e^+e^- annihilation at NLO

Reference: Peskin, An Introduction to QFT; One-Loop calculations: See R.D. Field, Application of Perturbative QCD which does the calculation in different gauges and with different regularizations. also Schwartz, Quantum Field Theory and the Standard Model.

Analogous to the QED process $e^+e^- \to \mu^+\mu^-$, which has been computed in QED, we can study the simplest QCD process

$$e^+e^- \to q\bar{q} \to \text{hadrons}.$$

The only modification that we need to make is as follows

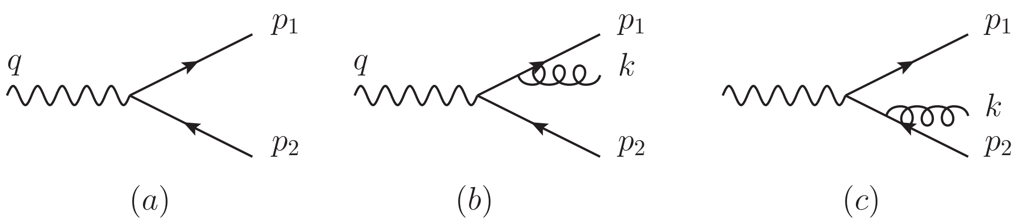
- Replace the muon charge e with the quark charge Qe.
- Count each quark three times, one for each color.
- Properly take into account the effects of strong interactions between $q\bar{q}$. Assume that $q\bar{q} \to hadrons$ process does not charge the total cross sections.

1 Leading order cross section

The process we consider is

$$e^+ + e^- \to \gamma^* \to q\bar{q}(\text{LO}) \quad \text{or} \quad q + \bar{q} + g(\text{NLO}). \tag{1}$$

To simplify the calculation we can compute this process as the decay of the virtual photon γ into $q\bar{q}$ and $q\bar{q}g$ as shown in the following figure.



Choose the centre of the mass frame of the virtual photon which gives $q = (Q, \vec{0})$ and neglect all masses for the quarks and gluon.

- According to energy momentum conservation $q = p_1 + p_2$. We can define $q^2 = Q^2 = s$ as the center of mass energy. In fact, we can view Q as the mass of the virtual photon.
- Define the leading order cross section $\gamma^* \to q\bar{q}$, which is shown in Figure (a), as the product of the flux factor $\frac{1}{2\sqrt{s}}$, the amplitude square and the two body final state phase space R_2^n (see Page 107, Eq. 4.86 of Peskin)

$$\sigma_0 = \frac{1}{2Q} \sum_{q,c,s} |\mathcal{M}_{\gamma^* \to q\bar{q}}|^2 R_2^n \tag{2}$$

where R_2^n is defined as (Here we use $n = d = 4 - 2\epsilon$ as the number of dimension)

$$\begin{aligned}
R_2^n &= \int \frac{d^{n-1}p_1}{(2\pi)^{n-1}2p_1} \int \frac{d^{n-1}p_2}{(2\pi)^{n-1}2p_2} (2\pi)^n \delta^{(n)}(q - p_1 - p_2), && (3)\\
&= \frac{1}{4(2\pi)^{n-2}} \Omega_{n-1} \int_0^\infty \frac{dp_1 p_1^{n-2}}{E_1 E_2} \delta(Q - E_1 - E_2) && (4)\\
\Downarrow & \quad \text{with} \quad \frac{dw}{w} = \frac{dp_1 p_1}{E_1 E_2} \quad \text{and } w \equiv E_1 + E_2, && (5)\\
&= \frac{1}{8\pi} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} \left(\frac{Q^2}{4\pi}\right)^{-\epsilon}, && (6)
\end{aligned}$$

where in the last step, we have used the following gamma function identity

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z), \quad \text{with} \quad z = 1-\epsilon, \tag{7}$$

where $\Gamma(t) \equiv \int_0^\infty dx x^{t-1} e^{-x}$. More gamma function identity

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}, \quad \text{with } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \tag{8}$$

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \left(\frac{1}{2}\gamma_E^2 + \frac{\pi^2}{12}\right)\epsilon + \mathcal{O}(\epsilon^2), \tag{9}$$

$$\text{the Euler constant } \gamma_E \equiv \lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{1}{m} - \ln n\right] \simeq 0.577,$$

$$\Gamma(x) = (x-1)\Gamma(x-1). \tag{10}$$

In the massless case, it is very simple to see that $E_1 = E_2 = p_1 = p_2$, which makes the above calculation straightforward.

