Higher-order corrections to exclusive heavy vector meson production

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Exclusive vector meson production in deep inelastic scattering

- $\gamma_\lambda^* + A \to V_{\lambda'} + A$
- (Heavy) vector mesons: $V = \mathrm{J}/\psi, \Upsilon \ldots$
- Polarization mixing highly suppressed $\Rightarrow \lambda = \lambda' = L, T$

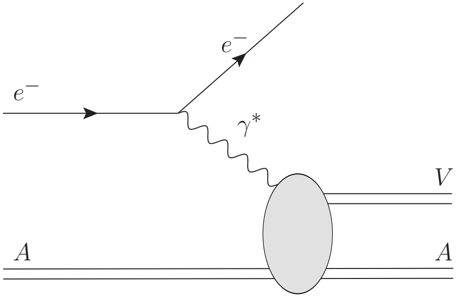
Why is this process interesting?

Provides information about the gluon structure of the nucleus:

- Requires an exchange of at least two gluons

  $\Rightarrow$ Cross section is sensitive to the squared gluon density
- The momentum transfer Δ can be measured
  - Conjugate of the impact parameter b

    $\Rightarrow$ Measures impact parameter dependent gluon distribution



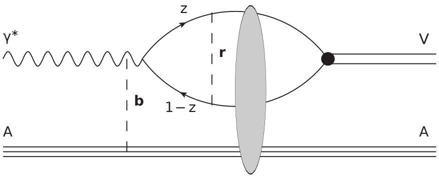
Vector meson production at the leading order in the dipole picture

- Calculation done using light-cone perturbation theory
- Factorizes at the high-energy limit:

$$\operatorname{Im} \mathcal{A}^{\lambda} = 2 \int \mathrm{d}^2\mathbf{b}\,\mathrm{d}^2\mathbf{r}\,\frac{\mathrm{d}z}{4\pi} e^{-i\left(\mathbf{b}+\left(\frac{1}{2}-z\right)\mathbf{r}\right)\cdot\Delta} \Psi_{\gamma^*}^{q\bar{q}}(\mathbf{r}, z) N(\mathbf{r}, \mathbf{b}, Y) \Psi_{V}^{q\bar{q}*}(\mathbf{r}, z)$$

- $\Psi_{\gamma^*}^{q\bar{q}}$: Photon light-front wave function
- N: Dipole-target scattering amplitude
- $\Psi_{V}^{q\bar{q}}$: Vector meson light-front wave function
- Mixed coordinate space:

  Transverse separation r, longitudinal momentum fraction z
- In the high-energy limit, the interaction of the dipole with the target is *eikonal*
  - The coordinates r and z are conserved in the interaction



NLO corrections in the dipole picture

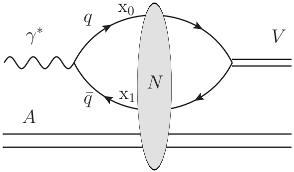
- At higher orders: perturbative corrections from gluon emission

  $\Rightarrow$ Need also the wave functions for the $q\bar{q}g$ state: $\Psi_{\gamma^*}^{q\bar{q}g}$ and $\Psi_{V}^{q\bar{q}g}$

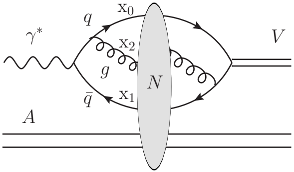
Both LO and NLO: three parts needed for the calculation

- Virtual photon light-front wave functions $\Psi_{\gamma^*}^{q\bar{q}}, \Psi_{\gamma^*}^{q\bar{q}g}$ from perturbative QCD [Beuf, Lappi, Paatelainen, 2103.14549]
- Rapidity $Y = \ln 1/x$ dependence of the dipole-target amplitude N described by perturbative evolution equations (Balitksy-Kovchegov equation)
- Meson light-front wave functions $\Psi_{V}^{q\bar{q}}, \Psi_{V}^{q\bar{q}g}$ nonperturbative – what can we say about these?

