1 Sudakov Resummation Homework problems:

Homework Problem 1: Read through the reading assignment and reproduce the following result for $e^+ + e^-$ annihilations at NLO with dim-reg.

$$\text{Consider the process} \quad e^+ + e^- \to \gamma^* \to q\bar{q}(\text{LO}) \quad \text{or} \quad q + \bar{q} + g(\text{NLO}). \tag{1}$$

Final results for the NLO real and virtual contributions are

$$\begin{aligned}
\sigma_r &= \sigma_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \frac{2\pi^2}{3}\right], \\
\sigma_v &= \sigma_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{2\pi^2}{3}\right],
\end{aligned}$$

respectively. Therefore, summing over the LO and NLO contributions to the total cross section yields

$$\lim_{\epsilon \to 0} \sigma^{\text{tot}}_{\gamma^* \to X} = \sigma_0 \left[1 + \frac{3}{4} C_F \frac{\alpha_s(\mu)}{\pi} + \mathcal{O}(\alpha_s^2)\right], \tag{2}$$

which is finite in 4-dimension when we take $\epsilon \to 0$.

Homework Problem 2: Consider the $2 \to 3$ ($e^+ + e^- \to q + \bar{q} + g$) process and derive the following thrust distribution for $T < 1$

$$\frac{d\sigma}{\sigma_0 dT} = \frac{C_F \alpha_s}{2\pi} \left[\frac{2(3T^2 - 3T + 2)}{T(1-T)} \ln \frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T}\right]. \tag{3}$$

Hint: Use $2 \to 3$ cross section, the thrust distribution can be cast into

$$\frac{1}{\sigma_0} \frac{d\sigma}{dT} = \frac{C_F \alpha_s}{2\pi} \int dx_1 \int dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \delta(T - \max[x_1, x_2, x_3]).$$

Solution:

Use the delta function trick, we can write the differential cross section of thrust as

$$\frac{1}{\sigma_0} \frac{d\sigma}{dT} = \frac{C_F \alpha_s}{2\pi} \int dx_1 \int dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \delta(T - \max[x_1, x_2, x_3]),$$

where $x_3 = 2 - x_1 - x_2$. We can perform the above integrations and find $\frac{1}{\sigma_0} \frac{d\sigma}{dT}$ as the function of T as follows

$$\begin{aligned}
\frac{d\sigma}{\sigma_0 dT} = \; & 2 \frac{C_F \alpha_s}{2\pi} \int_{1-T/2}^{T} \mathrm{d}x_2 \left[\frac{T^2 + x_2^2}{(1-T)(1-x_2)}\right]_{x_1 > x_2 > x_3;\, \text{or}, \, x_2 > x_1 > x_3} \\
& + 2 \frac{C_F \alpha_s}{2\pi} \int_{2-2T}^{1-T/2} \mathrm{d}x_2 \left[\frac{T^2 + x_2^2}{(1-T)(1-x_2)}\right]_{x_1 > x_3 > x_2;\, \text{or}, \, x_2 > x_3 > x_1} \\
& + 2 \frac{C_F \alpha_s}{2\pi} \int_{1-T/2}^{T} \mathrm{d}x_2 \left[\frac{(2-T-x_2)^2 + x_2^2}{(T+x_2-1)(1-x_2)}\right]_{x_3 > x_2 > x_1;\, \text{or}, \, x_3 > x_1 > x_2}
\end{aligned} \tag{4}$$

As shown above, first we consider the region in which $x_1 > x_2 > x_3$, and note that the delta function sets $T = x_1$ in this region, then determine the range of the integration for x_2 according to energy momentum conservation before we integrate over x_2. For example, in this region, $T > x_2 > x_3 = 2 - T - x_2$ which gives $T > x_2 > 1 - T/2$. When $x_1 > x_3 > x_2$, the integration ranges from $2 - 2T$ to $1 - T/2$. When $x_3 > x_2 > x_1$, we find $T > x_2 > 2 - T - x_2$. Since x_1 and x_2 are symmetric, we can take into account the other half contribution by multiplying a factor of 2. By summing all of six regions together, one obtains the final expression at the first non-trivial order in Eq. (3).