- It is straightforward to write down the amplitude for $\gamma^* \to q\bar{q}$ as follows

$$i\mathcal{M}^{(0)}_{\gamma^*\to q\bar{q}} = (-iee_q\mu^\epsilon)\delta_{ij}\bar{u}(p_1)\gamma^\mu v(p_2)\epsilon_\mu(q), \tag{11}$$

where e_q is the charge number of the quark and δ_{ij} indicates that the quark and anti-quark should have opposite color. Remeber now that the electric coupling carries dimension of μ^ϵ in the dimensional regularization. Summing over the polarizations of the virtual photon when we square the amplitude, we can obtain

$$\sum_{q,c,s}|\mathcal{M}^{(0)}_{\gamma^*\to q\bar{q}}|^2 = -g_{\mu\nu}\sum_{q,c,s}\mathcal{M}^{\mu}_{\gamma^*\to q\bar{q}}\mathcal{M}^{\nu *}_{\gamma^*\to q\bar{q}}, \tag{12}$$

$$= -e^2\sum_q e_q^2\mu^{2\epsilon}N_c \mathrm{Tr}\left[\not{p}_1\gamma^\mu\not{p}_2\gamma_\mu\right], \tag{13}$$

$$= 4(1-\epsilon)e^2\mu^{2\epsilon}N_c Q^2\sum_q e_q^2. \tag{14}$$

The evaluation of the above trace can be done directly or use the identity $\gamma^\mu\gamma^\nu\gamma_\mu = -(d-2)\gamma^\nu$.

- Therefore, we can obtain the LO cross section as follows

$$\sigma_0 = \alpha\sum_q e_q^2\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} N_c Q\sum_q e_q^2\frac{\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)}. \tag{15}$$

- Notice that in the case of $n = 4$ dimension, $R_2 = \frac{1}{8\pi}$ and $\sigma_0 = \alpha Q N_c \sum_q e_q^2$.

2 Next-to-leading order (NLO) Real Diagram

Now consider the process $\gamma^* \to q\bar{q}g$ as shown in Figure (b) and (c). Compute the corresponding cross section which is defined as

$$\sigma_3 = \frac{1}{2Q}|\mathcal{M}_{\gamma^*\to q\bar{q}g}|^2 R_3\,, \tag{16}$$

where R_3 is the three body final state phase space defined as

$$R_3^n = \int\frac{d^{n-1}p_1}{(2\pi)^{n-1}2p_1}\int\frac{d^{n-1}p_2}{(2\pi)^{n-1}2p_2}\int\frac{d^{n-1}k}{(2\pi)^{n-1}2k}(2\pi)^n\delta^{(n)}(q-p_1-p_2-k). \tag{17}$$

- Now let us compute R_3^n

$$
\begin{aligned}
R_3^n &= \int \frac{d^{n-1}p_1}{(2\pi)^{n-1}2p_1}\int \frac{d^{n-1}p_2}{(2\pi)^{n-1}2p_2}\frac{2\pi}{2E_k}\delta(Q-E_1-E_2-E_k). \\
&\Downarrow \quad \text{with} \quad x_1=\frac{2p_1^0}{Q},\, x_2=\frac{2p_2^0}{Q},\, x_3=\frac{2k^0}{Q}=\frac{2E_k}{Q} \\
&= \left(\frac{Q}{4\pi}\right)^{2n-3}\frac{1}{Q^3}\int dx_1 d\Omega_{n-1}\int dx_2 d\Omega_{n-1} \\
&\quad \times \frac{(x_1x_2)^{n-2}}{x_1x_2x_3}\delta(x_1+x_2+x_3-2) \qquad (18)
\end{aligned}
$$

It is also interesting to notice that due to momentum conservation (as compared to $x_1+x_2+x_3=2$, which is due to energy conservation)

$$
x_3=\sqrt{x_1^2+x_2^2+2x_1x_2\cos\theta}, \qquad (19)
$$

which implies that the angle θ between $\vec{p}_1$ and $\vec{p}_2$ is determined by the delta function $\delta(x_1+x_2+x_3-2)$. At this moment, we need to use the explicit form for differential solid angle on a d-dimensional unit sphere

$$
\begin{aligned}
d\Omega_d &= \sin^{d-2}\phi_{d-1}\sin^{d-3}\phi_{d-2}\cdots\sin\phi_2 d\phi_{d-1}\cdots d\phi_2 d\phi_1 \qquad (20)\\
d\Omega_d &= d\Omega_{d-1}\sin^{d-2}\phi_{d-1}d\phi_{d-1}. \qquad (21)
\end{aligned}
$$

Ω_d can be interpreted geometrically as the $d-1$-dimensional surface area of a unit d-dimensional sphere. For example, $\Omega_3=4\pi$.

