A new Wilson Line based Action for Gluodynamics

Based on H. Kakkad, P. Kotko, A. Stasto, 2021- arXiv:2102.11371

Hiren Kakkad
Faculty of Physics, AGH UST

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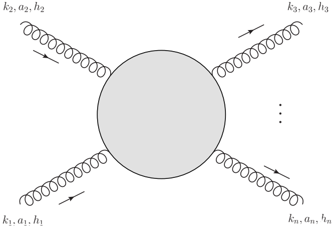
Pure Gluon Amplitudes

On-shell amplitudes

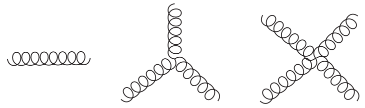
- Each gluon leg is characterized by three quantities.
- k_i = 4-momentum. It satisfies the on-shell relation $k_i^2 = E^2 - \vec{p}^2 = 0$.
- a_i = color index.
- $h_i = \pm$ represents helicities.



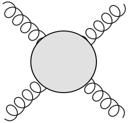
What does the blob represent ?

- Built from basic building blocks.
- Feynman rules.
- Sum of all contributing diagrams built from the Feynman rules with the subsequent integration over the internal loop momenta.

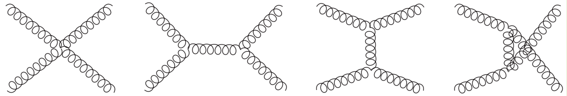
Building blocks



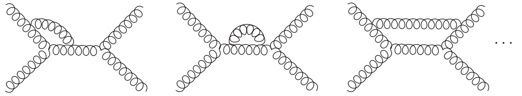
Example : Four point amplitude.



Tree Amplitude



Loop Amplitude



Problem with Feynman diagram technique.

The number of Feynman diagrams contributing to the amplitude of a gluon tree level ($g + g \longrightarrow ng$) grows factorily.

n	2	3	4	5	6	7	8
# of diagrams	4	25	220	2485	34300	559405	10525900

Color Decomposition

Color Decomposition

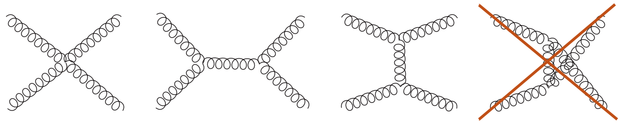
- Technique to disentangle the color and kinematical degrees of freedom in a gauge theory scattering amplitude.
- Lie Algebra structure constants in terms of generators T^a.

$$\tilde{f}^{abc} \equiv i\sqrt{2}f^{abc} = \mathrm{Tr}\left(T^a T^b T^c\right) - \mathrm{Tr}\left(T^a T^c T^b\right), \mathrm{Tr}(T^a T^b) = \delta^{ab}$$

- Fierz Identity systematically combines them into a single trace.
- n-gluon tree amplitudes :

$$\mathcal{A}_n^{tree}(\{k_i, h_i, a_i\}) = \sum_{\sigma \in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}) A_n^{tree}(\sigma(1^{h_1}), \ldots, \sigma(n^{h_n}))$$

Color ordered : Planar graphs with no leg-crossings allowed



Spinor Helicity Formalism

Helicity Spinors

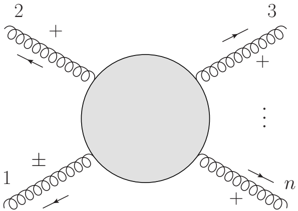
- Uniform description of the on-shell degrees of freedom (momentum and polarization).
- Spinors from massless Dirac equation.
- Kinematical DOF in terms of Spinors :
  - 4-Momentum in terms of Spinors.

$$k_i^{\mu}(\sigma_{\mu})_{\alpha\dot{\alpha}} = (\not{k}_i)_{\alpha\dot{\alpha}} = \begin{pmatrix} k_i^t + k_i^z & k_i^x - ik_i^y \\ k_i^x + ik_i^y & k_i^t - k_i^z \end{pmatrix} = (\lambda_i)_{\alpha}(\tilde{\lambda}_i)_{\dot{\alpha}} .$$

  - Polarization vectors also in terms of Spinors
-

$$\langle ij \rangle \equiv \epsilon^{\alpha\beta}(\lambda_i)_{\alpha}(\lambda_j)_{\beta} , [ij] \equiv \epsilon^{\dot{\alpha}\dot{\beta}}(\tilde{\lambda}_i)_{\dot{\alpha}}(\tilde{\lambda}_j)_{\dot{\beta}} .$$

- Renders the analytic expressions of scattering amplitudes in an often much more compact form compared to the standard four-vector notation.
- In order to uniformize the description we shall take all particles as outgoing.