LO:



NLO:

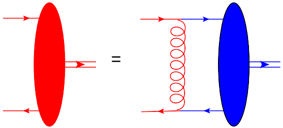


Vector meson wave function

- Heavy vector meson Fock states: $|V\rangle = \Psi_V^{q\bar{q}} |q\bar{q}\rangle + \Psi_V^{q\bar{q}g} |q\bar{q}g\rangle +$ higher orders
- Corrections in α_s and the heavy quark's velocity v
- Nonrelativistic expansion [Escobedo and Lappi, 1911.01136]:

$$\Psi_V^n = \sum_{m,k} C_{n\leftarrow m}^k \int_0^1 \frac{\mathrm{d}z'}{4\pi} \left(\frac{1}{m_q}\nabla\right)^k \phi^m(\mathrm{r}=0,z')$$

  - ϕ^m = leading-order wave function for Fock state m
  - α_s corrections in $C_{n\leftarrow m}^k$, relativistic corrections go as v^k



A loop correction to LFWF [1911.01136]

- We consider two types of higher-order corrections:

Relativistic corrections at LO, $\alpha_s^0 v^2$

$$\Psi_V^{q\bar{q}} = \sum_{k_1,k_2,k_3=0}^{2} \frac{1}{k_1!k_2!k_3!}(m_q r_1)^{k_1}(m_q r_2)^{k_2} 4\pi \left(-\frac{1}{2i}\partial_z\right)^{k_3} \delta(z-1/2)$$
$$\times \int_0^1 \frac{\mathrm{d}z'}{4\pi} \frac{1}{m_q^{k_1+k_2}} [2i(z'-1/2)]^{k_3} \partial_1^{k_1}\partial_2^{k_2} \phi^{q\bar{q}}(\mathrm{r}=0,z')$$

NLO corrections in the nonrelativistic limit, $\alpha_s v^0$

$$\Psi_V^{q\bar{q}} = C_{q\bar{q}\leftarrow q\bar{q}}^0 \int_0^1 \frac{\mathrm{d}z'}{4\pi} \phi^{q\bar{q}}(\mathrm{r}=0,z')$$

$$\Psi_V^{q\bar{q}g} = C_{q\bar{q}g\leftarrow q\bar{q}}^0 \int_0^1 \frac{\mathrm{d}z'}{4\pi} \phi^{q\bar{q}}(\mathrm{r}=0,z')$$

Relativistic corrections

- Nonperturbative constants $\int_0^1 \frac{\mathrm{d}z'}{4\pi}[2i(z'-1/2)]^{k_3}\partial_1^{k_1}\partial_2^{k_2}\phi^{q\bar{q}}$
- Plan: Determine these constants at the order v^2
- We can do this by looking at the wave function in the *rest frame*
- The rest frame wave function is described in terms of spin and 3-position $\vec{r} = (x^1, x^2, x^3)$
  - J^{PC} conservation $\Rightarrow$ a combination of S and D waves
- From NRQCD (*Non-Relativistic QCD*): D wave velocity-suppressed

  $\Rightarrow$ Need to consider only S wave:

$$\phi_{s\bar{s}}^{q\bar{q}}(\lambda, \vec{r}) = \frac{1}{\sqrt{2}}\xi_s^\dagger \epsilon^\lambda \cdot \sigma \chi_{\bar{s}}\,\phi(r)$$

  - ϵ^λ = the polarization vector of J/ψ
  - ξ_s and $\chi_{\bar{s}}$ = quark and antiquark spinors with spins s and $\bar{s}$
  - ϕ = "scalar" part of the wave function – depends only on $r = |\vec{r}|$

NRQCD-motivated wave function

- Our approach: Expand the wave function in the *rest frame* [Lappi, Mäntysaari, Penttala, 2006.02830]:

$$\phi(r) = \underbrace{\phi(0)}_{\mathcal{O}(v^0)} + \frac{1}{6} \underbrace{\nabla^2 \phi(0) r^2}_{\mathcal{O}(v^2)} + \mathcal{O}(v^4)$$

- Two unknown constants $\phi(0), \nabla^2\phi(0)$
  - Related to wave function and its derivative at origin $\vec{r} = 0$
  - Can be written in terms of NRQCD *long-distance matrix elements* (LDMEs)
  - For J/ψ: LDMEs determined from charmonium decay widths [Bodwin et al., 0710.0994]

$$\Rightarrow \phi(0) = 0.213 \ \mathrm{GeV}^{3/2}, \quad \frac{1}{6}\nabla^2\phi(0) = -0.0157 \ \mathrm{GeV}^{7/2}$$

From rest frame to light front

- Now we have our vector meson wave function with relativistic corrections in rest frame
  – But this is not the light-front wave function!

