Renormalization group flows of field dependent couplings

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ACHT 2021 meeting

April 23, 2021

GF & A. Patkos, Phys. Rev. D**103**, 056015 (2021) [arXiv:2011.08387]
GF, Symmetry **13**(3), 488 (2021) [arXiv:2012.08706]

Outline

Introduction of field dependent couplings

Functional Renormalization Group

Application I: Yukawa coupling

Application II: $U_A(1)$ 't Hooft coupling

Summary

Motivation

- What is a field dependent coupling?
- Classical theory:

$$S[\phi] = \int \mathcal{L} = \int \left[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + g\phi^4 + \eta\phi^6 + ...\right]$$

- Quantum effective action:

$$\Gamma[\phi] = \sum_n \int \Gamma^{(n)}(\{p_i\};\mu)\,\phi(p_1)...\phi(p_n)$$

$\longrightarrow$ promote $\Gamma^{(n)}(\{p_i\};\mu) \rightarrow \Gamma^{(n)}(\{p_i\};\mu,\phi)$?

$\longrightarrow$ does not make sense (Γ is perturbative in ϕ)

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$\longrightarrow$ does not make sense (Γ is perturbative in ϕ)

- Multicomponent ϕ^a + internal symmetries $\Rightarrow$ reorganize Γ!

$\longrightarrow$ linear symmetries of S are inherited by Γ

$\longrightarrow$ only certain (invariant) combinations appear: $I_1, I_2 ... I_N$

$$\Gamma[\phi] = \sum_{\{\alpha\}} \int \Gamma^{(\alpha)}(\{p_i\};\mu, I_1) I_2^{\alpha_2} I_3^{\alpha_3} ... I_N^{\alpha_N}$$

Motivation

- Why does it make sense to consider field dependent couplings?
  $\longrightarrow$ one can still expand them $\Rightarrow$ non-renormalizable terms

$$\Gamma^{(\alpha)}(I_1) = \sum_n \Gamma_n^{(\alpha)} I_1^n$$

- Continuum limit: non-renormalizable operators disappear (renormalizability!)
  $\longrightarrow$ BUT: they are important in the IR

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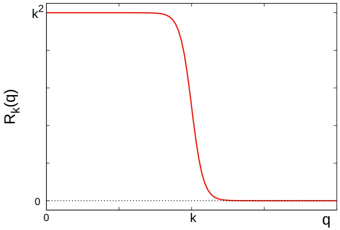
- Continuum limit: non-renormalizable operators disappear (renormalizability!)
  $\longrightarrow$ BUT: they are important in the IR
- Confusion: in the Wilsonian renormalization group
  irrelevance $\longleftrightarrow$ non-renormalizablilty
- Perturbatively non-renormalizable operators are not important only on a critical surface
  $\longrightarrow$ corresponding fixed point needs to be „close" to Gaussian
- $\Gamma_n^{(\alpha)}$ does have importance in the IR!
  $\longrightarrow$ resumming $\Gamma_n^{(\alpha)}$ can (actually) be a necessity

Functional Renormalization Group

- The Functional Renormalization Group is designed to resum field dependence
- Scale dependent partition function:

$$Z_k[J] = \int \mathcal{D}\phi e^{-(\mathcal{S}[\phi]+\int J\phi)} \times e^{-\frac{1}{2}\int \phi R_k \phi}$$



- Scale dependent effective action:

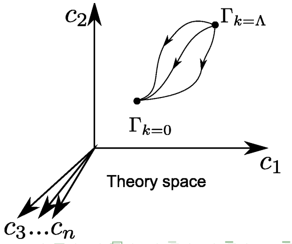
$$\Gamma_k[\phi] = -\log Z_k[J] - \int J\phi - \frac{1}{2}\int \phi R_k \phi$$

$\longrightarrow$ $k \approx \Lambda$: no fluctuations included

$\Rightarrow \Gamma_k[\phi]|_{k=\Lambda} = \mathcal{S}[\phi]$

$\longrightarrow$ $k = 0$: all fluctuations included

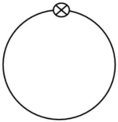
$\Rightarrow \Gamma_k[\phi]|_{k=0} = \Gamma[\phi]$



