1 Sudakov Resummation Homework problems:

Homework Problem 1: Read through the reading assignment and reproduce the following result for $e^+ + e^-$ annihilations at NLO with dim-reg.

$$\text{Consider the process} \quad e^+ + e^- \to \gamma^* \to q\bar{q}(\text{LO}) \quad \text{or} \quad q + \bar{q} + g(\text{NLO}). \tag{1}$$

Final results for the NLO real and virtual contributions are

$$\begin{aligned}
\sigma_r &= \sigma_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \frac{2\pi^2}{3}\right], \\
\sigma_v &= \sigma_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{2\pi^2}{3}\right],
\end{aligned}$$

respectively. Therefore, summing over the LO and NLO contributions to the total cross section yields

$$\lim_{\epsilon\to 0} \sigma^{\text{tot}}_{\gamma^*\to X} = \sigma_0 \left[1 + \frac{3}{4} C_F \frac{\alpha_s(\mu)}{\pi} + \mathcal{O}(\alpha_s^2)\right], \tag{2}$$

which is finite in 4-dimension when we take $\epsilon \to 0$.

Homework Problem 2: Consider the $2 \to 3$ ($e^+ + e^- \to q + \bar{q} + g$) process and derive the following thrust distribution for $T < 1$

$$\frac{d\sigma}{\sigma_0 dT} = \frac{C_F \alpha_s}{2\pi} \left[\frac{2(3T^2 - 3T + 2)}{T(1-T)} \ln \frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T}\right]. \tag{3}$$

Hint: Use $2 \to 3$ cross section, the thrust distribution can be cast into

$$\frac{1}{\sigma_0}\frac{d\sigma}{dT} = \frac{C_F \alpha_s}{2\pi} \int dx_1 \int dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \delta(T - \max[x_1, x_2, x_3]).$$

Homework Problem 3:

After including the Born and virtual as well as real contributions, the cross section becomes

$$\frac{d\sigma}{\sigma_0 dT} = \frac{C_F \alpha_s}{2\pi} \left[\frac{2(3T^2 - 3T + 2)}{T(1-T)} \ln \frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T}\right] + C\delta(1-T), \tag{4}$$

where C is a divergent constant, which can be determined by the following integral according to Eq. (2)

$$\int_{T_{\min}}^1 dT \frac{d\sigma}{\sigma_0 dT} = 1 + C_F \frac{3\alpha_s}{4\pi} + \mathcal{O}(\alpha_s^2). \tag{5}$$

1. Show that $T_{\min} = 2/3$ for $2 \to 3$ processes.
2. From Eq. (3), show that the following exact expression of the thrust distribution at one-loop satisfies Eq. (5)

$$\begin{aligned}
\frac{d\sigma}{\sigma_0 dT} &= \delta(1-T) + \frac{C_F \alpha_s}{2\pi} \left[\delta(1-T)\left(\frac{\pi^2}{3} - 1\right) - \frac{3(3T-2)(2-T)}{(1-T)_+}\right] \\
&\quad + \frac{C_F \alpha_s}{2\pi} \frac{2(3T^2 - 3T + 2)}{T} \left[\frac{\ln(2T-1)}{(1-T)_+} - \left(\frac{\ln(1-T)}{1-T}\right)_+\right],
\end{aligned} \tag{6}$$

where the plus distribution is defined as $\int_a^1 dx g(x)(f(x))_+ = \int_a^1 dx g(x) f(x) - g(1) \int_0^1 dx f(x)$.

Homework Problem 4: See Problem 5.6 in Peskin (also Problem 3.3 and 5.3 for more background)
This problem extends the spinor product formalism to the case with external photons.
Let k be the momentum of an external real photon ($k^2 = 0$), and let p be another lightlike vector, chosen so that $p \cdot k \neq 0$. Let $u_R(p)$ and $u_L(p)$ be the massless right-handed/left-handed spinors with the lightlike momentum p defined with the following properties (see Problems 3.3 and 5.3)

$$u_L(p)\bar{u}_L(p) = \frac{1-\gamma^5}{2}\not{p}, \quad \text{and} \quad u_R(p)\bar{u}_R(p) = \frac{1+\gamma^5}{2}\not{p}. \tag{7}$$

Now define photon polarization vectors as follows:

$$\epsilon_+^\mu(k) = \frac{1}{\sqrt{4k\cdot p}}\bar{u}_R(k)\gamma^\mu u_R(p), \quad \epsilon_-^\mu(k) = \frac{1}{\sqrt{4k\cdot p}}\bar{u}_L(k)\gamma^\mu u_L(p). \tag{8}$$