Two things to take into account:

1. Different coordinates:

   $\vec{r}$ in rest frame vs (r, z) in light front

$$\vec{r} \xrightarrow{\text{3D Fourier}} \vec{k} \xrightarrow{k^3 \to z} (\mathrm{k}, z) \xrightarrow{\text{2D Fourier}} (\mathrm{r}, z)$$

2. Different spinors in rest frame and light front:

   Bjorken-Drell u_s (rest frame) vs Lepage-Brodsky u_h (light-front)

   $\Rightarrow$ Change of spinor basis (also called *Melosh rotation*)

   $\Psi^{\lambda}_{h\bar{h}}(\mathrm{k}, z) = \sum_{s\bar{s}} R^{*sh}(\mathrm{k}, z) R^{*\bar{s}\bar{h}}(-\mathrm{k}, 1-z) \Psi^{\lambda}_{s\bar{s}}(\mathrm{k}, z)$, where $R^{sh}(\mathrm{k}, z) = \frac{1}{2m_c} \bar{u}_s(\mathrm{k}, z) u_h(\mathrm{k}, z)$

After all this we have a light-front wave function with relativistic corrections of order v^2!

Meson light-front wave function with relativistic corrections

$$\Psi^{\lambda=0}_{+-}(\mathrm{r},z) = \Psi^{\lambda=0}_{-+}(\mathrm{r},z) = \frac{\pi\sqrt{2}}{\sqrt{m_c}}\left[\phi(0)\delta(z-1/2) + \frac{\nabla^2\phi(0)}{6m_c^2}\left(\left(\frac{5}{2}+\mathrm{r}^2 m_c^2\right)\delta(z-1/2) - \frac{1}{4}\partial_z^2\delta(z-1/2)\right)\right]$$

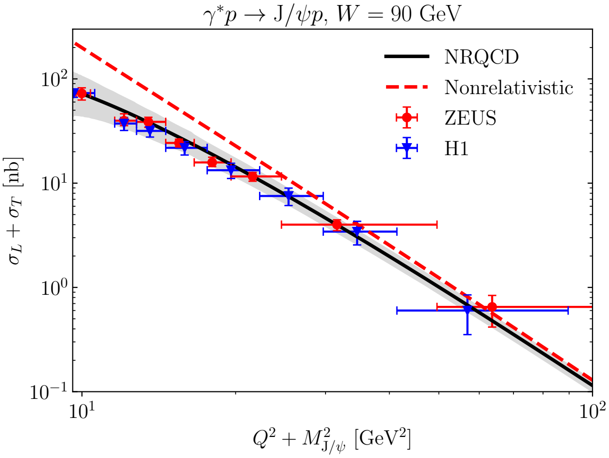
$$\Psi^{\lambda=1}_{++}(\mathrm{r},z) = \Psi^{\lambda=-1}_{--}(\mathrm{r},z) = \frac{2\pi}{\sqrt{m_c}}\left[\phi(0)\delta(z-1/2) + \frac{\nabla^2\phi(0)}{6m_c^2}\left(\left(\frac{7}{2}+\mathrm{r}^2 m_c^2\right)\delta(z-1/2) - \frac{1}{4}\partial_z^2\delta(z-1/2)\right)\right]$$

$$\Psi^{\lambda=1}_{+-}(\mathrm{r},z) = -\Psi^{\lambda=1}_{-+}(\mathrm{r},z) = \left(\Psi^{\lambda=-1}_{-+}(\mathrm{r},z)\right)^* = \left(-\Psi^{\lambda=-1}_{+-}(\mathrm{r},z)\right)^* = -\frac{\pi i}{3m_c^{3/2}}\nabla^2\phi(0)\delta(z-1/2)(r_1+ir_2)$$

$$\Psi^{\lambda=1}_{--}(\mathrm{r},z) = \Psi^{\lambda=-1}_{++}(\mathrm{r},z) = \Psi^{\lambda=0}_{++}(\mathrm{r},z) = \Psi^{\lambda=0}_{--}(\mathrm{r},z) = 0$$