Now we let $\phi_{d-1}=\theta$ to be the angle between quark and antiquark, define $z\equiv\cos\theta$, and set $d=n-1$ in the above formula, which gives $d\Omega_{n-1}=d\Omega_{n-2}(1-z^2)^{(n-4)/2}$, therefore we can obtain

$$
\begin{aligned}
R_3^n &= \left(\frac{Q}{4\pi}\right)^{2n-3}\frac{\Omega_{n-1}\Omega_{n-2}}{Q^3}\int dx_1\int dx_2 (x_1x_2)^{n-3} \\
&\quad \times\int_{-1}^{+1}dz(1-z^2)^{(n-4)/2}\frac{1}{x_3}\delta(x_1+x_2+x_3-2) \qquad (22)\\
&\Downarrow \quad \text{note that} \int dz\delta(f(z))=\frac{1}{|f'(z)|}, \text{and} \quad \left|\frac{dx_3}{dz}\right|=\frac{x_1x_2}{x_3} \\
&= \left(\frac{Q}{4\pi}\right)^{2n-3}\frac{\Omega_{n-1}\Omega_{n-2}}{Q^3}\int dx_1dx_2\left[x_1x_2(1-z^2)^{1/2}\right]^{(n-4)}. \qquad (23)
\end{aligned}
$$

In addition, from Eq. (19) and energy conservation $x_1+x_2+x_3=2$, we also have

$$
z=\frac{x_3^2-x_1^2-x_2^2}{2x_1x_2}\Rightarrow x_1^2x_2^2(1-z^2)=4(1-x_1)(1-x_2)(x_1+x_2-1), \qquad (24)
$$

which eventually gives ($x_3<1$, $x_1+x_2>1$)

$$
\begin{aligned}
R_3^n &= \left(\frac{Q}{4\pi}\right)^{2n-3}\frac{\Omega_{n-1}\Omega_{n-2}}{Q^3}\int_0^1 dx_1\int_{1-x_1}^1 dx_2 \\
&\quad \times\left[4(1-x_1)(1-x_2)(x_1+x_2-1)\right]^{(n-4)/2}, \qquad (25)\\
&= \frac{Q^2\left(\frac{Q^2}{4\pi}\right)^{-2\epsilon}}{128\pi^3\Gamma(2-2\epsilon)}\int_0^1 dx_1\int_{1-x_1}^1 dx_2\frac{1}{[(1-x_1)(1-x_2)(1-x_3)]^{\epsilon}}.
\end{aligned}
$$

In the special case of $n=4$, one can show that $R_3=\frac{Q^2}{128\pi^3}\int_0^1 dx_1\int_{1-x_1}^1 dx_2$.

- Then our next task is to compute $|\mathcal{M}_{\gamma^*\to q\bar{q}g}|^2$. In this note, we use the Feynman gauge, which gives $-g_{\mu\nu}$ for the gluon propagator. The total result in the end is gauge independent, although the contribution from each diagram may change with different gauge choice (See Field's book). **The real diagram**

amplitudes can be written as

$$\begin{aligned} i\mathcal{M}_{\gamma^*\to q\bar{q}g} &= (-iee_q)(-ig)\mu^{2\epsilon}\epsilon^{\mu}(q)\epsilon^{*\nu}(k)t^a_{ij} \\ &\quad \times \bar{u}(p_1)\left[\gamma_\nu \frac{i(\not{p}_1+\not{k})}{(p_1+k)^2}\gamma_\mu + \gamma_\mu \frac{i(-\not{p}_2-\not{k})}{(p_2+k)^2}\gamma_\nu\right]v(p_2). \end{aligned} \tag{26}$$

It is important to remember that we have a minus sign when the momentum flow direction is in the opposite direction of the fermion charge flow. This results in a minus sign for the second term coming from figure (c). Let us call the first term in the square brackets as $i\mathcal{M}_1$ and the second as $i\mathcal{M}_2$. It is then straightforward to find that

$$|\mathcal{M}_1|^2 = (ee_q g\mu^{2\epsilon})^2 C_F N_c \mathrm{Tr}\left[\gamma_\nu \frac{(\not{p}_1+\not{k})}{(p_1+k)^2}\gamma_\mu \not{p}_2 \gamma^\mu \frac{(\not{p}_1+\not{k})}{(p_1+k)^2}\gamma^\nu \not{p}_1\right], \tag{27}$$

$$\Downarrow \qquad \gamma_\mu \not{p}_2 \gamma^\mu = -2(1-\epsilon)\not{p}_2 \text{ and } \gamma_\nu \not{p}_1 \gamma^\nu = -2(1-\epsilon)\not{p}_1$$

$$= \frac{(ee_q g\mu^{2\epsilon})^2 C_F N_c}{D_1^2} 4(1-\epsilon)^2 \mathrm{Tr}\left[(\not{p}_1+\not{k})\not{p}_2(\not{p}_1+\not{k})\not{p}_1\right] \tag{28}$$

$$= 8(ee_q g\mu^{2\epsilon})^2 C_F N_c (1-\epsilon)^2 \frac{1-x_1}{1-x_2}. \tag{29}$$

Here we have used

$$D_1 \equiv (p_1+k)^2 = (q-p_2)^2 = q^2 - 2p_2\cdot q = Q^2(1-x_2),$$

and we will also use $D_2 \equiv (p_2+k)^2 = Q^2(1-x_1)$. Note that we can have similar relations as follows:

$$\begin{aligned} (p_1+k)^2 &= 2p_1\cdot k = Q^2(1-x_2), \\ (p_2+k)^2 &= 2p_2\cdot k = Q^2(1-x_1), \\ (p_1+p_2)^2 &= 2p_1\cdot p_2 = Q^2(1-x_3). \end{aligned}$$

It is important notice that the case $x_1 \to 1$ corresponds to $k \| p_2$. This is to say that the radiated gluon and antiquark are parallel to each other, or they are collinear with each other. Furthermore, $x_2 \to 1$ implies to $k \| p_1$ and $x_3 \to 1$ indicates $p_1 \| p_2$. In addition, the soft limit (gluon has vanishing energy) $x_3 \to 0$ is reached when both $x_1 \to 1$ and $x_2 \to 1$.