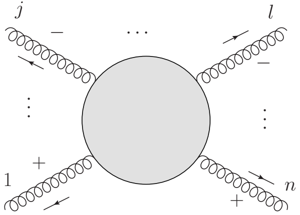
Vanishing Amplitudes

$$A_n^{tree}(1^{\pm}, 2^{+}, \ldots, n^{+}) = 0\,.$$

MHV Amplitudes

Maximally Helicity Violating

$$A_n^{tree}(\ldots, j^{-}, \ldots, l^{-}, \ldots) \;=\; \frac{\langle jl\rangle^4}{\langle 12\rangle \cdots \langle n1\rangle}.$$



[S.J.Parke, T.R Taylor, 1986]

Why so simple ?

Amplitudes have additional hidden symmetries/structure that constrain their form.

[F. Cachazo, P. Svrcek, E. Witten, 2004]

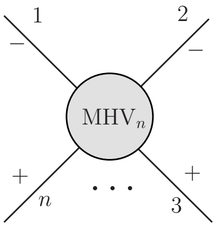
Basic idea

- Method truly motivated by the geometry.
- MHV amplitudes continued off-shell are used as interaction vertices.
- Any amplitude can be constructed by combining such vertices using scalar propagators.

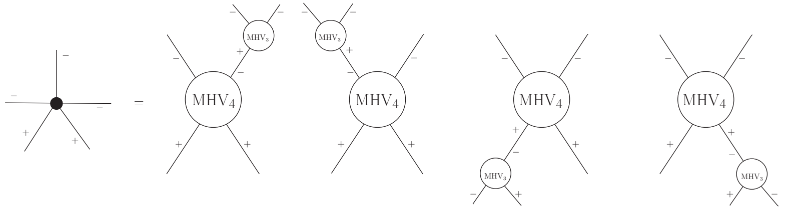
Important feature

This technique gives a simple and systematic method of computing amplitudes of gluons.

Building blocks



5 point $(- - - + +)$ in CSW method



Lagrangian origin of MHV rules.

[P. Mansfield, 2006]

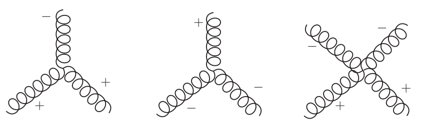
Basic Idea

$$S^{(\mathrm{LC})}_{\mathrm{Y-M}}\left[A^{+},A^{-}\right]=\left(\mathcal{L}^{(\mathrm{LC})}_{+-}+\mathcal{L}^{(\mathrm{LC})}_{++-}+\mathcal{L}^{(\mathrm{LC})}_{+--}+\mathcal{L}^{(\mathrm{LC})}_{++--}\right).$$

- Only plus-helicity and minus-helicity gluon fields.

$$\left\{A^{+},A^{-}\right\}\rightarrow\left\{B^{+},B^{-}\right\}$$

Interaction vertices



Transformation

$$\mathcal{L}^{(\mathrm{LC})}_{+-}+\mathcal{L}^{(\mathrm{LC})}_{++-}\longrightarrow\mathcal{L}^{(\mathrm{LC})}_{+-}$$

MHV action : Action with MHV vertices

$$S^{(\mathrm{LC})}_{\mathrm{Y-M}}\left[B^{+},B^{-}\right]=\left(\mathcal{L}^{(\mathrm{LC})}_{+-}+\mathcal{L}^{(\mathrm{LC})}_{--+}+\cdots+\mathcal{L}^{(\mathrm{LC})}_{--+\cdots+}+\dots\right)$$

MHV Vertices (color stripped)

$$\mathcal{V}\left(1^-,2^-,3^+,\ldots,n^+\right) \equiv \left(\frac{p_1\cdot\eta}{p_2\cdot\eta}\right)^2 \frac{\widetilde{v}_{21}^{*4}}{\widetilde{v}_{1n}^{*}\widetilde{v}_{n(n-1)}^{*}\widetilde{v}_{(n-1)(n-2)}^{*}\ldots\widetilde{v}_{21}^{*}}$$