Relativistic corrections at LO in α_s: phenomenology

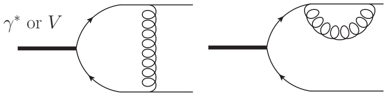
- *Nonrelativistic*: only the v^0 part
- *NRQCD*: v^2 corrections included
  - Error band from the uncertainty of the NRQCD matrix elements
- W = center-of-mass energy for the $\gamma^* p$ system
- Q^2 = photon virtuality
- Relativistic corrections are important at small Q^2
  - Large Q^2: v^2 corrections become almost negligible
- Including v^2 corrections results in a good agreement with the data



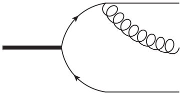
[Lappi, Mäntysaari, Penttala, 2006.02830]
HERA data from [hep-ex/0510016] and [hep-ex/0404008]

NLO in the nonrelativistic limit

$q\bar{q}$ (virtual corrections):



$q\bar{q}g$ (real corrections):



- Corrections from real and virtual gluons to the wave functions
- UV divergences between the $q\bar{q}$ and $q\bar{q}g$ parts of the calculation cancel
- IR divergences cancel when one takes into account:
  - Renormalization of the leading-order wave function $\phi^{q\bar{q}}$
    - Can be written in terms of the leptonic decay width
  - The rapidity dependence of the dipole amplitude which can be described in terms of the Balitsky-Kovchegov (BK) equation:

$$\frac{\partial}{\partial Y} N(\mathrm{x}_{01}) = \frac{N_c \alpha_s}{2\pi^2} \int \mathrm{d}^2 \mathrm{x}_2 \, \frac{\mathrm{x}_{01}^2}{\mathrm{x}_{20}^2 \mathrm{x}_{21}^2} \left[N(\mathrm{x}_{02}) + N(\mathrm{x}_{12}) - N(\mathrm{x}_{01}) - N(\mathrm{x}_{02}) N(\mathrm{x}_{12}) \right], \quad \mathrm{x}_{ij} = \mathrm{x}_i - \mathrm{x}_j$$

$\Rightarrow$ The total production amplitude is finite and can be numerically evaluated

Final expression (longitudinal production)

$$-i\mathcal{A}^L = -Q\sqrt{\Gamma(V\to e^-e^+)\frac{3M_V}{16\pi^2\alpha_{\mathrm{em}}}}\int \mathrm{d}^2\mathbf{x}_{01}\int \mathrm{d}^2\mathbf{b}\left\{\mathcal{K}^{\mathrm{LO}}_{q\bar q}(Y_0)+\frac{\alpha_s C_F}{2\pi}\mathcal{K}^{\mathrm{NLO}}_{q\bar q}(Y_{\mathrm{dip}})+\frac{\alpha_s C_F}{2\pi}\int \mathrm{d}^2\mathbf{x}_{20}\int_{z_{\min}}^{1/2}\mathrm{d}z_2\,\mathcal{K}_{q\bar q g}(Y_{\mathrm{qqg}})\right\}$$

where $\mathcal{K}^{\mathrm{LO}}_{q\bar q}(Y_0) = K_0(\zeta)N_{01}(Y_0)$, $\zeta = |\mathbf{x}_{01}|\sqrt{\frac{1}{4}Q^2+m_q^2}$,

$$\mathcal{K}^{\mathrm{NLO}}_{q\bar q}(Y_{\mathrm{dip}}) = \left[\mathcal{K}+\tilde{\mathcal{I}}_\nu\left(z=\frac{1}{2},\mathbf{x}_{01}\right)+K_0(\zeta)\left(6-\frac{\pi^2}{3}+\Omega_{\mathcal{V}}\left(\gamma;z=\frac{1}{2}\right)+L\left(\gamma;z=\frac{1}{2}\right)-3\log\left(\frac{|\mathbf{x}_{10}|m}{2}\right)-3\gamma_E\right)\right]N_{01}(Y_{\mathrm{dip}})$$

