ANALOGIES BETWEEN LATTICE QCD and the TRUNCATED NAMBU – JONA-LASINIO MODEL

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ABSTRACT:

A modified Nambu--Jona-Lasinio Model with lattice structure is very instructive.

It shows several similar problems and their solutions as the Lattice QCD.

We study the limits of the large box size, small cell size and realistic pion mass.

In particular, we study the relation of the discrete (bound state) solutions to the physical scattering states, for example the pion-pion scattering.

OUTLINE

1. THE TWO-LEVEL NAMBU – JONA-LASINIO MODEL
2. THE QUASISPIN NJL-like MODEL
3. RELATION TO LATTICE CALCULATIONS
4. THE SPECTRUM OF 0^- and 0^+ EXCITATIONS
5. EMERGENCE OF THE σ MESON
6. EXTRACTION OF THE $\pi - \pi$ SCATTERING LENGTH
7. THE WIDTH OF THE σ MESON
8. THE EQUATION OF STATE AT ZERO BARYON NUMBER

1 THE TWO-LEVEL NAMBU JONA–LASINIO MODEL

M.Rosina and B.T.Oblak, Few-Body Syst.47 (2010)117-123

ASSUMPTIONS:

1. Sharp 3-momentum cutoff $0 \leq |\vec{p}_i| \leq \Lambda$.
2. Box of volume $\mathcal{V}$ with periodic boundary conditions.
   This gives a finite number of discrete momentum states, $\mathcal{N} = N_h N_c N_f \mathcal{V}\Lambda^3/6\pi^2$ occupied by N quarks.
3. $|\vec{p}_i| \to P = \frac{3}{4}\Lambda$.
4. Each quark conserves its momentum.
5. Restriction to one flavour (temporarily).

$$
\begin{aligned}
H &= \sum_{k=1}^{N}\Big(\gamma_5(k)h(k)P + m_0\beta(k)\Big) + \\
&- \frac{g}{2}\Big(\sum_{k=1}^{N}\beta(k)\sum_{l=1}^{N}\beta(l) + \sum_{k=1}^{N}\mathrm{i}\beta(k)\gamma_5(k)\sum_{l=1}^{N}\mathrm{i}\beta(l)\gamma_5(l)\Big) \quad .
\end{aligned}
$$

Here γ_5 and β are Dirac matrices, $h = \vec{\sigma}\cdot\vec{p}/|\vec{p}|$, m_0 is the bare quark mass and $g = 4G/\mathcal{V}$.