Functional Renormalization Group

- Flow equation of the effective action:

$$\partial_k \Gamma_k = \frac{1}{2}\int_{q,p} \mathrm{Tr}\left[\partial_k R_k(q,p)(\Gamma_{k,2}+R_k)^{-1}(p,q)\right] = \frac{1}{2}$$



- If one is interested in zero momentum couplings:
(Local Potential Approximation - LPA)

$$\Gamma_k[\phi^a] = \int \left[\frac{1}{2}(\partial_i \phi^a)^2 + V_k(\phi^a)\right]$$

$\longrightarrow$ no expansion in ϕ^a $\Rightarrow$ non-perturbativity

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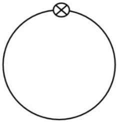

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- Flow equation for the effective potential V_k:

$$\partial_k V_k[I_1, I_2, \ldots] = \frac{1}{2}\int_p \mathrm{Tr}\left[\partial_k R_k(p)\big(p^2 + V_{k,2} + R_k(p)\big)^{-1}\right]$$

- Field dependent couplings:

$$V_k[I_1, I_2, \ldots] = \sum_{\{\alpha\}} \underline{\underline{V_k^{(\alpha)}(I_1)}}\, I_2^{\alpha_2} I_3^{\alpha_3} \ldots I_N^{\alpha_N}$$

Functional Renormalization Group

- Identification of the field dependent couplings:

$$\sum_{\{\alpha\}} \partial_k V_k^{(\alpha)}(I_1)\, I_2^{\alpha_2} I_3^{\alpha_3} \dots I_N^{\alpha_N} = \frac{1}{2}\int_p \tilde{\partial}_k \,\mathrm{Tr}\, \log(p^2 + V_{k,2} + R_k(p))$$

$\longrightarrow$ problem: entries of V''_k are not functions of the I_i invariants but are expressed in terms of background ϕ^a

$\longrightarrow$ symmetry ensures that the above expansion must contain invariant combinations

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$\longrightarrow$ symmetry ensures that the above expansion must contain invariant combinations

- Task: find a background through which the above expansion is naturally realized
- Remember: reorganized expansion is physically motivated

$\longrightarrow$ one is interested in non-perturbativity/resummation in one particular invariant (I_1)

$\longrightarrow$ e.g. in the vacuum $I_1 \neq 0$ but $I_i = 0$ for $2 \leq i \leq N$

Application I: Yukawa coupling

- Three flavor quark-meson model: [M - mesons, ψ - quarks]

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}\,[\partial_i M^\dagger \partial_i M] + \bar{\psi}(\not\partial + g_Y M_5)\psi + V[M]$$

$$V[M] = \frac{1}{2}m^2\,\mathrm{Tr}\,[M^\dagger M] + g_1(\,\mathrm{Tr}\,[M^\dagger M])^2 + g_2\,\mathrm{Tr}\,(M^\dagger M M^\dagger M)$$

- Eff. action (Γ) depends on chirally invariant combinations!
- Symmetry breaking: $U_L(3)\times U_R(3) \longrightarrow U_V(3)$

Pure meson:

$$I_1 = \mathrm{Tr}\,(M^\dagger M) \longrightarrow \text{nonzero}$$
$$I_2 = \mathrm{Tr}\,(M^\dagger M - \mathrm{Tr}\,(M^\dagger M)/3)^2 \longrightarrow 0$$
$$I_3 = \mathrm{Tr}\,(M^\dagger M - \mathrm{Tr}\,(M^\dagger M)/3)^3 \longrightarrow 0$$

Quark-meson:

$$\tilde{I}_1 = \bar{\psi}M_5\psi \longrightarrow 0$$
$$\tilde{I}_2 = \bar{\psi}M_5(M_5^\dagger M_5 - \mathrm{Tr}\,(M_5^\dagger M_5)/3)\psi \longrightarrow 0$$

...