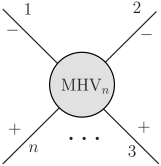
The $\widetilde{v}_{ij}$, $\widetilde{v}_{ij}^{\star}$ are off shell extension of spinor products $\langle ij\rangle$, $[ij]$.

$$\widetilde{v}_{ij}^{*} = p_i\cdot\eta\left(\frac{p_j\cdot\epsilon_\perp^+}{p_j\cdot\eta} - \frac{p_i\cdot\epsilon_\perp^+}{p_i\cdot\eta}\right)$$

$$\eta = (1,0,0,-1)/\sqrt{2}, \quad \epsilon_\perp^+ = (0,1,+i,0)/\sqrt{2}$$

Important points

- The transformation results in only MHV vertices.
- The presence of one triple gluon vertex $(--+)$.
- Interpretation of $B^{\pm}[A^{\pm}]$?



$B^{\pm}[A^{\pm}]$ Fields

Wilson Line

$$\mathcal{W}[A]\,(x,y) = \mathbb{P}\exp\left[ig\int_C dz_\mu\,\hat{A}^\mu\,(z)\right]$$

$B^{\pm}[A^{\pm}]$ as Wilson lines

[P. Kotko, 2014], [P. Kotko, A. Stasto, 2017]

$B^{+}[A^{\pm}]$

$$B_a^{+}[A]\,(x) = \int_{-\infty}^{\infty} d\alpha\,\mathrm{Tr}\left\{\frac{1}{2\pi g}t^a\partial_-\,\mathbb{P}\exp\left[ig\int_{-\infty}^{\infty} ds\,\varepsilon_\alpha^{+}\cdot\hat{A}\left(x+s\varepsilon_\alpha^{+}\right)\right]\right\}$$

$$\varepsilon_\alpha^{+} = \epsilon_\perp^{+} - \alpha\eta, \quad \hat{A} = A_a t^a$$

[H. Kakkad, P. Kotko, A. Stasto, 2020]

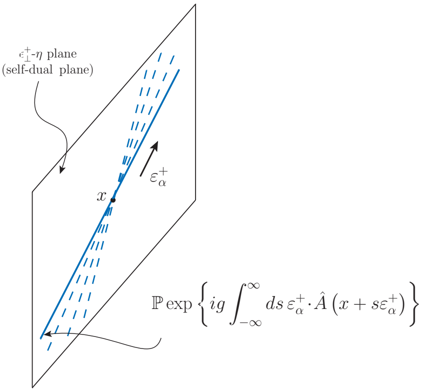
$B^{-}[A^{\pm}]$

$$B_a^{-}(x) = \int d^3\mathbf{y}\left[\frac{\partial_-^2(y)}{\partial_-^2(x)}\frac{\delta B_a^{+}(x^+;\mathbf{x})}{\delta A_c^{+}(x^+;\mathbf{y})}\right]A_c^{-}(x^+;\mathbf{y})$$

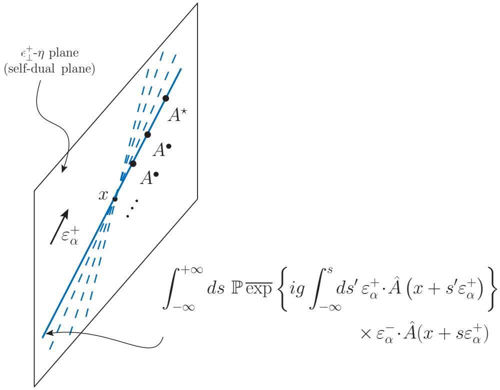
$B^{\pm}[A^{\pm}]$ Fields

Geometrical Representation.

$B^{+}[A^{+}]$



$B^{-}[A^{\pm}]$



$B^{-}[A^{\pm}]$

Cut through a bigger structure spanning two planes ?