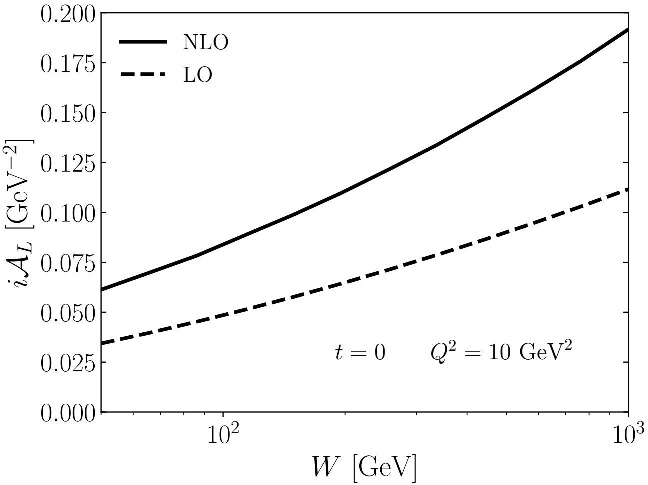
and

$$\mathcal{K}_{q\bar q g}(Y_{\mathrm{q\bar q g}}) = -32\pi m_q\left\{\frac{i\mathbf{x}^i_{20}}{|\mathbf{x}_{20}|}K_1(2m_q z_2|\mathbf{x}_{20}|)\left[\left((1-z_2)^2+z_2^2\right)\mathcal{I}^i_{(f)}+(2z_2^2-1)(1-2z_2)\mathcal{I}^i_{(g)}\right]N_{012}(Y_{\mathrm{q\bar q g}})\right.$$
$$\left.+4m_q z_2^3 K_1(2m_q z_2|\mathbf{x}_{20}|)\left[\mathcal{I}_{(f)}-\frac{1-2z_2}{1+2z_2}\mathcal{I}_{(g)}\right]N_{012}(Y_{\mathrm{q\bar q g}})+\frac{1}{8\pi^2}\left((1-z_2)^2+z_2^2\right)\frac{1}{m_q z_2|\mathbf{x}_{20}|^2}K_0(\zeta)e^{-\mathbf{x}^2_{20}/(\mathbf{x}^2_{10}e^{\gamma_E})}N_{01}(Y_{\mathrm{q\bar q g}})\right\}.$$

Equation for transverse production similar but more complicated.

NLO amplitude for $\gamma_L^* + p \to \mathrm{J}/\psi + p$

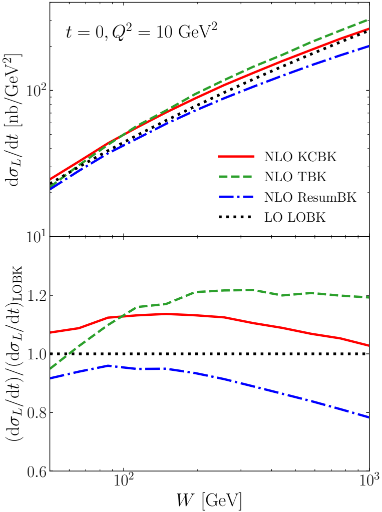
- We have evaluated longitudinal production numerically
  - Transverse production a work-in-progress
- NLO corrections are large
  - $\sim$ 75% of the LO result
- Significant increase of the amplitude
- However, the production amplitude depends on the dipole amplitude used
  - Here the same dipole amplitude was used for both cases
  - But this is not actually consistent as we'll see. . .



[Mäntysaari, Penttala, 2104.02349]

NLO cross section for $\gamma_L^* + p \to \mathrm{J}/\psi + p$ with different dipole amplitudes

- Compare dipole amplitudes from different fits
- The dipole amplitudes need a fitted initial condition
  - Usually done using HERA structure function data
  - *LOBK* = LO fit [Lappi and Mäntysaari, 1309.6963]
  - *KCBK, TBK, ResumBK* = NLO fits [Beuf et al., 2007.01645]
- The difference between the LO and NLO results is smaller than what the amplitude plot indicates
  - LO fit compensates for NLO effects
    $\Rightarrow$ Important to use the fit with the same order!
- Some variation between the different NLO fits
  - Complementary information to structure function analyses
  - Probe target structure at different length scales than structure functions



[Mäntysaari, Penttala, 2104.02349]

Summary

- Relativistic corrections to exclusive heavy vector meson production at order v^2 were determined
- NLO corrections to longitudinal production were calculated at the nonrelativistic limit
- Relativistic corrections are important for small Q^2
  - Good agreement with the data
- NLO corrections found to be important in the calculation
  - However, LO dipole amplitude fit can capture most of the NLO effects
- Some deviations between different NLO fits

  $\Rightarrow$ Complementary probe to structure functions
- Future: transverse production at NLO
  - Calculations very similar to the longitudinal case
  - Will allow comparison of the NLO results to the data
- Important developments: precise measurements expected at ultra-peripheral collisions at the LHC and the future Electron-Ion Collider