Noticing that $|\mathcal{M}_2|^2$ has the same structure as $|\mathcal{M}_1|^2$, it is then pretty easy to find

$$|\mathcal{M}_2|^2 = 8(ee_q g\mu^{2\epsilon})^2 C_F N_c (1-\epsilon)^2 \frac{1-x_2}{1-x_1}, \tag{30}$$

by interchanging $x_1 \Leftrightarrow x_2$, and therefore find

$$|\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 = 8(ee_q g\mu^{2\epsilon})^2 C_F N_c (1-\epsilon)^2 \left[\frac{1-x_1}{1-x_2} + \frac{1-x_2}{1-x_1}\right]. \tag{31}$$

Next, let us compute the interference term $\mathcal{M}_1^*\mathcal{M}_2 + \mathcal{M}_2^*\mathcal{M}_1 = 2|\mathcal{M}_1^*\mathcal{M}_2|$ as follows

$$\begin{aligned} 2|\mathcal{M}_1^*\mathcal{M}_2| &= -\frac{2(ee_q g\mu^{2\epsilon})^2 C_F N_c}{Q^4(1-x_1)(1-x_2)} \mathrm{Tr}\left[\gamma_\mu(\not{p}_2+\not{k})\gamma_\nu \not{p}_2 \gamma^\mu(\not{p}_1+\not{k})\gamma^\nu \not{p}_1\right] \\ &\quad \Downarrow \quad \text{use } \gamma^\mu\gamma^\alpha\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\alpha + 2\epsilon\gamma^\alpha\gamma^\rho\gamma^\sigma; \\ &\quad \Downarrow \quad \text{and } \gamma^\mu\gamma^\rho\gamma^\sigma\gamma_\mu = 4g^{\rho\sigma} - 2\epsilon\gamma^\rho\gamma^\sigma; \\ &\Downarrow \quad \mathrm{Tr}[\cdots] = -16(2-\epsilon-\epsilon^2)p_1\cdot p_2(p_1+k)\cdot(p_2+k) \\ &\qquad\qquad + 16\epsilon(1-\epsilon)[(p_1+k)\cdot p_1(p_2+k)\cdot p_2] \\ &\qquad\qquad + 16\epsilon(1-\epsilon)[(p_2+k)\cdot p_1(p_2+k)\cdot p_1]\ ; \\ &= -\frac{2(ee_q g\mu^{2\epsilon})^2 C_F N_c}{Q^4(1-x_1)(1-x_2)} \\ &\quad \times 8Q^4\left[-(1-x_3) + \epsilon x_1 x_2 - \epsilon^2(1-x_1)(1-x_2)\right] \\ &= -16(ee_q g\mu^{2\epsilon})^2 C_F N_c \left[\frac{1-x_1-x_2+\epsilon x_1 x_2}{(1-x_1)(1-x_2)} - \epsilon^2\right]. \end{aligned} \tag{32}$$

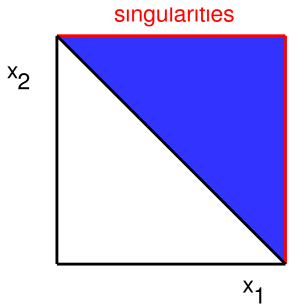
Therefore, at the end of the day, one gets for the total square of the amplitudes as

$$|\mathcal{M}|^2 = 8(ee_q g\mu^{2\epsilon})^2 C_F N_c (1-\epsilon)\frac{x_1^2+x_2^2-\epsilon x_3^2}{(1-x_1)(1-x_2)}. \tag{33}$$

Here $x_3 = 2-x_1-x_2$.

- Let us put the results back to $n=4(\epsilon=0)$ dimension, which gives

$$|\mathcal{M}_{\gamma^*\to q\bar{q}g}|^2 = 8(ee_q g)^2 C_F N_c \frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}, \tag{34}$$



with the integration range indicated in the coloured area. When $x_1 \to 1$ or/and $x_2 \to 1$, there are divergences appearing in the cross section. When $x_3 \to 0$, we have the soft divergence, and when $x_1 \to 1$ or $x_2 \to 1$, we have the collinear divergence. Of course, these two types of IR divergences can coincide and happen at the same time.

