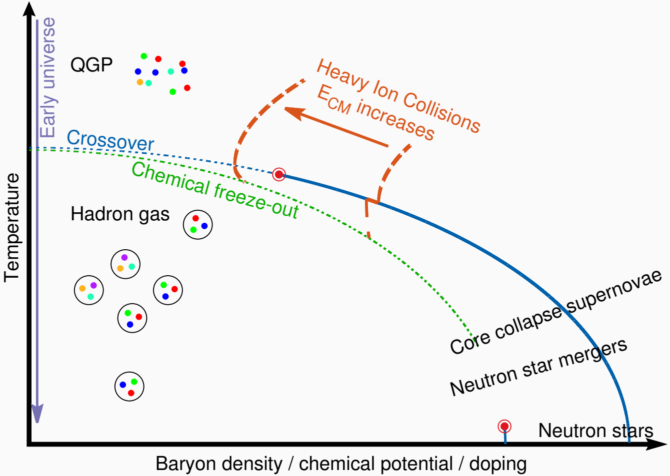
The QCD phase diagram from the analytic continuation of lattice QCD data

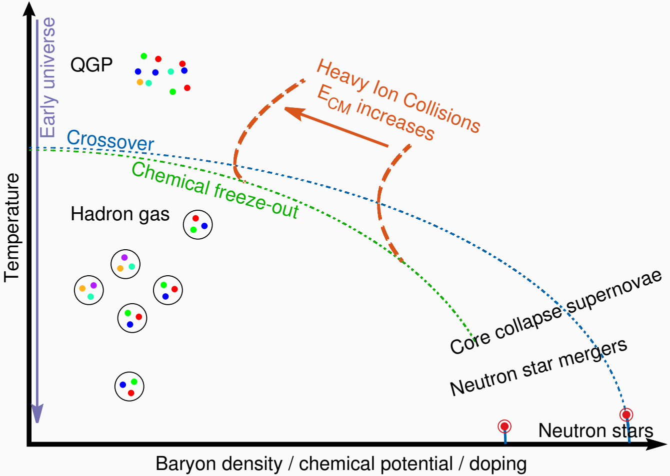
Attila Pásztor
Eötvös University, Budapest

ACHT 2021, Zagreb/Zoom

The phase diagram of QCD - public relations version



The phase diagram of QCD - an other version



QCD in the grand canonical ensemble

Grand canonical partition function:

$$Z = \mathrm{Tr}\left[e^{-(H_{QCD}-\mu_u N_u-\mu_d N_d-\mu_s N_s)/T}\right] = \mathrm{Tr}\left[e^{-(H_{QCD}-\mu_B B-\mu_Q Q-\mu_S S/T}\right]$$

$$p = \frac{T}{V}\log Z$$

Change of basis:

$$\mu_u = \frac{1}{3}\mu_B + \frac{2}{3}\mu_Q \quad \mu_d = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q \quad \mu_s = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q - \mu_S$$

Generalized susceptibilities:

$$\chi^{BSQ}_{i,j,k} = \frac{\partial^{i+j+k}\left(p/T^4\right)}{(\partial\hat{\mu}_B)^i(\partial\hat{\mu}_S)^j(\partial\hat{\mu}_Q)^k} \qquad \chi^{uds}_{i,j,k} = \frac{\partial^{i+j+k}\left(p/T^4\right)}{(\partial\hat{\mu}_u)^i(\partial\hat{\mu}_d)^j(\partial\hat{\mu}_s)^k}$$

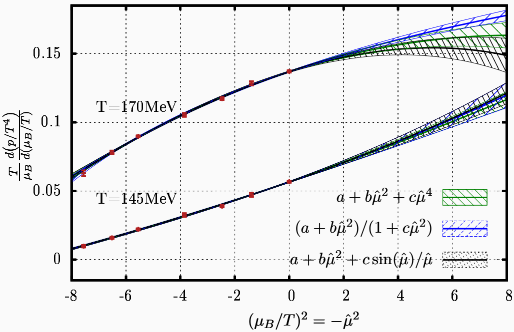
where $\hat{\mu} = \mu/T$ are similarly related to higher order cumulants.