Homework Problem 3:

After including the Born and virtual as well as real contributions, the cross section becomes

$$\frac{d\sigma}{\sigma_0 dT} = \frac{C_F\alpha_s}{2\pi}\left[\frac{2(3T^2-3T+2)}{T(1-T)}\ln\frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T}\right] + C\delta(1-T), \tag{5}$$

where C is a divergent constant, which can be determined by the following integral according to Eq. (2)

$$\int_{T_{\min}}^{1} dT\frac{d\sigma}{\sigma_0 dT} = 1 + C_F\frac{3\alpha_s}{4\pi} + \mathcal{O}(\alpha_s^2). \tag{6}$$

1. Show that $T_{\min} = 2/3$ for $2 \to 3$ processes.
2. From Eq. (3), show that the following exact expression of the thrust distribution at one-loop satisfies Eq. (6)

$$\begin{aligned}\frac{d\sigma}{\sigma_0 dT} &= \delta(1-T) + \frac{C_F\alpha_s}{2\pi}\left[\delta(1-T)\left(\frac{\pi^2}{3}-1\right) - \frac{3(3T-2)(2-T)}{(1-T)_+}\right] \\ &\quad + \frac{C_F\alpha_s}{2\pi}\frac{2(3T^2-3T+2)}{T}\left[\frac{\ln(2T-1)}{(1-T)_+} - \left(\frac{\ln(1-T)}{1-T}\right)_+\right],\end{aligned} \tag{7}$$

where the plus distribution is defined as $\int_a^1 dx g(x)(f(x))_+ = \int_a^1 dx g(x)f(x) - g(1)\int_0^1 dx f(x)$.

Solution:

1. Since $T = \max[x_1, x_2, x_3]$ and $x_1 + x_2 + x_3 = 2$, one finds $3T \leq 2$ and $T_{\min} = 2/3$ for the Mercedes Benz event shape.
2. With the help of computing software, one finds

$$\int_0^{\frac{2}{3}} \frac{3}{1-t}dt + \int_{\frac{2}{3}}^1 \frac{3-3(3t-2)(2-t)}{1-t}dt = \frac{5}{2} + \log(27) \tag{8}$$

$$\begin{aligned}&\int_{\frac{2}{3}}^1\left(\frac{\left(2\left(3t^2-3t+2\right)\right)\log(2t-1)}{t(1-t)} - \frac{\left(2\left(3t^2-3t+2\right)-4t\right)\log(1-t)}{t(1-t)}\right)dt + \int_0^{\frac{2}{3}}\frac{4\log(1-t)}{1-t}dt \\ &= -8\text{Li}_2\left(\frac{2}{3}\right) - 4\text{Li}_2\left(-\frac{1}{3}\right) + \frac{\pi^2}{3} + (\log 3)(-3 + 8\log 2 - 6\log 3)\end{aligned} \tag{9}$$

$$\left[\frac{5}{2} + \log(27)\right] + \left[-8\text{Li}_2\left(\frac{2}{3}\right) - 4\text{Li}_2\left(-\frac{1}{3}\right) + \frac{\pi^2}{3} + (\log 3)(-3+8\log 2 - 6\log 3)\right] + \left[\frac{\pi^2}{3} - 1\right] = \frac{3}{2}. \tag{10}$$

This agrees with Eq. (6).

Homework Problem 4: See Problem 5.6 in Peskin (also Problem 3.3 and 5.3 for more background)

This problem extends the spinor product formalism to the case with external photons.