Application I: Yukawa coupling

- Based on symmetry breaking, pure mesonic part is:

$$V_m = \sum_{\{\alpha\}} V^{(\alpha)}(I_1)\; I_2^{\alpha_2} I_3^{\alpha_3} \approx \underline{\underline{U(I_1)}} + \underline{\underline{C(I_1)}}\, \mathrm{Tr}\left(M^\dagger M - \frac{1}{3}\mathrm{Tr}\left(M^\dagger M\right)\right)^2 \ldots$$

- Similarly, the fermion-meson interaction is approximated via

$$V_{fm} \approx \underline{\underline{g_Y(I_1)}}\, \bar{\psi} M_5 \psi + \underline{\underline{g_{Y,2}(I_1)}}\, \bar{\psi} M_5 \left(M_5^\dagger M_5 - \frac{1}{3}\mathrm{Tr}\left(M_5^\dagger M_5\right)\right)\psi + \ldots$$

$\longrightarrow$ task: calculate mass matrices and identify operators in the rhs of the RG flow eq.

$\longrightarrow$ working with a general background field is hopeless

$\longrightarrow$ problem: how to distinguish each term from each other?

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- One needs an expansion in terms of M generating I_2, $\tilde{I}_2$, ... but keeps I_1 non-perturbative!
- Solution: $M = (s_a + i\pi_a)T_a \equiv s_0 T_0 + s_8 T_8$, $s_8 \ll s_0$!

Application I: Yukawa coupling

- In the background $M = (s_a + i\pi_a)T_a \equiv s_0 T_0 + s_8 T_8$:

$$
\begin{aligned}
I_1 &= (s_0^2 + s_8^2)/2 \\
I_2 &\sim s_0^2 s_8^2 + \mathcal{O}(s_8^4) \\
I_3 &\sim s_0^3 s_8^3 + \mathcal{O}(s_8^6) \\
\tilde{I}_1 &= \psi(s_0 T_0 + s_8 T_8)\bar{\psi} \\
\tilde{I}_2 &\sim \psi s_0 s_8^2 \bar{\psi} + \mathcal{O}(s_8^3) \\
&\dots
\end{aligned}
$$

- An expansion in terms of s_8 and ψ realizes the invariant expansion that keeps I_1 non-perturbative!
- Recipe: 1.) Calculate particle masses in terms of M and ψ
  2.) Expand the RG flow equation in terms of s_8 and ψ
  3.) Identify all <u>invariants</u> using the above expressions
  4.) The coefficients give the non-perturbative flows of the field dependent couplings

Application I: Yukawa coupling

- Common mistake in the literature:

$$g_Y \neq \frac{\delta^3 V}{\delta\bar{\psi}\delta\psi\delta M_5}$$

$\longrightarrow$ this includes contributions from higher order couplings (e.g. $g_{Y,2}$)

$\longrightarrow$ to identify g_Y one carefully needs to project out these „contaminations"

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- Yukawa term in the RG eq: $(p_R^2 = p^2 + R_k,\ \tilde\partial_k$ acts on $R_k)$

$$\int_p \tilde\partial_k \Bigg[\frac{3g_{Y,k}^3/2}{(p_R^2 + g_{Y,k}^2\frac{1}{3}I_1)(p_R^2+U_k')} - \frac{\frac{4}{3}g_{Y,k}^3}{(p_R^2+\frac{1}{3}g_{Y,k}^2 I_1)(p_R^2+U_k'+\frac{4}{3}C_k I_1)}$$

$$-\frac{\frac{1}{6}g_{Y,k}^3+\frac{2}{3}I_1 g_{Y,k}^2 g_{Y,k}'+\frac{2}{3}I_1^2 g_{Y,k}g_{Y,k}'^2}{(p_R^2+\frac{1}{3}g_{Y,k}^2 I_1)(p_R^2+U_k'+2I_1U_k'')} + \frac{9g_{Y,k}'/2}{p_R^2+U_k'}$$

$$+\frac{4g_{Y,k}'}{p_R^2+U_k'+\frac{4}{3}C_k I_1} + \frac{3g_{Y,k}'/2+\rho g_{Y,k}''}{p_R^2+U_k'+2I_1U_k''}\Bigg] \bar\psi M_5 \psi$$