First, <u>show</u> $\epsilon_\pm^\mu(k)k_\mu = 0$.
(Hint: Simply use the Dirac equation for $\bar{u}_{R,L}(k)\not{k} = 0$)
Using the identity $\bar{u}_L(p)\gamma^\mu u_R(k) = 0$ and

$$\sum u(p)\bar{u}(p) = u_L(p)\bar{u}_L(p) + u_R(p)\bar{u}_R(p) = \not{p} \tag{9}$$

to <u>compute</u> the polarization sum and show

$$\epsilon_+^\mu\epsilon_+^{\nu*} + \epsilon_-^\mu\epsilon_-^{\nu*} = \frac{\mathrm{Tr}[\gamma^\mu\not{p}\gamma^\nu\not{k}]}{4p\cdot k} = -g^{\mu\nu} + \frac{k^\mu p^\nu + k^\nu p^\mu}{p\cdot k}. \tag{10}$$

It is important to note that the second on the right hand side of the above equation gives zero when dotted with any photon emission amplitude $\mathcal{M}^\mu$ due to the Ward identity, so we have

$$|\epsilon_+\cdot\mathcal{M}|^2 + |\epsilon_-\cdot\mathcal{M}|^2 = -g^{\mu\nu}\mathcal{M}_\mu\mathcal{M}_\nu^*; \tag{11}$$

thus, we can use the vectors $\epsilon_\pm^\mu(k)$ to compute photon polarization sums.
Despite the dependence on p in the above expression of $\epsilon_\pm^\mu(k)$, it is important to note that we can choose the auxiliary variable p to be any four vector in the definition of polarization vectors as long as $k\cdot p \neq 0$. The final squared amplitude does not depend on the choice of the auxiliary p vector.

Hint:

$$\begin{aligned}
\epsilon_+^\mu\epsilon_+^{\nu*} + \epsilon_-^\mu\epsilon_-^{\nu*} &= \frac{1}{4p\cdot k}\left[\bar{u}_R(k)\gamma^\mu u_R(p)\bar{u}_R(p)\gamma^\nu u_R(k) + (R\leftrightarrow L)\right] \\
(\text{N.B.}\, \bar{u}_R\gamma^\mu u_L = 0) &= \frac{1}{4p\cdot k}\left[\bar{u}_R(k)\gamma^\mu(u_R(p)\bar{u}_R(p) + u_L(p)\bar{u}_L(p))\gamma^\nu u_R(k) + (R\leftrightarrow L)\right]
\end{aligned}$$

Homework Problem 5:
Use the dimensional regularization ($\overline{MS}$ scheme, multiplying a factor of $S_\epsilon^{-1} = (4\pi e^{-\gamma_E})^{-\epsilon}$ with $\gamma_E \simeq 0.577$ the Euler constant) and show the following identities

$$S_\epsilon^{-1}\mu^{2\epsilon}\int\frac{d^{2-2\epsilon}k_\perp}{(2\pi)^{2-2\epsilon}}e^{ik_\perp\cdot b_\perp}\frac{1}{k_\perp^2} = \frac{1}{4\pi}\left[-\frac{1}{\epsilon} + \ln\frac{c_0^2}{\mu^2 b_\perp^2}\right], \tag{12}$$

$$S_\epsilon^{-1}\mu^{2\epsilon}\int\frac{d^{2-2\epsilon}k_\perp}{(2\pi)^{2-2\epsilon}}e^{ik_\perp\cdot b_\perp}\frac{1}{k_\perp^2}\ln\frac{Q^2}{k_\perp^2} = \frac{1}{4\pi}\left[\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{Q^2}{\mu^2} + \frac{1}{2}\ln^2\frac{Q^2}{\mu^2} - \frac{1}{2}\ln^2\frac{Q^2 b_\perp^2}{c_0^2} - \frac{\pi^2}{12}\right], \tag{13}$$

$$S_\epsilon^{-1}\,\mu^{2\epsilon}\int\frac{d^{2-2\epsilon}l_\perp}{(2\pi)^{2-2\epsilon}}\frac{1}{l_\perp^2}\ln\frac{Q^2}{l_\perp^2}\bigg|_{l_\perp<Q} = \frac{1}{4\pi}\left[\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{Q^2}{\mu^2} + \frac{1}{2}\ln^2\frac{Q^2}{\mu^2} - \frac{\pi^2}{12}\right], \tag{14}$$

where $c_0 \equiv 2e^{-\gamma_E}$. Hints: see the appendix in [$arXiv:1308.2993$].