So e.g. $\chi^B_1 \sim \langle B\rangle \quad \chi^B_2 \sim \left\langle B^2\right\rangle - \langle B\rangle^2 \quad \chi^{BQ}_{11} \sim \langle BQ\rangle - \langle B\rangle\langle Q\rangle$

Imaginary chemical potential method

At real $\mu_B > 0$: sign problem, cannot do lattice simulations; At $\mu_B^2 \leq 0$: sign problem is absent

Analytical continuation on $N_t = 12$ raw data



Two uses:

- Numerical differentiation at $\mu = 0$: safe (for low orders)
- Extrapolation: risky

Alternative: calculate derivatives directly at $\mu_B = 0$ (**Taylor method**)

Outline

Two questions for lattice QCD

1. Can we draw the transition line at finite μ_B?
2. Can we at least give a lower bound on where the CEP is on this line?

Answers

1. Yes, under some reasonable smoothness assumptions. Depending on the assumptions, up to $\mu_B = 300$MeV or even 600MeV.
2. No. No useful constraints directly from the lattice. Yet?

Mostly based on

- 2002.02821 [hep-lat]; PRL 125 (2020); Borsanyi, Fodor, Guenther, Kara, Katz, Parotto, Pasztor, Ratti, Szabo
- 2010.00394 [hep-lat]; PRD 103 (2021); Pasztor, Szep, Marko
- 2102.06625 [hep-lat]; Bellwied, Borsanyi, Fodor, Guenther, Katz, Parotto, Pasztor, Pesznyak, Ratti, Szabo

THE TRANSITION LINE

2002.02821 [hep-lat]; PRL 125 (2020); Borsanyi, Fodor, Guenther, Kara, Katz, Parotto, Pasztor, Ratti, Szabo

How to get per mille accuracy on T_c at imaginary μ?

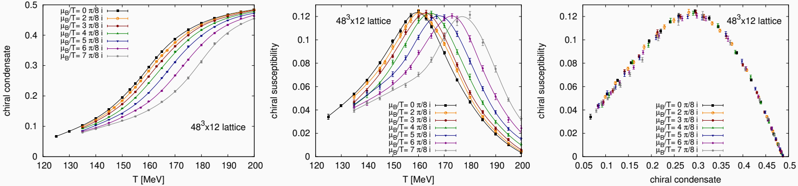
Basic observables:

chiral condensate: $\langle\bar{\psi}\psi\rangle = \frac{T}{V}\frac{\partial \ln Z}{\partial m_{ud}}$ chiral susceptibility: $\chi = \frac{T}{V}\frac{\partial^2 \ln Z}{\partial m_{ud}^2}$

renormalization:

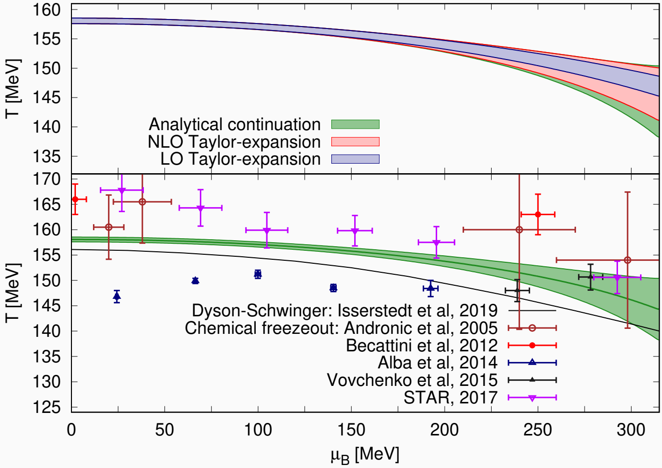
$$\langle\bar{\psi}\psi\rangle_R = -\left[\langle\bar{\psi}\psi\rangle_T - \langle\bar{\psi}\psi\rangle_0\right]\frac{m_{ud}}{f_\pi^4} \qquad \chi_R = \left[\chi_T - \chi_0\right]\frac{m_{ud}^2}{f_\pi^4}$$

An empirical observation:



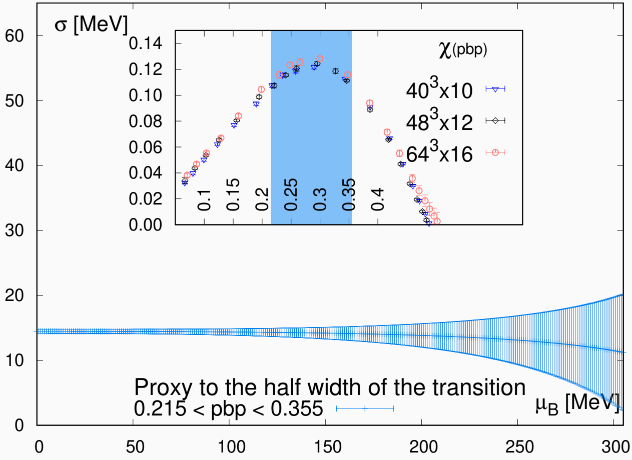
PRL 125 (2020); Borsanyi, Fodor, Guenther, Kara, Katz, Parotto, Pasztor, Ratti, Szabo

Results of a polynomial extrapolation



PRL 125 (2020); Borsanyi, Fodor, Guenther, Kara, Katz, Parotto, Pasztor, Ratti, Szabo

Extrapolating the width of the transition



PRL 125 (2020); Borsanyi, Fodor, Guenther, Kara, Katz, Parotto, Pasztor, Ratti, Szabo

THE TRANSITION LINE AGAIN

2010.00394 [hep-lat]; PRD 103 (2021); Pasztor, Szep, Marko

Beyond polynomials: Padé approximants

The $[n/m]$-th order Padé approximant of $f(x)$ is the rational function:

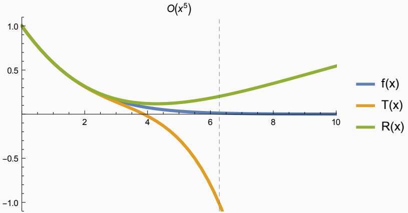
$$R_m^n(x) = \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{j=0}^{m} b_j x^j}$$

s.t. $K+1 := n+m+1$ Taylor-series coeff.-s of the two agree:

Example: $f(x) = x/(\mathrm{e}^x - 1)$

$$\begin{aligned} f(0) &= R_m^n(0) \\ f^{'}(0) &= R_m^{n'}(0) \\ f^{''}(0) &= R_m^{n''}(0) \\ &\vdots \\ f^{(n+m)}(0) &= R_m^{n(n+m)}(0). \end{aligned}$$

(Taylor-series convergence radius $=2\pi$)



Wikipedia: the "best" approximation of a fn. by a given order rational fn.

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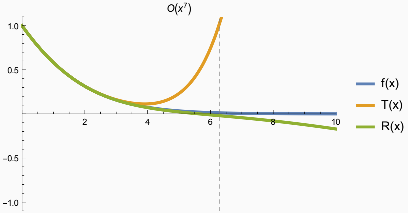

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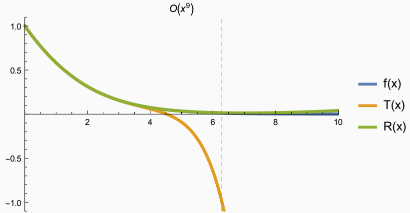

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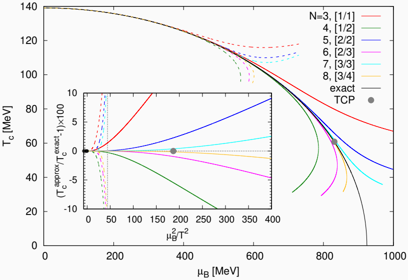
Example without noise: a chiral effective model

Chiral limit of the $N_f = 2$ constituent quark-meson model in a leading order large-N expansion. See: Jakovác et al., PLB **582**, 179 (2004).

- The model exhibits a **line of second order** phase transitions for $\mu^2 > 0$, which ends in a **tricritical point**.
- Both the transition line and the location of the tricritical point can be determined **analytically**.



- Alternating convergence of the Padé approximants beyond the radius of convergence of the Taylor series.
- The tricritical point is **not** a special point of the transition line.

How to apply it to noisy lattice QCD data?