Application I: Yukawa coupling

- Is it really necessary to make couplings depend on the field?
  $\longrightarrow$ expansion in I_1 would lead to ordinary flowing couplings
- Problem! A typical contribution contains a propagator

$$\frac{1}{p_R^2 + U_k'} = \frac{1}{p_R^2 + m_k^2 + g_{1,k} I_1 + \dots} = \frac{1}{p_R^2 + m_k^2} + \mathcal{O}(I_1)$$

- IF the potential is symmetry breaking ($m^2 < 0$), this blows up!
  $\longrightarrow$ singular RG flow
  $\longrightarrow$ resummation in I_1 is a necessity

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- IF the potential is symmetry breaking ($m^2<0$), this blows up!
  $\longrightarrow$ singular RG flow
  $\longrightarrow$ resummation in I_1 is a necessity
- Why no such problems do not occur in the field theoretical RG? (e.g. in MS or $\bar{M}S$ schemes)
  $\longrightarrow$ field theoretical RG provides a massless scheme
  $\longrightarrow$ running couplings are determined via UV divergences
  $\Rightarrow$ mass parameters never appear in denominators!
- Wilsonian RG is more general

Application I: Yukawa coupling

- Parametrization:

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}\,[\partial_i M^\dagger \partial_i M] + \bar{\psi}(\partial\!\!\!/ + g_Y M_5)\psi + V[M] - h_0 s_0 - h_8 s_8$$

$$V[M] = U(I_1) + g_2\,\mathrm{Tr}\,(M^\dagger M M^\dagger M); \quad U(I_1) = \frac{1}{2}m^2\,\mathrm{Tr}\,[M^\dagger M] + g_1(\,\mathrm{Tr}\,[M^\dagger M])^2$$

- PCAC relations: $h_{\mathrm{ns}} = m_\pi^2 f_\pi$, $h_{\mathrm{s}} = \frac{1}{\sqrt{2}}(2m_K^2 f_K - m_\pi^2 f_\pi)$
- Ward identities: $s_{\mathrm{ns}} = f_\pi$, $s_{\mathrm{s}} = \sqrt{2}(f_K - f_\pi/2)$
- m_π and m_K determines U' and g_2
  $\longrightarrow$ changing U'' allows for tuning m_σ
  $\longrightarrow$ we choose $450\,\mathrm{MeV} \lesssim m_\sigma \lesssim 600\,\mathrm{MeV}$
  ($\Rightarrow 10 \lesssim U'' \lesssim 20$)

- Dressed Yukawa coupling as a function of the bare one (the UV scale was set to $\Lambda = 1\,\mathrm{GeV}$)

U''	$g_{Y,k=\Lambda}$	$g_{Y,k=0}$	Δg
10	5	6.0	16%
10	10	14.4	31%
10	15	22.7	34%
10	20	30.3	34%

U''	$g_{Y,k=\Lambda}$	$g_{Y,k=0}$	Δg
20	5	6.0	16%
20	10	14.1	29%
20	15	22.0	32%
20	20	29.2	32%

- Note: one-loop Yukawa β-function is zero
$\longrightarrow$ no flow without field dependence

Application II: $U_A(1)$ anomaly

- Meson model with anomaly: [M - mesons]

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}\,[\partial_i M^\dagger \partial_i M] + \underline{a(\det M^\dagger + \det M)} + V[M]$$
$$V[M] = \frac{1}{2}m^2\,\mathrm{Tr}\,[M^\dagger M] + g_1(\,\mathrm{Tr}\,[M^\dagger M])^2 + g_2\,\mathrm{Tr}\,(M^\dagger M M^\dagger M)$$

- Ansatz for the effective action:

$$\Gamma_k = \int \left[\frac{1}{2}\mathrm{Tr}\,[\partial_i M^\dagger \partial_i M] + \underline{\underline{A_k[I_1]}}(\det M^\dagger + \det M) + V_k[M]\right]$$
$$V_k[M] = \underline{\underline{U_k[I_1]}} + \underline{\underline{C_k[I_1]}}\, I_2$$