Backup

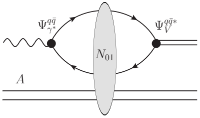
Backup - $q\bar{q}$ and $q\bar{q}g$ parts of the calculation

LO result: $-\frac{ee_f Q}{2\pi}\sqrt{\frac{N_c}{2}}2\int_0^1 \frac{\mathrm{d}z'}{4\pi}\phi^{q\bar{q}}(\mathrm{r}=0,z')\int \mathrm{d}^2\mathrm{x}_{01}\, N_{01} K_0(\zeta),\ \zeta = |\mathrm{x}_{01}|\sqrt{\frac{1}{4}Q^2 + m_q^2}$

NLO, $q\bar{q}$ (dipole) part:

$$-\frac{ee_f Q}{2\pi}\sqrt{\frac{N_c}{2}}\frac{\alpha_s C_F}{2\pi}2\int_0^1 \frac{\mathrm{d}z'}{4\pi}\phi^{q\bar{q}}(\mathrm{r}=0,z')\int \mathrm{d}^{D-2}\mathrm{x}_{01}\, N_{01}\Bigg\{-\frac{2}{D-4}\left(4\log(2\alpha)+3\right)K_{D/2-2}(\zeta)+\mathcal{K}_0+\tilde{\mathcal{I}}_\mathcal{V}\left(z_0=\frac{1}{2},\mathrm{x}_{01}\right)$$

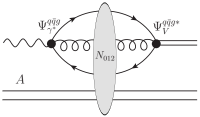
$$+K_0(\zeta)\left[\frac{1}{\alpha}+3-\frac{\pi^2}{3}+L\left(\gamma;z_0=\frac{1}{2}\right)+\Omega_\mathcal{V}\left(\gamma;z_0=\frac{1}{2}\right)+4\gamma_E\log(2\alpha)+4\log(2\alpha)\log\left(\frac{2\pi^2|\mathrm{x}_{01}|^3\mu^2}{\sqrt{\frac{1}{4}Q^2+m^2}}\right)+3\log\left(\frac{4\pi^2|\mathrm{x}_{01}|^2\mu^2}{m\sqrt{\frac{1}{4}Q^2+m_q^2}}\right)\right]\Bigg\}$$



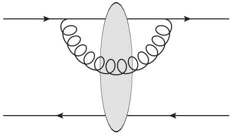
NLO, $q\bar{q}g$ part:

$$-\frac{ee_f Q}{2\pi}\sqrt{\frac{N_c}{2}}\frac{\alpha_s C_F}{2\pi}2\int_0^1 \frac{\mathrm{d}z'}{4\pi}\phi^{q\bar{q}}(\mathrm{r}=0,z')\int \mathrm{d}^{D-2}\mathrm{x}_{01}\,\mathrm{d}^{D-2}\mathrm{x}_{20}\int_\alpha^{1/2}\mathrm{d}z_2\,(-32\pi m_q)N_{012}\left(\frac{m_q z_2}{\pi|\mathrm{x}_{20}|\mu}\right)^{D/2-2}$$

$$\times\Bigg\{i\frac{\mathrm{x}_{20}^i}{|\mathrm{x}_{20}|}K_{D/2-1}\left(2m_q z_2|\mathrm{x}_{02}|\right)\left[(1-z_2)^2+(D-3)z_2^2\right]\mathcal{I}^i_{(f)}+i\frac{\mathrm{x}_{20}^i}{|\mathrm{x}_{20}|}K_1\left(2m_q z_2|\mathrm{x}_{02}|\right)\left[(2z_2^2-1)(1-2z_2)\right]\mathcal{I}^i_{(g)}+4m_q z_2^3 K_0\left(2m_q z_2|\mathrm{x}_{02}|\right)\left[\mathcal{I}_{(f)}-\frac{1-2z_2}{1+2z_2}\mathcal{I}_{(g)}\right]\Bigg\}$$