- The logarithmic divergence $\int_0^1 \frac{dx}{1-x}$ in 4-d has been converted into a pole in ϵ in the $4-2\epsilon$ dimension as follows

$$\int_0^1 \frac{dx}{(1-x)^{1+\epsilon}} = \frac{-1}{\epsilon}(1-x)^{-\epsilon}\Big|_0^1 = \frac{-1}{\epsilon}, \tag{35}$$

provided that $\epsilon < 0$. As we shall see immediately, we will obtain poles from integrations over x_1 and x_2 for σ_3.

- The appearance of the IR divergence in the real cross section is due to the fact that we are trying to distinguish the NLO $q\bar{q}g$ state from the LO $q\bar{q}$ state. According to the KLN theorem, we are going to get the IR divergences as expected.

Now eventually, we can compute the cross section of $\gamma^* \to q\bar{q}g$ by putting everything altogether according to Eq. (16)

$$\sigma_3 = \frac{Q\left(\frac{Q}{4\pi}\right)^{-2\epsilon}}{256\pi^3\Gamma(2-2\epsilon)}\int_0^1 dx_1 \int_{1-x_1}^1 dx_2 \frac{|\mathcal{M}|^2}{[(1-x_1)(1-x_2)(1-x_3)]^\epsilon}. \tag{36}$$

The direct evaluation of the above integrals are not easy. One needs to use a trick to simplify the integrand. Let us define $x_2 = 1-vx_1$, which gives

$$1-x_3 = x_1+x_2-1 = x_1(1-v), \tag{37}$$

$$1-x_2 = vx_1, \quad \Rightarrow \int_{1-x_1}^1 dx_2 = \int_0^1 x_1 dv. \tag{38}$$

It is then clear that the integration can be separated into $\int dv$ and $\int dx_1$ completely. The objective of this change of variable is to be able to factorize the denominator of the integrand in Eq (36) into factors with independent separate variables.

Using the definition of β function (to save time, one always uses computer programs to do these integrations), one can now compute the three body final state real cross section σ_3 analytically and expand it in terms of ϵ up to constant terms as follows

$$\sigma_3 = \frac{Q\left(\frac{Q^2}{4\pi}\right)^{-2\epsilon}}{256\pi^3\Gamma(2-2\epsilon)}\int_0^1 dx_1\int_0^1 dv\frac{x_1|\mathcal{M}|^2}{[(1-x_1)(1-x_2)(1-x_3)]^\epsilon}, \tag{39}$$

$$= \frac{\sigma_0\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}}{\Gamma(2-\epsilon)}\frac{\alpha_s C_F}{2\pi}\int_0^1 dx_1\int_0^1 dv\frac{x_1|\mathcal{M}|^2}{[(1-x_1)(1-x_2)(1-x_3)]^\epsilon}, \tag{40}$$

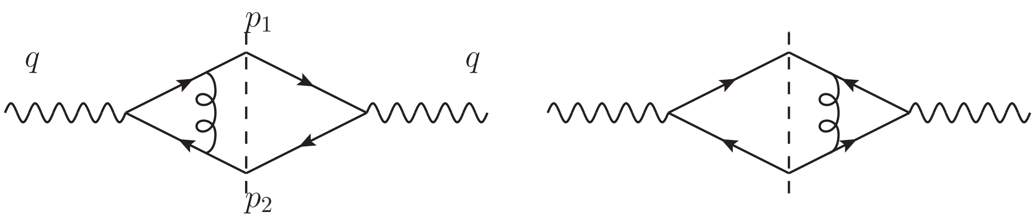
$$= \sigma_0\frac{\alpha_s}{2\pi}C_F\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+\frac{19}{2}-\frac{2\pi^2}{3}+\mathcal{O}(\epsilon)\right]. \tag{41}$$

Several comments are in order:

1. As commented before, we are expecting to encounter singularities when we perform the above calculation in $n=4$-dimension. That is to say we are getting divergent real NLO cross sections when we take $\epsilon\to 0$ limit as shown above.
2. The $\frac{2}{\epsilon^2}$ term comes from the kinematical region where the radiated gluon is soft ($x_3\to 0$) and collinear (parallel to q or $\bar{q}$). The pole of $\frac{3}{\epsilon}$ comes from the collinear singularities when the radiated gluon is parallel to q or $\bar{q}$.
3. Of course, σ_3 itself is not an experimental observable, therefore it is acceptable to be divergent. In high energy experiments, subject to the sensitivity of detectors, we can only measure particles whose energy is above some certain value E_0 and we can only distinguish particles which are separated at least by some minimum angle θ_0. Namely, we can never measure an arbitrary soft gluon or distinguish a quark from a quark plus a collinear gluon.