2 THE QUASISPIN NJL-like MODEL

We introduce the quasispin operators which obey the spin commutation relations

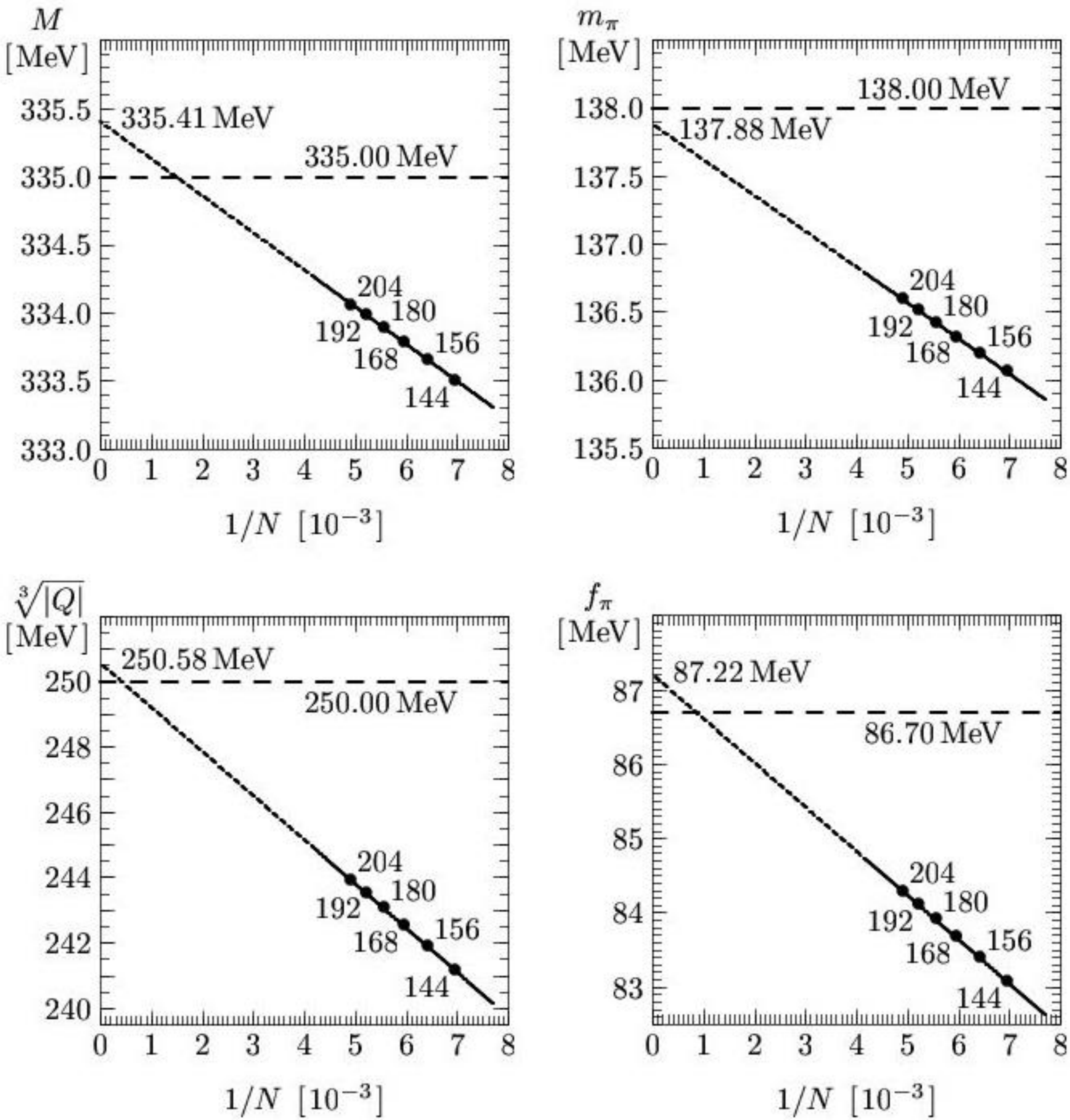
$$j_x = \frac{1}{2}\beta\,, \quad j_y = \frac{1}{2}\mathrm{i}\beta\gamma_5\,, \quad j_z = \frac{1}{2}\gamma_5\,,$$

$$R_\alpha = \sum_{k=1}^{N} \frac{1+h(k)}{2} j_\alpha(k)\,,\; L_\alpha = \sum_{k=1}^{N} \frac{1-h(k)}{2} j_\alpha(k)\,,\; J_\alpha = R_\alpha + L_\alpha = \sum_{k=1}^{N} j_\alpha(k)\,.$$

The model Hamiltonian can then be written as $H = 2P(R_z - L_z) + 2m_0 J_x - 2g(J_x^2 + J_y^2)$.
The MODEL PARAMETERS are determined by fitting

$$\begin{aligned} M &= \sqrt{\Big(E_g(N) - E_g(N-1)\Big)^2 - P^2} = 335\,\mathrm{MeV} \\ Q &= \langle g|\bar{\psi}\psi|g\rangle = \frac{1}{\mathcal{V}}\langle g|\sum_i \beta(i)|g\rangle = \frac{1}{\mathcal{V}}\langle g|J_x|g\rangle = 250^3\,\mathrm{MeV}^3 \\ m_\pi &= E_1(N) - E_g(N) = 138\,\mathrm{MeV}\,. \end{aligned}$$

The values $\Lambda = 648\,\mathrm{MeV}$, $G = 40.6\,\mathrm{MeV\,fm}^3$, $m_0 = 4.58\,\mathrm{MeV}$ are very close to full Nambu-Jona Lasinio (HF + RPA) of the Coimbra group ans of Buballa.