Reminder

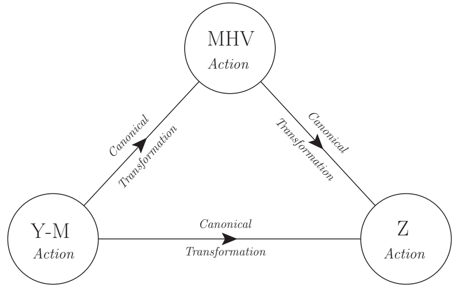
The presence of one triple gluon vertex $(--+)$ in the MHV action.

Motivation

- New classical action which does not involve any triple-gluon vertices.
- Extension of the geometric structure.

[H. Kakkad, P. Kotko, A. Stasto, 2021]- arXiv :2102.11371

Z Action



$$\left\{B^+, B^-\right\} \to \left\{Z^+, Z^-\right\}$$

Z Action

Structure of the new action

$$S_{\mathrm{Y-M}}^{(\mathrm{LC})}\left[Z^{+},Z^{-}\right]=\left\{\mathcal{L}_{-+}^{(\mathrm{LC})}+\mathcal{L}_{--++}^{(\mathrm{LC})}+\mathcal{L}_{--+++}^{(\mathrm{LC})}+\mathcal{L}_{--++++}^{(\mathrm{LC})}+\dots\right.$$
$$+\mathcal{L}_{---++}^{(\mathrm{LC})}+\mathcal{L}_{---+++}^{(\mathrm{LC})}+\mathcal{L}_{---++++}^{(\mathrm{LC})}+\dots$$
$$\vdots$$
$$\left.+\mathcal{L}_{---\dots-++}^{(\mathrm{LC})}+\mathcal{L}_{---\dots-+++}^{(\mathrm{LC})}+\mathcal{L}_{---\dots-++++}^{(\mathrm{LC})}+\dots\right\}$$

Example : $\mathcal{L}_{----++}^{(\mathrm{LC})}$

$$=\left(\frac{p_1\cdot\eta}{p_2\cdot\eta}\right)^2\frac{\widetilde{v}_{21}^{*4}}{\widetilde{v}_{16}^{*}\widetilde{v}_{6(345)}^{*}\widetilde{v}_{(345)2}^{*}\widetilde{v}_{21}^{*}}\times\left(\frac{p_5\cdot\eta}{p_{345}\cdot\eta}\right)^2\frac{\widetilde{v}_{(345)3}}{\widetilde{v}_{54}\widetilde{v}_{43}\widetilde{v}_{3(345)}}$$
$$+\left(\frac{p_3\cdot\eta}{p_4\cdot\eta}\right)^2\frac{\widetilde{v}_{43}^{*4}}{\widetilde{v}_{3(612)}^{*}\widetilde{v}_{(612)5}^{*}\widetilde{v}_{54}^{*}\widetilde{v}_{43}^{*}}\times\left(\frac{p_6\cdot\eta}{p_{612}\cdot\eta}\right)^2\frac{\widetilde{v}_{(612)6}}{\widetilde{v}_{21}\widetilde{v}_{16}\widetilde{v}_{6(612)}}+\dots$$

Z Action

Important features

- There are no three point interaction vertices.
- At the classical level there are no all-plus, all-minus, as well as $(-+\cdots+)$, $(-\cdots-+)$ vertices.
- There are MHV vertices, $(--+\cdots+)$, corresponding to MHV amplitudes in the on-shell limit.

$$\mathcal{A}\left(1^-,2^-,3^+,\ldots,n^+\right) \equiv \left(\frac{p_1\cdot\eta}{p_2\cdot\eta}\right)^2 \frac{\widetilde{v}^{*4}_{21}}{\widetilde{v}^{*}_{1n}\widetilde{v}^{*}_{n(n-1)}\widetilde{v}^{*}_{(n-1)(n-2)}\cdots\widetilde{v}^{*}_{21}}$$

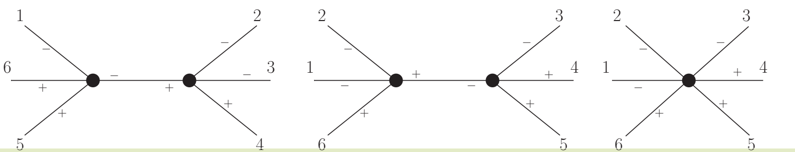
- There are $\overline{\mathrm{MHV}}$ vertices, $(-\cdots-++)$, corresponding to $\overline{\mathrm{MHV}}$ amplitudes in the on-shell limit.