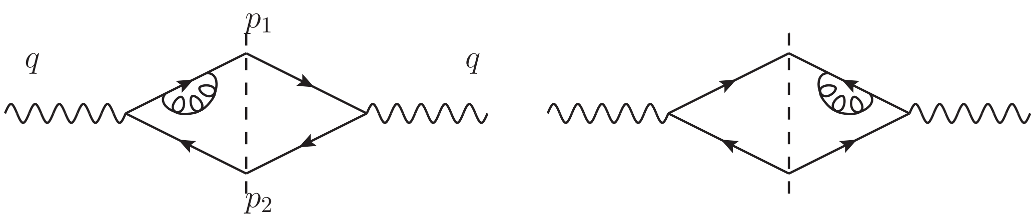
3 NLO virtual diagram

The virtual diagrams at NLO is shown as follows[1]



(d) (e)

[1]In principle, we should also consider the following two virtual diagrams as well.



(f) (g)

However, in the dimensional regularization, the contributions from these two diagrams vanish. In other regularisations, one should of course consider the figures (f) and (g). We will discuss this issue in the next chapter when we cover the topic of renormalization.

Now consider the virtual process $\gamma^* \to q\bar{q}$ as shown in Figure (d) and (e). Compute the corresponding cross section which is defined as

$$\sigma_V = \frac{1}{2Q} R_2^n |\mathcal{M}^{(1)*}_{\gamma^*\to q\bar{q}} \mathcal{M}^{(0)}_{\gamma^*\to q\bar{q}} + \mathcal{M}^{(0)*}_{\gamma^*\to q\bar{q}} \mathcal{M}^{(1)}_{\gamma^*\to q\bar{q}}| . \tag{42}$$

We have studied the two body Lorentz invariant phase space R_2^n before. Now the only task left is to compute the corresponding amplitude square. The way to understand the virtual graph is to write the two body final state amplitude as

$$i\mathcal{M}_{\gamma^*\to q\bar{q}} = i\mathcal{M}^{(0)}_{\gamma^*\to q\bar{q}}(\text{No}\, g) + i\mathcal{M}^{(1)}_{\gamma^*\to q\bar{q}}(1g) + i\mathcal{M}^{(2)}_{\gamma^*\to q\bar{q}}(2g) + \cdots, \tag{43}$$

the square of the above amplitudes can be viewed as an expansion in terms of α_s^m (m is the number of virtual gluons). Therefore, at NLO, the amplitude square gives $2\text{Re}\left[\mathcal{M}^{(1)*}_{\gamma^*\to q\bar{q}} \mathcal{M}^{(0)}_{\gamma^*\to q\bar{q}}\right]$. [2]

According to Feynman rules, summing over the virtual photon polarisations $\sum \epsilon_\mu \epsilon^*_\nu \Rightarrow -g_{\mu\nu}$, we can write it down as follows

$$\begin{aligned}
& 2\mathcal{M}^{(1)*}_{\gamma^*\to q\bar{q}} \mathcal{M}^{(0)}_{\gamma^*\to q\bar{q}} \\
= \ & 2(ee_q g\mu^{2\epsilon})^2 C_F N_c (i)^{3+2}(-i)^2(-1) \\
\times \ & \int \frac{d^n k}{(2\pi)^n k^2} \text{Tr}\left[\not{p}_1 \gamma^\alpha \frac{(\not{p}_1 + \not{k})}{(p_1+k)^2} \gamma^\mu \frac{-(\not{p}_2 - \not{k})}{(p_2-k)^2} \gamma_\alpha \not{p}_2 \gamma_\mu\right] \\
= \ & 2i(ee_q g\mu^{2\epsilon})^2 C_F N_c \int \frac{d^n k}{(2\pi)^n} \frac{N}{k^2(p_1+k)^2(p_2-k)^2} \\
& \quad \text{with}\, N \equiv \text{Tr}\left[\not{p}_1 \gamma^\alpha (\not{p}_1 + \not{k}) \gamma^\mu (\not{k} - \not{p}_2) \gamma_\alpha \not{p}_2 \gamma_\mu\right].
\end{aligned} \tag{44}$$

The loop integral has the denominator of the form $k^2(p_1+k)^2(p_2-k)^2$, which vanishes when $k \to 0$ (soft) or when k is collinear with either p_1 (quark) or p_2 (antiquark). The singularities of the denominator corresponds the same types of singularities as observed in the real contribution for the $q\bar{q}g$ final state. **It is important to notice that, when the gluon is soft or collinear, the NLO contribution from real and virtual diagrams become the same but with opposite sign.** This is the reason that the singularity shall cancel between real and virtual diagrams for the total cross section as we shall see later. Graphically, the minus sign between the real and virtual diagrams can be easily seen by comparing the gluon vertices in these two cases ($i^2 = -1$ for virtual graphs, while $-i \times i = 1$ for real graphs.). When the gluon is soft in virtual graphs, it becomes on-shell as in the real graphs. When the gluon is collinear with quark or antiquark in virtual graphs, physically it is indistinguishable from the collinear real graphs.