N→ ∞

Limit

- - - =

Hartree - Fock+ RPA for N= ∞

3. RELATION TO LATTICE CALCULATIONS

The model assumption $0 \leqslant |\mathbf{p}| \leqslant \infty$ corresponds to the cell size

$$a = 6^{1/3}\pi^{2/3}/\Lambda = 1.2 \text{ fm}$$

(for $N_c = 3$ colours, $N_f = 1$ flavours, 2 helicities, $\Lambda = 650$MeV, N=144).

The periodic boundary condition in *V* corresponds to the block size

$$L = \sqrt[3]{V} = \sqrt[3]{N/6}\, a \approx 3\, a = 3.6 \text{ fm}.$$

(Note that in one dimension L = 30 a and that the model is not sensitive on the number of dimensions.)

In the quasispin model it is very instructive that the number of colours N_c and the number of spatial states $V\Lambda^3/6\pi^2$ appear on equal footing in the product $\mathbf{N = 2\, N_c \times} \boldsymbol{V\Lambda^3/6\pi^2}$. Therefore it is the same limit $N \rightarrow \infty$ whether we take the large N_c limit or a large block *V*. This explains why even with 3 colours the quasispin model behaves similarly as the theorems for large N_c limit suggest.

We shall give examples how to exploit the variation of Λ and N (**a** and **L**), as well as the limit $m_{\text{“quark“}} \rightarrow m$ **($m_{\text{“}\pi\text{“}} \rightarrow m_\pi$).**

4. THE SPECTRUM OF 0^- and 0^+ EXCITATIONS

n	Parity	$(E - E_0)$[MeV] $N = 144$	$(E - E_0)$[MeV] $N = 192$	$\bar{V}$[MeV] $N = 144$	$\bar{V}$[MeV] $N = 192$
10	+	932	(942)	-9.5	(-5.4)
9	−	803	(805)	-11.7	(-7.2)
8	+	771	861	-11.3	-8.3
7	−	767	802	-8.8	-7.3
6	+	646	709	-11.4	-7.3
6	+	634	655	-12.2	-10.9
5	−	580	611	-10.0	-7.2
4	+	482	503	-10.5	-7.1
3	−	378	388	-10.1	-7.1
2	+	261	266	-10.3	-7.1
1	−	136	137		
0	+	0	0		

n_π parity		E [MeV]	ΔE [MeV]	Intruder
8	+	866	63	
	-	816		$\sigma(667)+\pi(136)+13$
7	-	803	93	
6	+	710	99	
	+	667		$\sigma(667)$
5	-	611	108	
4	+	503	115	
3	-	388	123	
2	+	265	129	
1	-	136	136	
0	+	0	0	

5. EMERGENCE OF THE **sigma** MESON

+	81	8
-	93	7
+	99	6
-	108	5
+	115	4
-	122	3
+	129	2
-	136	1
+	0	0

-	3	sigma+pi(655+150)

+	56	sigma(655)

6. EXTRACTION OF THE π – π SCATTERING LENGTH

Energy levels of n-pion states → average effective pion-pion potential $\bar{V}$:

$$E_{n\pi} = n\,m_\pi + \frac{n(n-1)}{2}\bar{V}.$$

N-dependence: In a larger volume, pions are more dilute → proportionally smaller $\bar{V}$.

The ratio of $\bar{V}$ for $N = 144$ and $N = 192$ is $10.3/7.1 = 1.45$, close to $192/144 = 1.33$.

