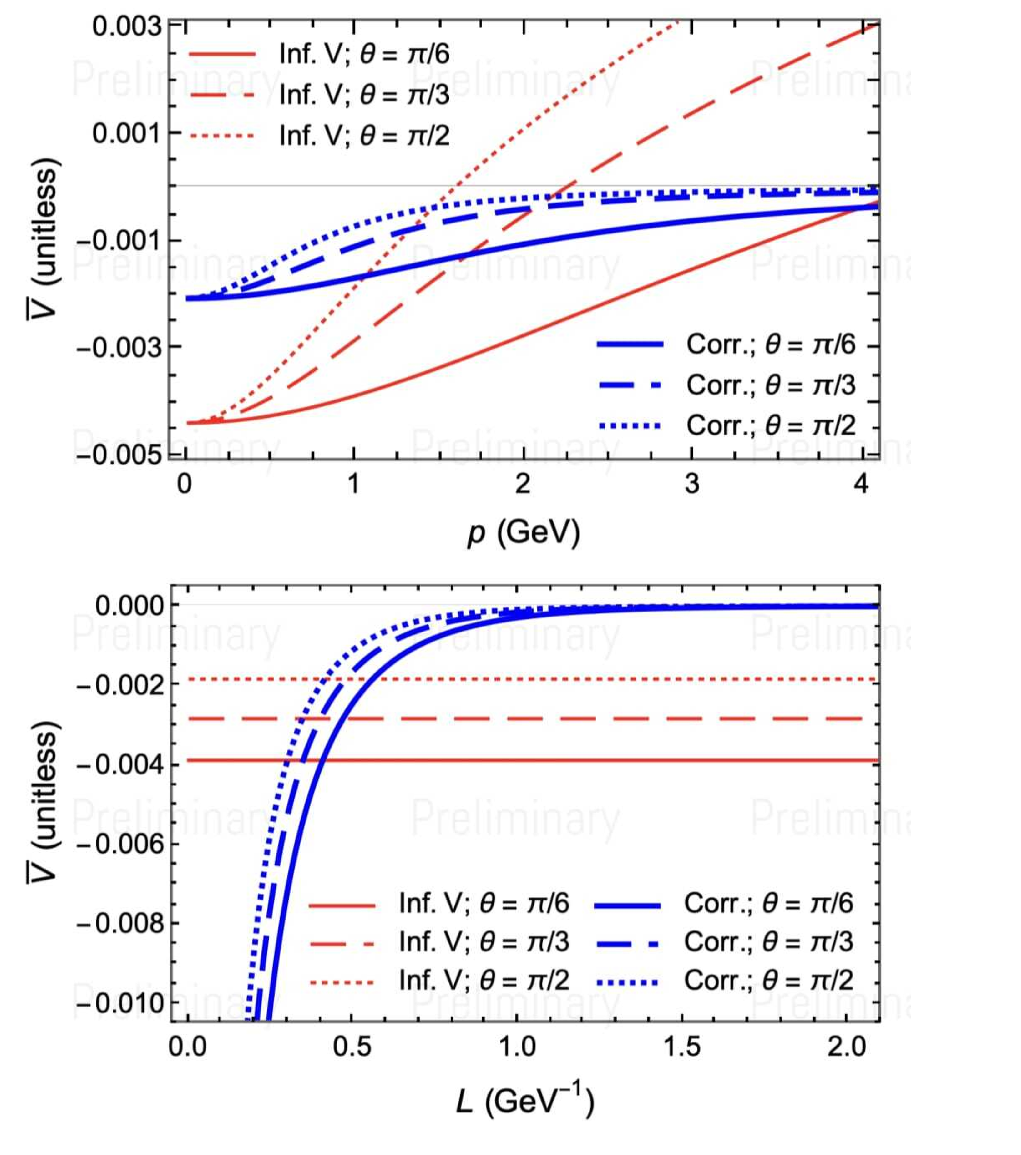


Figure 4: The total cross section (neglecting the scaling factor of $\frac{\lambda^2}{16\pi}$) as a function of the incoming spatial momentum p. Here used are incoming particles each with mass 0.5 GeV and a length scale of 1 GeV^{-1}.

REFERENCES

[1] Gordon H. Hardy. *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work.* Cambridge University Press, 1940.

[2] W. A. Horowitz and J. F. Du Plessis. Finite system size correction to NLO scattering in ϕ^4 theory. *Phys. Rev. D*, 105(9):L091901, 2022.

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