Let k be the momentum of an external real photon ($k^2 = 0$), and let p be another lightlike vector, chosen so that $p \cdot k \neq 0$. Let $u_R(p)$ and $u_L(p)$ be the massless right-handed/left-handed spinors with the lightlike momentum p defined with the following properties (see Problems 3.3 and 5.3)

$$u_L(p)\bar{u}_L(p) = \frac{1-\gamma^5}{2}\not{p}, \quad \text{and} \quad u_R(p)\bar{u}_R(p) = \frac{1+\gamma^5}{2}\not{p}. \tag{11}$$

Now define photon polarization vectors as follows:

$$\epsilon_+^\mu(k) = \frac{1}{\sqrt{4k\cdot p}}\bar{u}_R(k)\gamma^\mu u_R(p), \quad \epsilon_-^\mu(k) = \frac{1}{\sqrt{4k\cdot p}}\bar{u}_L(k)\gamma^\mu u_L(p). \tag{12}$$

First, show $\epsilon^{\mu}_{\pm}(k)k_{\mu} = 0$.

(Hint: Simply use the Dirac equation for $\bar{u}_{R,L}(k)\not{k} = 0$)

Using the identity $\bar{u}_L(p)\gamma^{\mu}u_R(k) = 0$ and

$$\sum u(p)\bar{u}(p) = u_L(p)\bar{u}_L(p) + u_R(p)\bar{u}_R(p) = \not{p} \tag{13}$$

to compute the polarization sum and show

$$\epsilon^{\mu}_{+}\epsilon^{\nu *}_{+} + \epsilon^{\mu}_{-}\epsilon^{\nu *}_{-} = \frac{\mathrm{Tr}[\gamma^{\mu}\not{p}\gamma^{\nu}\not{k}]}{4p\cdot k} = -g^{\mu\nu} + \frac{k^{\mu}p^{\nu} + k^{\nu}p^{\mu}}{p\cdot k}. \tag{14}$$

It is important to note that the second on the right hand side of the above equation gives zero when dotted with any photon emission amplitude $\mathcal{M}^{\mu}$ due to the Ward identity, so we have

$$|\epsilon_{+}\cdot\mathcal{M}|^2 + |\epsilon_{-}\cdot\mathcal{M}|^2 = -g^{\mu\nu}\mathcal{M}_{\mu}\mathcal{M}^{*}_{\nu}; \tag{15}$$

thus, we can use the vectors $\epsilon^{\mu}_{\pm}(k)$ to compute photon polarization sums.

Despite the dependence on p in the above expression of $\epsilon^{\mu}_{\pm}(k)$, it is important to note that we can choose the auxiliary variable p to be any four vector in the definition of polarization vectors as long as $k\cdot p \neq 0$. The final squared amplitude does not depend on the choice of the auxiliary p vector.

Hint:

$$\begin{aligned}
\epsilon^{\mu}_{+}\epsilon^{\nu *}_{+} + \epsilon^{\mu}_{-}\epsilon^{\nu *}_{-} &= \frac{1}{4p\cdot k}\left[\bar{u}_R(k)\gamma^{\mu}u_R(p)\bar{u}_R(p)\gamma^{\nu}u_R(k) + (R\longleftrightarrow L)\right] \\
(\text{N.B.}\,\bar{u}_R\gamma^{\mu}u_L = 0) &= \frac{1}{4p\cdot k}\left[\bar{u}_R(k)\gamma^{\mu}(u_R(p)\bar{u}_R(p) + u_L(p)\bar{u}_L(p))\gamma^{\nu}u_R(k) + (R\longleftrightarrow L)\right]
\end{aligned}$$

Solution:

(1) $\epsilon^{\mu}_{\pm}(k)k_{\mu} = \frac{1}{\sqrt{4k\cdot p}}\bar{u}_{R,L}(k)\not{k}u_{R,L}(p) = 0$, since $\bar{u}_{R,L}(k)\not{k} = 0$ according to the Dirac equation. The definition of $\epsilon^{\mu}_{\pm}(k)$ also tells us the corresponding spin sum of the spinors gives ± 1 accordingly.