In first-order Born approximation (" Lüscher formula")

$$a\,m_\pi = \frac{m_\pi^2}{4\pi}\int V(\vec{r})\,\mathrm{d}^3r = \frac{m_\pi^2}{4\pi}\bar{V}\mathcal{V} = -0.0768\,.$$

(For $N = 192$ we have $\bar{V} = -7.1\,\mathrm{MeV}$ and $\mathcal{V} = \pi^2 N/\Lambda^3 = 53\,\mathrm{fm}^3$.)

To be compared with Lesniak: -0.034 ("non-uniform fit") or -0.044 ("uniform fit").

The chiral perturbation theory (soft pions) suggests in leading order

$a_0^{I=2}\,m_\pi = -m_\pi^2/16\pi f_\pi^2 = -0.0445.$

7. THE WIDTH OF THE σ MESON

Discrete eigenvalues → we get a discrete sigma resonance *energy*, but not its *width.*

The two-level model does not have the flexibility to adapt the two-pion energy to the sigma energy (to describe the correct relative motion of the two pions) and to "scan" the resonance curve.
The extension in this direction is in progress.

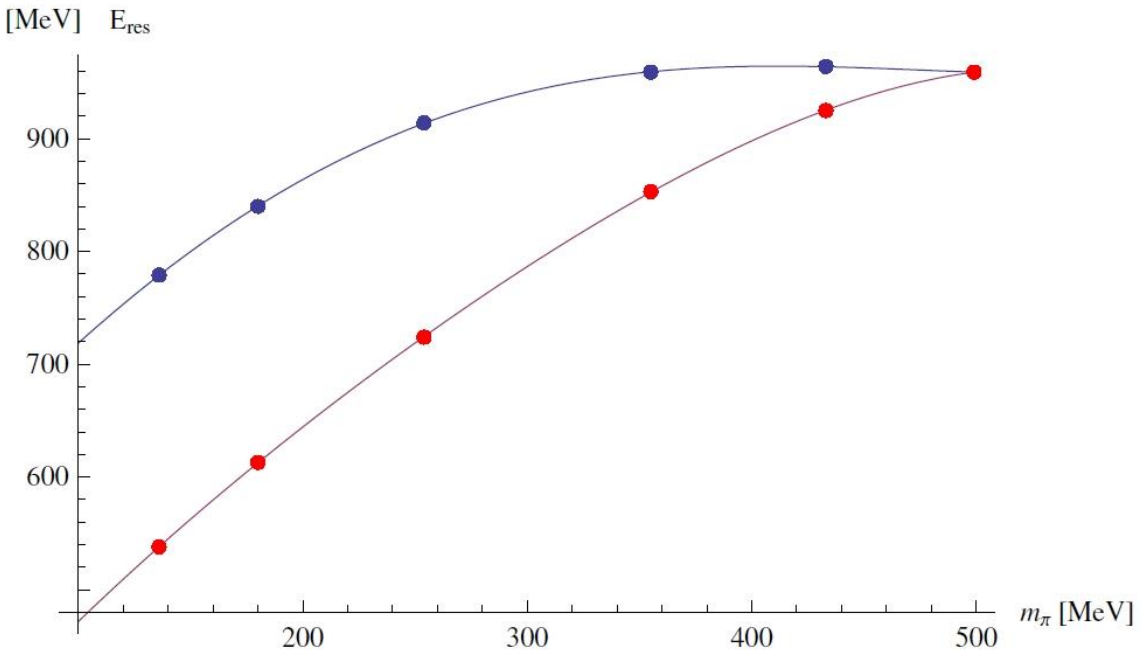
In the meantime, we explore the method of analytic continuation from the bound state [V.M. Krasnopolsky and V.I.Kukulin, Phys. Lett, **69A** (1978) 251, V.M. Krasnopolsky and V.I.Kukulin, Phys. Lett, **96B** (1980) 4, N. Tanaka et al. Phys. Rev. **C59** (1999) 1391.]