Using the same technique for γ-matrices, it is straightforward to cast N into

$$\begin{aligned}
N \ = \ & -16(2-\epsilon)(p_1+k)\cdot p_2 (k-p_2)\cdot p_1 - 16\epsilon^2 k^2 p_1 \cdot p_2 \\
& + 16\epsilon\left[(p_1+k)\cdot(k-p_2) p_1\cdot p_2 + p_1\cdot k p_2 \cdot k\right] \\
= \ & 8(1-\epsilon)\left[Q^4 - 2Q^2(k\cdot p_1 - k\cdot p_2) - 4k\cdot p_1 k\cdot p_2 + \epsilon Q^2 k^2\right].
\end{aligned} \tag{45}$$

It is also worth mentioning that these four terms in N has different types of singularities. The first term has both soft and collinear (IR) singularities, while the second and third term which is proportional to k only have collinear singularity. The fourth term not only has collinear singularity (IR) but also have UV divergence. On the other hand, the UV and IR divergences cancel each other in the dimensional regularization which makes the fourth term finite. (For further discussion on this issue, see Chapter 20 of Schwartz.)

[2] For convenience, we will just compute $2\mathcal{M}^{(1)*}_{\gamma^*\to q\bar{q}} \mathcal{M}^{(0)}_{\gamma^*\to q\bar{q}}$ directly in the following, and take the real part of the final results in the end.

Next, let us start to work on the integrations, which can be achieved by using the following identities

$$\begin{aligned}
\mathcal{I}_1 &= \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2(p_1+k)^2(p_2-k)^2} \\
&= \frac{-i\Gamma(3-\frac{n}{2})}{(4\pi)^{n/2}} \left(-\frac{1}{Q^2}\right)^{3-\frac{n}{2}} \frac{B(\frac{n}{2}-2,\frac{n}{2}-1)}{\frac{n}{2}-2}, \qquad (46)\\
\mathcal{I}_2 &= \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu}{k^2(p_1+k)^2(p_2-k)^2} \\
&= \frac{i\Gamma(3-\frac{n}{2})}{(4\pi)^{n/2}} \left(-\frac{1}{Q^2}\right)^{3-\frac{n}{2}} \frac{B(\frac{n}{2}-1,\frac{n}{2}-1)}{\frac{n}{2}-2} \left(p_{1\mu}-p_{2\mu}\right), \qquad (47)\\
\mathcal{I}_3 &= \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu k_\nu}{k^2(p_1+k)^2(p_2-k)^2} \\
&= \frac{-i\Gamma(3-\frac{n}{2})}{(4\pi)^{n/2}} \left(-\frac{1}{Q^2}\right)^{3-\frac{n}{2}} \frac{B(\frac{n}{2}-1,\frac{n}{2})}{\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)} \qquad (48)\\
&\times \left[\frac{n-2}{2}\left(p_{1\mu}p_{1\nu}+p_{2\mu}p_{2\nu}\right)-\frac{n-4}{2}\left(p_{1\mu}p_{2\nu}+p_{1\nu}p_{2\mu}\right)-p_1\cdot p_2 g_{\mu\nu}\right].
\end{aligned}$$

Let me discuss the derivation of $\mathcal{I}_1$ in detail, and leave the proof of $\mathcal{I}_2$ and $\mathcal{I}_3$ to you if you wish to practice what you have learnt so far. [3]

Using the so-called Feynman parameter technique, we can cast $\mathcal{I}_1$ into the following form ($x,y,z \in [0,1]$)

$$\begin{aligned}
\mathcal{I}_1 &= \int_0^1 dxdydz\delta(x+y+z-1)\int \frac{d^n k}{(2\pi)^n}\frac{2}{D^3} \qquad (49)\\
&\text{with} \quad D \equiv xk^2+y(p_1+k)^2+z(p_2-k)^2 \\
&\qquad = l^2-(yp_1-zp_2)^2 = l^2+yzQ^2, \qquad (50)
\end{aligned}$$

where $l \equiv k+yp_1-zp_2$ allows us to shift the integration from k to l. Let us also define $\Delta = -yzQ^2$, and then use the Wick rotation to write $\mathcal{I}_1$ as

$$\begin{aligned}
\mathcal{I}_1 &= \int_0^1 dydz \int \frac{d^n l}{(2\pi)^n}\frac{2}{(l^2-\Delta)^3} \qquad (51)\\
&= -i\int_0^1 dydz \int \frac{d^n l_E}{(2\pi)^n}\frac{2}{(l_E^2+\Delta)^3} \qquad (52)\\
&= \int_0^1 dydz \frac{-2i}{(4\pi)^{n/2}}\frac{\Gamma(3-\frac{n}{2})}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{3-\frac{n}{2}}, \qquad (53)
\end{aligned}$$

where we have use Eq. (**??**) in the last step. In the end, it is straightforward to finish the rest of the $dydz$ integrations by noticing that $0<x<1 \Rightarrow y+z<1$ which sets the upper limit of dz integration to be $1-y$. Therefore, we arrive at the $\mathcal{I}_1$ identity after taking into account $\Delta = -yzQ^2$ and the result

$$\int_0^1 dy \int_0^{1-y} dz y^{\frac{n}{2}-3} z^{\frac{n}{2}-3} = \int_0^1 dy y^{\frac{n}{2}-3}\frac{(1-y)^{\frac{n}{2}-2}}{\frac{n}{2}-2} = \frac{B(\frac{n}{2}-2,\frac{n}{2}-1)}{\frac{n}{2}-2}. \qquad (54)$$