(2) Now compute the polarization sum

$$\begin{aligned}
\epsilon^{\mu}_{+}\epsilon^{\nu *}_{+} + \epsilon^{\mu}_{-}\epsilon^{\nu *}_{-} &= \frac{1}{4p\cdot k}\left[\bar{u}_R(k)\gamma^{\mu}u_R(p)\bar{u}_R(p)\gamma^{\nu}u_R(k) + (R\longleftrightarrow L)\right] \\
(\text{N.B.}\,\bar{u}_R\gamma^{\mu}u_L = 0) &= \frac{1}{4p\cdot k}\left[\bar{u}_R(k)\gamma^{\mu}(u_R(p)\bar{u}_R(p) + u_L(p)\bar{u}_L(p))\gamma^{\nu}u_R(k) + (R\longleftrightarrow L)\right] \\
&= \frac{1}{4p\cdot k}\left[\bar{u}_R(k)\gamma^{\mu}\not{p}\gamma^{\nu}u_R(k) + (R\longleftrightarrow L)\right] \\
&= \frac{\mathrm{Tr}[\gamma^{\mu}\not{p}\gamma^{\nu}\not{k}]}{4p\cdot k} = -g^{\mu\nu} + \frac{k^{\mu}p^{\nu} + k^{\nu}p^{\mu}}{p\cdot k}.
\end{aligned} \tag{16}$$

Therefore, it is equivalent to the usually polarization sum in light of the Ward identity

$$|\epsilon_{+}\cdot\mathcal{M}|^2 + |\epsilon_{-}\cdot\mathcal{M}|^2 = -g^{\mu\nu}\mathcal{M}_{\mu}\mathcal{M}^{*}_{\nu}, \tag{17}$$

with $k\cdot\mathcal{M} = 0$. It is important to note that we can choose the auxiliary variable p to be any four vector in the definition of polarization vectors as long as $k\cdot p \neq 0$.

Homework Problem 5:

Use the dimensional regularization ($\overline{MS}$ scheme, multiplying a factor of $S_{\epsilon}^{-1} = (4\pi e^{-\gamma_E})^{-\epsilon}$ with $\gamma_E \simeq 0.577$

the Euler constant) and show the following identities

$$S_\epsilon^{-1}\mu^{2\epsilon}\int\frac{d^{2-2\epsilon}k_\perp}{(2\pi)^{2-2\epsilon}}e^{ik_\perp\cdot b_\perp}\frac{1}{k_\perp^2}=\frac{1}{4\pi}\left[-\frac{1}{\epsilon}+\ln\frac{c_0^2}{\mu^2 b_\perp^2}\right], \tag{18}$$

$$S_\epsilon^{-1}\mu^{2\epsilon}\int\frac{d^{2-2\epsilon}k_\perp}{(2\pi)^{2-2\epsilon}}e^{ik_\perp\cdot b_\perp}\frac{1}{k_\perp^2}\ln\frac{Q^2}{k_\perp^2}=\frac{1}{4\pi}\left[\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{Q^2}{\mu^2}+\frac{1}{2}\ln^2\frac{Q^2}{\mu^2}-\frac{1}{2}\ln^2\frac{Q^2b_\perp^2}{c_0^2}-\frac{\pi^2}{12}\right], \tag{19}$$

$$S_\epsilon^{-1}\,\mu^{2\epsilon}\int\frac{d^{2-2\epsilon}l_\perp}{(2\pi)^{2-2\epsilon}}\frac{1}{l_\perp^2}\ln\frac{Q^2}{l_\perp^2}\bigg|_{l_\perp<Q}=\frac{1}{4\pi}\left[\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{Q^2}{\mu^2}+\frac{1}{2}\ln^2\frac{Q^2}{\mu^2}-\frac{\pi^2}{12}\right], \tag{20}$$

where $c_0\equiv 2e^{-\gamma_E}$. Hints: see the appendix in [$arXiv:1308.2993$].