In order to get the complex pole, we vary the bare quark mass m from the region where the meson would be bound, ($E < E_{2\pi}$) to the physical value (where $E >> E_{2\pi}$).

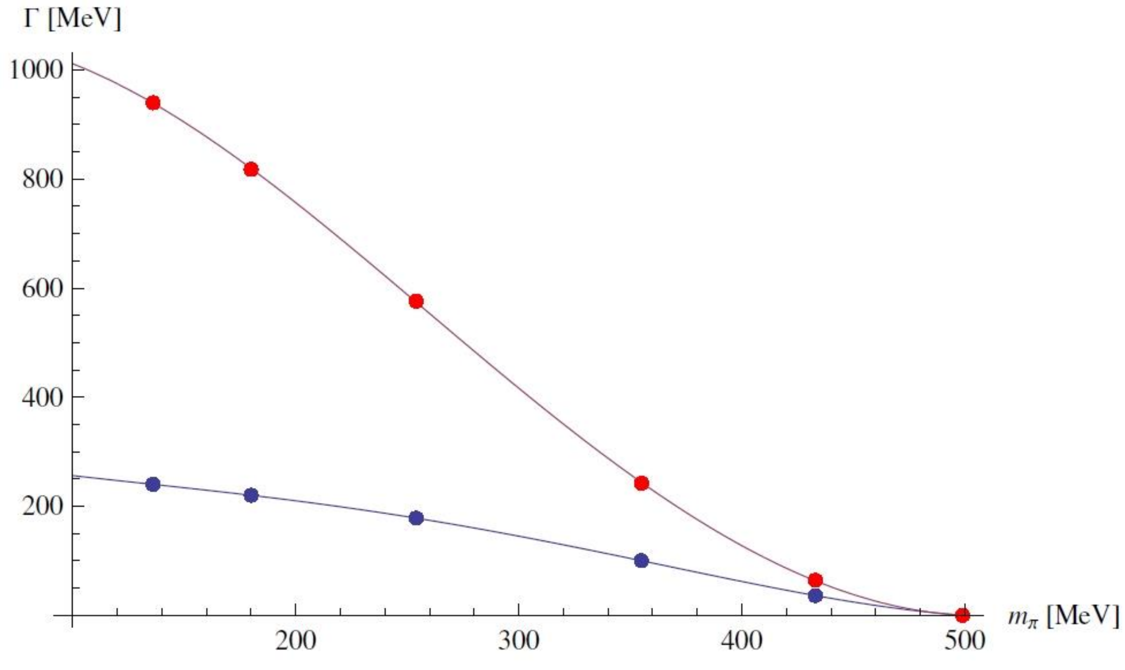
- Determine the threshold value m_{th} and calculate $\epsilon = E_\sigma - E_{2\pi}$ as a function of m for $m > m_{th}$.
- Introduce a variable $x = \sqrt{m - m_{th}}$; calculate $k(x) = i\sqrt{-\epsilon}$ in the bound state region (Fig.2).
- Fit $k(x)$ by a polynomial $k(x) = i(c_0 + c_1 x + c_2 x^2 + \ldots + c_{2M} x^{2M})$.
- Construct a Padé approximant:

$$k(x) = i\,\frac{a_0 + a_1 x + \ldots + a_M x^M}{1 + b_1 x + \ldots + b_M x^M} \,.$$

- Analytically continue $k(x)$ to the region $m < m_{th}$ (i.e. to imaginary x) where $k(x)$ becomes complex.
- Determine the position and the width of the resonance as analytic continuation in m (Fig.3 and Fig.4): $E_{res} = \mathrm{Recont}_{m \to m_0} k^2$, $\qquad \Gamma = -2\,\mathrm{Imcont}_{m \to m_0} k^2$.