$$\mathcal{A}\left(1^-,\ldots,n-2^-,n-1^+,n^+\right) \equiv \left(\frac{p_{n-1}\cdot\eta}{p_n\cdot\eta}\right)^2 \frac{\widetilde{v}^{4}_{n(n-1)}}{\widetilde{v}_{1n}\widetilde{v}_{n(n-1)}\widetilde{v}_{(n-1)(n-2)}\cdots\widetilde{v}_{21}}$$

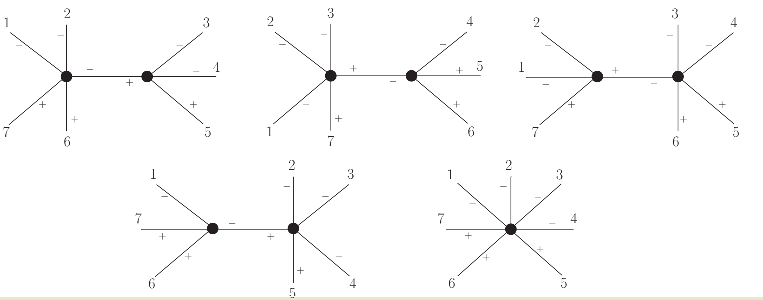
- All vertices have the form which can be easily calculated.

Calculating scattering amplitudes in Z theory

6 point NMHV $(- - - + + +)$

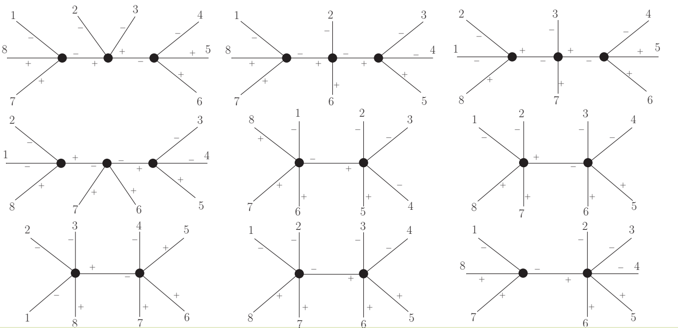


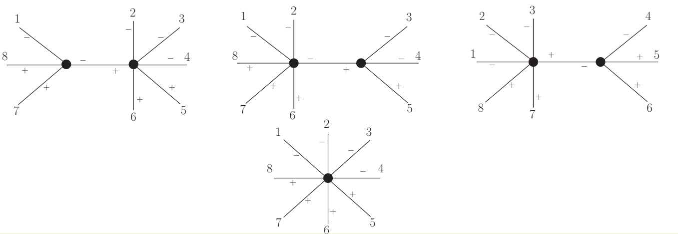
7 point NNMHV $(- - - - + + +)$



Amplitudes

8 point NNMHV $(- - - - + + ++)$





Amplitudes Overview

# legs	helicity	# diagrams
4 point	MHV	1
	$\overline{\text{MHV}}$	1
5 point	MHV	1
	$\overline{\text{MHV}}$	1
6 point	MHV	1
	NMHV	3
	$\overline{\text{MHV}}$	1
7 point	MHV	1
	NMHV	5
	NNMHV	5
	$\overline{\text{MHV}}$	1
8 point	MHV	1
	NMHV	7
	NNMHV	13
	NNNMHV	7
	$\overline{\text{MHV}}$	1