Solution:

(1) First, by rewriting $(q_\perp^2)^{-\alpha}$ in the following

$$(q_\perp^2)^{-\alpha}=\frac{1}{\Gamma(\alpha)}\int_0^\infty dx\,x^{\alpha-1}e^{-xq_\perp^2}, \tag{21}$$

and shifting coordinate in the $q_\perp$ integral, one can show

$$\int\frac{d^{2-2\epsilon}q_\perp}{(q_\perp^2)^\alpha}e^{-iq_\perp\cdot R_\perp}=\int_0^\infty\frac{dx\,x^{\alpha-1}}{\Gamma(\alpha)}\int d^{2-2\epsilon}q_\perp e^{-x(q_\perp+\frac{iR_\perp}{2x})^2-\frac{R_\perp^2}{4x}}=\frac{\pi^{1-\epsilon}\Gamma(1-\alpha-\epsilon)}{\Gamma(\alpha)}\left(\frac{R_\perp^2}{4}\right)^{\alpha+\epsilon-1}. \tag{22}$$

After setting $\alpha=1$ and using $\overline{\text{MS}}$, it is straightforward to see

$$\mu^{2\epsilon}\int\frac{\mathrm{d}^{2-2\epsilon}q_\perp}{(2\pi)^{2-2\epsilon}}e^{-iq_\perp\cdot R_\perp}\frac{1}{q_\perp^2}=\frac{1}{4\pi}\left(-\frac{1}{\epsilon}+\ln\frac{c_0^2}{\mu^2R_\perp^2}\right). \tag{23}$$

(2) By setting $\alpha=1+a$, one can easily find

$$\mu^{2\epsilon}\int\frac{d^{2-2\epsilon}q_\perp}{(2\pi)^{2-2\epsilon}q_\perp^2}\left(\frac{M^2}{q_\perp^2}\right)^a e^{-iq_\perp\cdot R_\perp}=\frac{1}{(4\pi)^{1-\epsilon}}\left(\frac{R_\perp^2\mu^2}{4}\right)^\epsilon\frac{\Gamma(-a-\epsilon)}{\Gamma(1+a)}\left(\frac{R_\perp^2M^2}{4}\right)^a. \tag{24}$$

By differentiating over a on both sides of the above equation and setting $a=0$ afterwards, one can obtain

$$\begin{aligned}&\mu^{2\epsilon}\int\frac{d^{2-2\epsilon}q_\perp}{(2\pi)^{2-2\epsilon}q_\perp^2}\ln\left(\frac{M^2}{q_\perp^2}\right)e^{-iq_\perp\cdot R_\perp}\\ =\ &\frac{1}{(4\pi)^{1-\epsilon}}\left(\frac{R_\perp^2\mu^2}{4}\right)^\epsilon\left[\gamma_E+\ln\frac{R_\perp^2M^2}{4}-\psi^{(0)}(-\epsilon)\right]\Gamma(-\epsilon),\end{aligned} \tag{25}$$

where $\psi^{(0)}(x)=\dfrac{\Gamma'(x)}{\Gamma(x)}$ is the zeroth order polygamma function. In the convention of $\overline{\text{MS}}$ subtraction scheme, we should also multiply a factor of $S_\epsilon^{-1}=(4\pi e^{-\gamma_E})^{-\epsilon}$ to the right hand side of the above results to convert from MS scheme to $\overline{\text{MS}}$ scheme. At last, by expanding ϵ around 0, we can find the result in Eq. (19).

(3) Use $\int d\Omega_d=2\pi^{d/2}/\Gamma[d/2]$ together the following identity

$$\int_0^{Q^2}\frac{d\,l_\perp^2}{l_\perp^2}\left(\frac{\mu^2}{l^2}\right)^\epsilon\ln\left(\frac{Q^2}{l^2}\right)=\frac{1}{\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^\epsilon, \tag{26}$$

the last integral can be evaluated in $d=2-2\epsilon$ dimension.