Now we are ready to evaluate all the integrals with $\mathcal{I}_{1,2,3}$ identities, and assemble all the pieces altogether

[3] In principle, you do not have to start from the beginning to compute $\mathcal{I}_2$ and $\mathcal{I}_3$. There is a trick from the book Quantum Field Theory by Lewis H. Ryder. (See page 382, chapter 9, Appendix A). The idea is that one can always get factors of k_μ by differentiating with respect to p_1^μ or p_2^μ.

and find the virtual contribution to the NLO cross section can be explicitly written as

$$
\begin{aligned}
\sigma_V &= \sigma_0 \frac{2\mathcal{M}^{(1)*}_{\gamma^*\to q\bar{q}}\mathcal{M}^{(0)}_{\gamma^*\to q\bar{q}}}{|\mathcal{M}^{(0)}_{\gamma^*\to q\bar{q}}|^2}, & (55)\\
&= -\sigma_0 \frac{\alpha_s C_F}{\pi}\left(-\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\Gamma(1+\epsilon) \\
&\quad \times\left[-\frac{1}{\epsilon}\frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}+\frac{2}{\epsilon}\frac{\Gamma(1-\epsilon)\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}-\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}+2\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}\right], & (56)\\
&= \sigma_0 \frac{\alpha_s C_F}{\pi}\left(-\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\times\left[-\frac{1}{\epsilon^2}-\left(\frac{3}{2\epsilon}+1\right)\frac{1}{1-2\epsilon}\right]. & (57)
\end{aligned}
$$

The last step is to take the real part of the above expression as mentioned earlier in the beginning of this subsection. With the assistance of

$$
\mathrm{Re}(-1)^{-\epsilon} = \mathrm{Re}[e^{-i\epsilon\pi}] = 1 - \frac{1}{2}\pi^2\epsilon^2 + \mathcal{O}(\epsilon^4), \tag{58}
$$

$$
\Gamma(1-\epsilon)\Gamma(1+\epsilon) = 1 + \frac{1}{6}\pi^2\epsilon^2 + \mathcal{O}(\epsilon^4), \tag{59}
$$

we can obtain

$$
\sigma_V = \sigma_0 \frac{\alpha_s C_F}{2\pi}\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}-8+\frac{2\pi^2}{3}\right]. \tag{60}
$$

4 Total cross section

At the end of the day, one finds that the NLO real and virtual contributions are

$$
\begin{aligned}
\sigma_3 &= \sigma_0 \frac{\alpha_s}{2\pi} C_F\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+\frac{19}{2}-\frac{2\pi^2}{3}\right],\\
\sigma_V &= \sigma_0 \frac{\alpha_s}{2\pi} C_F\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}-8+\frac{2\pi^2}{3}\right],
\end{aligned}
$$

respectively. Therefore, by summing over the LO and NLO contributions to the cross section of $\gamma^* \to X$, we can obtain

$$
\lim_{\epsilon\to 0}\sigma^{\mathrm{tot}}_{\gamma^*\to X} = \sigma_0\left[1+\frac{3}{4}C_F\frac{\alpha_s(\mu)}{\pi}+\mathcal{O}(\alpha_s^2)\right], \tag{61}
$$

which is finite in 4-dimension when we take $\epsilon \to 0$. [4] In general, we can use the following standard procedures to compute all the diagrams

Some more comments are in order:

- The ratio between the $e^+e^- \to$ hadrons total cross section and the $e^+e^- \to \mu^+\mu^-$ cross section. $N_c \sum_{u,d,s} e_i^2 = 2$, $N_c \sum_{u,d,s,c} e_i^2 = \frac{10}{3}$, $N_c \sum_{u,d,s,c,b} e_i^2 = \frac{11}{3}$.

$$
R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum_{u,d,s,\dots} e_i^2\left[1+\frac{\alpha_s(Q^2)}{\pi}\right]
$$

- The cancellation of soft and collinear singularities between the real and virtual gluon diagrams for total cross section is no accident. We have the **KLN theorem**, which states that suitably defined inclusive quantities will indeed be free of singularities in the massless limit. The total hadronic cross section of $e^+e^- \to \gamma^* \to X$(hadrons) is an example of such **infrared safe** quantities. In fact, any experimental observable must be infrared safe in order to be sensibly computed in field theory.

[4] For this specific process, one can give gluons a small mass m_g and use it as a regulator for IR divergence, which allows us to obtain exactly the same total cross section. See Application of pQCD by R. D. Field for details.