Backup - UV subtraction



- Divergences at the limit $D \to 4$ from virtual gluon loops
- Also present in the $q\bar{q}g$ part
- Solution: subtract the divergence from $q\bar{q}g$ and add it to the dipole part

$$N_{012} x_{20}^i \mathcal{I}^i_{(f)} K_{D/2-1}\left(2mz_2|x_{20}|\right) = \underbrace{\left\{ N_{012} x_{20}^i \mathcal{I}^i_{(f)} K_{D/2-1}\left(2mz_2|x_{20}|\right) - N_{01}\left[x_{20}^i \mathcal{I}^i_{(f)} K_{D/2-1}\right]_{\mathrm{UV}} \right\}}_{\text{UV finite}} + \underbrace{N_{01}\left[x_{20}^i \mathcal{I}^i_{(f)} K_{D/2-1}\right]_{\mathrm{UV}}}_{\text{UV divergent, combine with } q\bar{q} \text{ part}}$$

where

$$\left[x_{20}^i \mathcal{I}^i_{(f)} K_{D/2-1}\right]_{\mathrm{UV}} = \Gamma(D/2-1)\frac{i\mu^{2-D/2}}{4\pi^{D/2}}|x_{20}|^{4-D}\left(\frac{\sqrt{\frac{1}{4}Q^2+m^2}}{2\pi|x_{10}|}\right)^{D/2-2} K_{D/2-2}\left(|x_{10}|\sqrt{\frac{1}{4}Q^2+m^2}\right) e^{-x_{20}^2/(x_{10}^2 e^{\gamma_E})} \cdot \frac{\Gamma(D/2-1)}{2}\left(mz_2|x_{20}|\right)^{-D/2+1}$$

- Subtraction scheme from [Hänninen, Lappi and Paatelainen, 1711.08207] and [Beuf, Lappi and Paatelainen, 2103.14549]

$\Rightarrow$ UV divergences cancel in the dipole term!

Backup - Renormalization of the leading-order wave function

- We need to take into account the renormalization of the leading-order wave function $\phi^{q\bar{q}}$
- Easiest done using the NLO expression for the leptonic width [Escobedo, Lappi, 1911.01136]:

$$\Gamma(V \to e^- e^+) = \frac{2 N_c e_f^2 e^4}{3\pi M_V} \sum_{h_0' h_1'} \left| \int \frac{\mathrm{d}z'}{4\pi} \phi^{q\bar{q}}_{h_0' h_1'} \right|^2 \left[1 + \frac{2\alpha_s C_F}{\pi} \left(\frac{1}{2\alpha} - 2 \right) \right].$$

- Solve $\int \mathrm{d}z' \, \phi^{q\bar{q}}$ from this and plug it into the equation for the production amplitude

  $\Rightarrow$ Cancels the divergence $1/\alpha$ from the dipole part
- Also allows us to replace the nonperturbative leading-order wave function with the leptonic width for which we can use the experimental value

- The $q\bar{q}g$ part is singular at $\alpha \to 0$
- This is related to the rapidity evolution of the dipole amplitude, described by the Balitsky-Kovchegov (BK) equation:

$$\frac{\partial}{\partial Y} N_{01} = \frac{N_c \alpha_s}{2\pi^2} \int \mathrm{d}^2 \mathbf{x}_2 \, \frac{\mathbf{x}_{01}^2}{\mathbf{x}_{20}^2 \mathbf{x}_{21}^2} \left[N_{02} + N_{12} - N_{01} - N_{02} N_{12} \right]$$

- In fact, we can write:

$$\begin{aligned} \frac{\alpha_s}{2\pi} \int \mathrm{d}^2 \mathbf{x}_2 \int_{\alpha}^{1/2} \mathrm{d} z_2 \, \mathcal{K}_{q\bar{q}g} &= K_0(\zeta) \int \mathrm{d}^2 \mathbf{x}_2 \int_{\alpha}^{1/2} \mathrm{d} z_2 \, \frac{N_c \alpha_s}{2\pi^2 z_2} \frac{\mathbf{x}_{01}^2}{\mathbf{x}_{20}^2 \mathbf{x}_{21}^2} \left[N_{02} + N_{12} - N_{01} - N_{02} N_{12} \right] + \text{nonsingular part} \\ &= K_0(\zeta) \left[N_{01}\Big(Y(z_2 = 1/2) \Big) - N_{01}\Big(Y(z_2 = \alpha) \Big) \right] + \text{nonsingular part} \end{aligned}$$

- Combining this with the LO result, we get $Y(z_2 = \alpha) \to Y(z_2 = 1/2)$ for the evolution rapidity in the LO term