The resonance energy E_{res}
of the σ meson as a
function of the pion mass,
using Padé approximants
of order 1 (below)
and 2 (above)



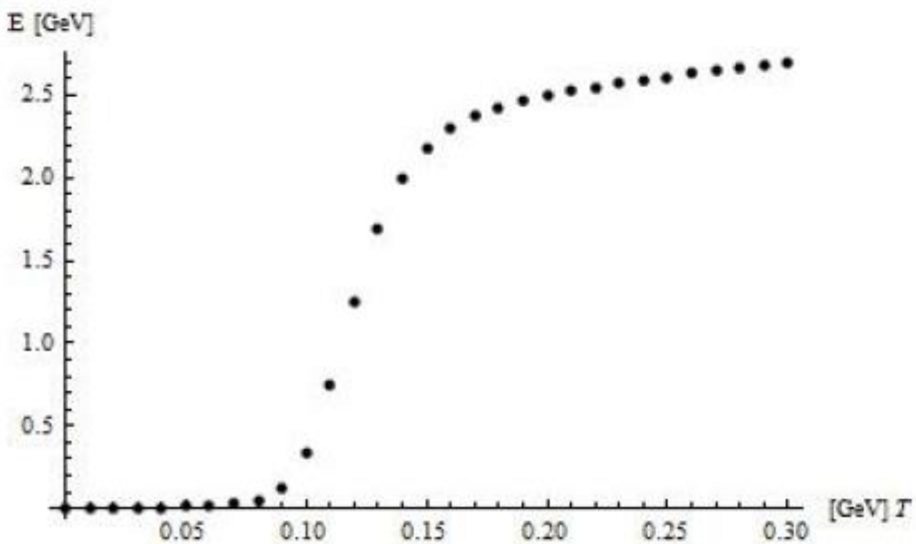
The width Γ
of the σ meson as a
function of the pion mass,
using Padé approximants
of order 1 (below)
and 2 (above)

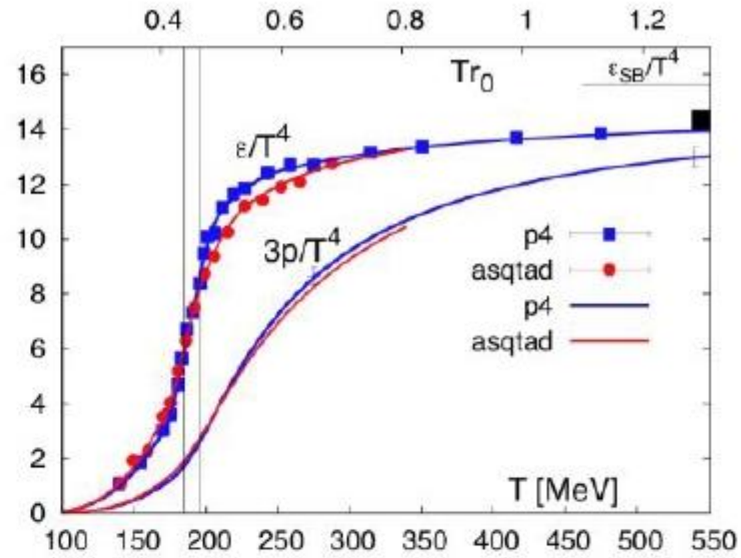
8. THE EQUATION OF STATE AT ZERO BARYON NUMBER