Z field Wilson Line

$Z^{\pm}[B^{\pm}]$ as Wilson Line functionals

$Z^{-}[B^{\pm}]$

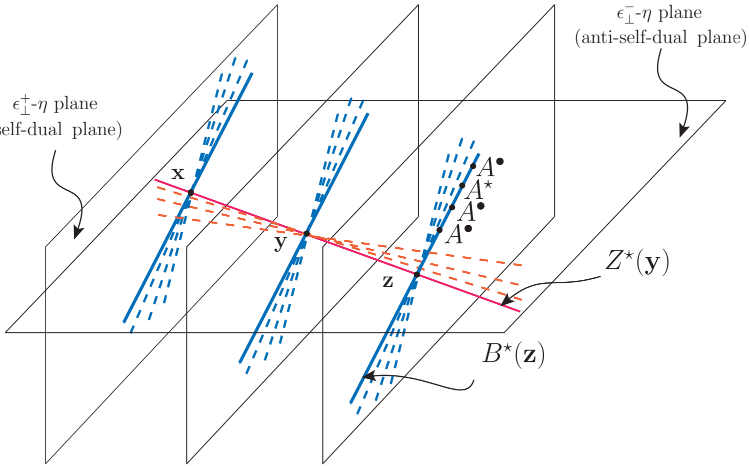
$$Z_a^{-}[B](x) = \int_{-\infty}^{\infty} d\alpha \,\mathrm{Tr} \left\{ \frac{1}{2\pi g} t^a \partial_- \, \mathbb{P} \exp \left[ig \int_{-\infty}^{\infty} ds \, \varepsilon_{\alpha}^{-} \cdot \hat{B} \left(x + s\varepsilon_{\alpha}^{-} \right) \right] \right\}$$

$$\varepsilon_{\alpha}^{-} = \epsilon_{\perp}^{-} - \alpha\eta, \quad \hat{B} = B_a t^a$$

$Z^{+}[B^{\pm}]$

$$Z_a^{+}(x) = \int d^3\mathbf{y} \left[\frac{\partial_-^2(y)}{\partial_-^2(x)} \frac{\delta Z_a^{-}(x^+;\mathbf{x})}{\delta B_c^{-}(x^+;\mathbf{y})} \right] B_c^{+}(x^+;\mathbf{y})$$

Geometrical Representation $Z^{-}[B^{-}]$



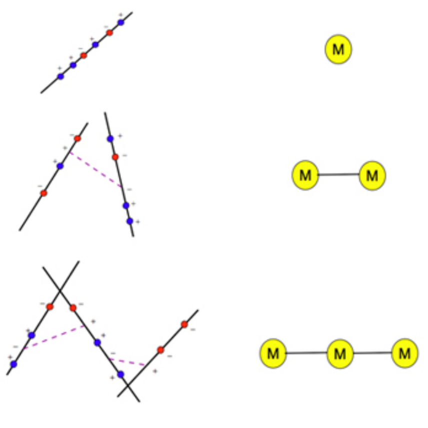
Summary

- Feynman diagram technique is not the best way to calculate amplitudes.
- Simplicity of MHV amplitudes led to the development of different techniques.
- The CSW action can be derived using field transformation whose solutions are given by certain Wilson Lines.
- Z-theory action has no triple-gluon vertices. The starting point is 4-point MHV.
- Vertices in Z-action have easy calculable form.
- Pure gluonic scattering amplitudes can be calculated conviniently with very few number of diagrams in Z-theory.
- The Z-theory is geometrically rich and intriguing.

Thank You for your Time !

Back-up

Geometrical Origin



Correspondence

Lines in Twistor space $\Longleftrightarrow$ Points in Minkowski Space

BCFW

BCFW Technique

- Complex shift the momentum of two consecutive legs such that the total momentum remains conserved.

$$\begin{array}{lll} \tilde{\lambda}_n & \rightarrow & \hat{\tilde{\lambda}}_n = \tilde{\lambda}_n - z\tilde{\lambda}_1 \,, \qquad \lambda_n \rightarrow \lambda_n \,, \\ \lambda_1 & \rightarrow & \hat{\lambda}_1 = \lambda_1 + z\lambda_n \,, \qquad \tilde{\lambda}_1 \rightarrow \tilde{\lambda}_1 \end{array}$$

- Identify the location of complex poles.
- At poles the amplitude factorizes and the off-shell particle goes on-shell.
- Express amplitude in terms of residues.
- Apply Cauchy's residue theorem under the limit that the residue for large 'z' vanishes'.

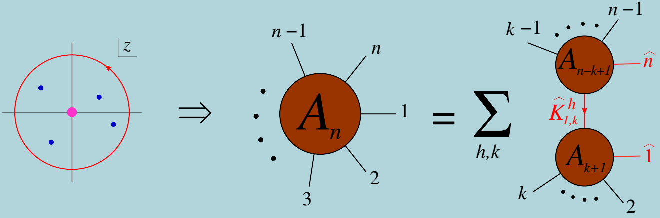


FIGURE – Illustration of how Cauchy's theorem leads to the BCFW recursion relation

THE END !