- Invariant identification:

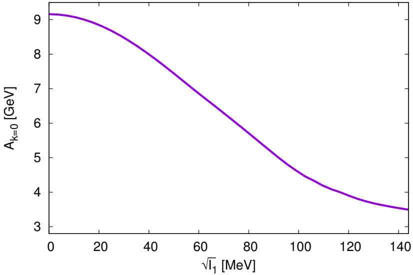
$$I_1 = \mathrm{Tr}\,[M^\dagger M], \quad I_2 = \mathrm{Tr}\,[(M^\dagger M - \mathrm{Tr}\,[M^\dagger M]/3)^2]$$
$$I_{\det} = (\det M^\dagger + \det M)|_{s_0, s_8} \sim s_0^3 + 3 s_8^2 s_0/2$$

Application II: $U_A(1)$ anomaly

- Identification of the field dependent couplings:

$$\sum_{\{\alpha\}} \partial_k V_k^{(\alpha)}(I_1)\, I_2^{\alpha_2} I_3^{\alpha_3} \ldots I_N^{\alpha_N} = \frac{1}{2}\int_p \tilde{\partial}_k \,\mathrm{Tr}\, \log(p^2 + V_{k,2} + R_k(p))$$

- Expanding in terms of s_8 will generate invariants in the rhs: $\Longrightarrow \partial_k U_k, \partial_k C_k, \underline{\underline{\partial_k A_k}}$
- Anomaly coefficient $A < 0$ but $|A|$ decreases with I_1!
- As the chiral condensate evaporates, the anomaly wants to go up!



Application II: $U_A(1)$ anomaly

- Coupling the nucleon field to the linear sigma model

$$\mathcal{L}_{\rm int} = g_Y \bar{\psi} M_5 \psi, \quad \psi^T = (p, n)$$

- Normal nuclear density, $n_N \approx 0.17\,\mathrm{fm}^{-3} \approx (109.131\,\mathrm{MeV})^3$ determines the Fermi momentum:
  $\longrightarrow p_F \approx 267.9\,\mathrm{MeV} \approx 1.36\,\mathrm{fm}^{-1}$ (mean field value)
- The quasiparticle mass in the medium (Landau mass) is $M_L \approx 0.8 m_N \Rightarrow s_{ns,N} \approx 69.52\,\mathrm{MeV}$
- As a result, the anomaly strengthens at the nuclear liquid-gas transition:

$$\frac{|A(s_{\rm ns} = s_{{\rm ns},N})| - |A(s_{\rm ns} = f_\pi)|}{|A(s_{\rm ns} = f_\pi)|} \approx 20\%$$

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- No Yukawa flow is taken into account
  $\longrightarrow$ interplay between the anomaly and the Yukawa coupling could be important (work in progress)

Application II: $U_A(1)$ anomaly

- Renormalization group flows of field dependent couplings
  $\longrightarrow$ multicomponent fields
  $\longrightarrow$ resummation of invariant operator(s)
  $\longrightarrow$ realized naturally by the FRG framework
- Application I.: flowing Yukawa coupling (3-flavor QM model)
  $\longrightarrow$ the naive $\delta^3\Gamma/\delta\bar{\psi}\delta\psi\delta M_5$ definition is invalid
  $\longrightarrow$ one needs to carefully project out „contaminations"
  $\longrightarrow$ $\sim$ 30% difference is obtained compared to bare value
- Application II.: 't Hooft coupling (3-flavor meson model)
  $\longrightarrow$ field dependent anomaly function decreases with χ-cond.
  $\longrightarrow$ as the condensate evaporates, the anomaly increases
  $\longrightarrow$ nuclear liquid-gas transition: $\sim$ 20% jump in the anomaly

More details: Eötvös Theor. Phys. seminar series via zoom
@ 27 April, 14:15 (CET+1), https://bodri.elte.hu