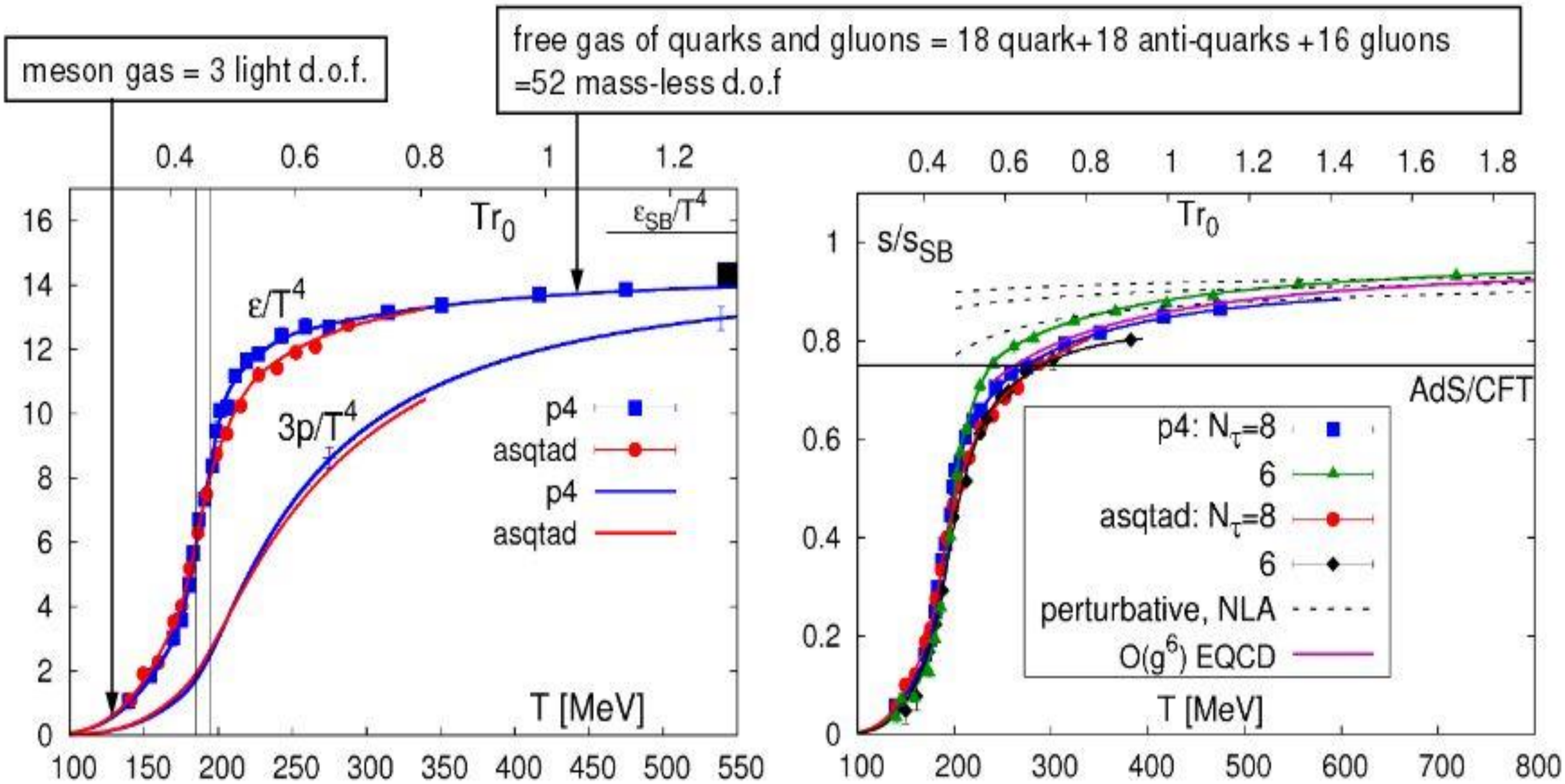
We calculate the canonical ensemble for a $N = \mathcal{N}$ quark system as a function of T.

$$Z = \sum D(i) \cdot \exp(-E(i)/T),$$

$$E = \sum D(i) \cdot \exp(-E(i)/T) \cdot E(i)/Z.$$







Bazavov et al (HotQCD), PRD 80 (09) 14504

Petreczky, NPA 830 (10) 11c

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THANKS FOR YOUR ATTENTION!

I SHALL APPRECIATE YOUR CRITICISM AND SUGGESTIONS