Data: Taylor coeff.s from HotQCD: Bazavov et al., PLB **795**, 15 (2019)
T_c at Imμ_B from WB: Borsányi et al., PRL **125**, 052001 (2020)
Obstacle: Padé approximants are fragile in the presence of noise: fake poles with small residue or fake pole-zero pairs (Froissart doublets)
Bayesian approach:

$$\mathcal{P}(a_i, b_j|\mathrm{data}) = \frac{1}{Z}\mathcal{P}(\mathrm{data}|a_i, b_j)\mathcal{P}_{prior}(a_i, b_j).$$

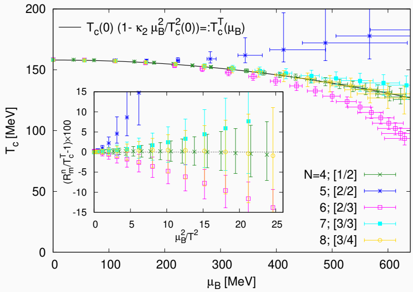
$$\mathcal{P}(\mathrm{data}|a_i, b_j) = \exp\left[-\frac{1}{2}\left(\chi^2_{i\mu} + \chi^2_{\mathrm{Taylor}}\right)\right]$$

$$\chi^2_{i\mu} = \sum_{i=1}^{L} \frac{\left(f_i - R^n_m(i\mu^I_{B,i};\vec{a},\vec{b})\right)^2}{\sigma^2_{f_i}}$$

$$\chi^2_{\mathrm{Taylor}} = \sum_{i=1}^{T} \frac{\left(c_i - \frac{\partial^i R^n_m(\mu_B;\vec{a},\vec{b})}{\partial\left(\mu_B^2\right)^i}\Big|_{\mu_B=0}\right)^2}{\sigma^2_{c_i}}$$

Without details: the prior restricts the function space from all rational functions of a given order to those that have **no poles** in a wide range of the real axis

Results of the Padé analysis



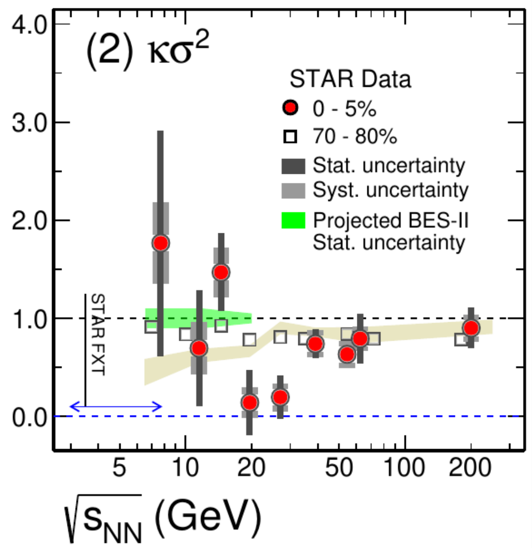
- Based on posterior distribution of $T_c = R_m^n(\mu_B^2/T^2)$.
- μ_B coordinate evaluated as $\mu_B = \sqrt{\mu_B^2/T^2} \times T_c$ also has a distribution.
- Distributions are skewed for higher $\mu_B \rightarrow$ we indicate the median and the most likely 68% in our results.

- Apparent alternating convergence.
- Consistent with the previous analysis, but goes higher in μ_B
- Highest orders are consistent with $T_c(0) - \kappa_2 \mu_B^2$.
- Consistent e.g. with the DSE endpoint in 1906.11644 [hep-ph], but **no information** on placement

CONTRIBUTIONS FROM THE $B = 2$ HILBERT SUBSPACE AND BARYON FLUCTUATIONS

2102.06625 [hep-lat]; Bellwied, Borsanyi, Fodor, Guenther, Katz, Parotto, Pasztor, Pesznyak, Ratti, Szabo

Experimental data on net-proton fuctuations



STAR: 2001.02852 [nucl-ex] Phys.Rev.Lett. 126 (2021) 9, 092301

The Hadron Resonance Gas Model

Hadrons are free particles in a heat bath, their interactions are introduced by adding all their resonances to the heat bath, as free particles.

$$\frac{p^{\mathrm{HRG}}}{T^4} = \frac{1}{VT^3}\left(\sum_{i\in\mathrm{mesons}} \log \mathcal{Z}^M\left(T, V, m_i, \{\mu\}\right) + \sum_{i\in\mathrm{baryons}} \log \mathcal{Z}^B\left(T, V, m_i, \{\mu\}\right)\right)$$

$$\log \mathcal{Z}^{M/B} = \mp \frac{Vd_i}{2\pi^2}\int_0^\infty dk k^2 \log\left(1 \mp z_i e^{-\sqrt{m_i^2+k^2}/T}\right)$$

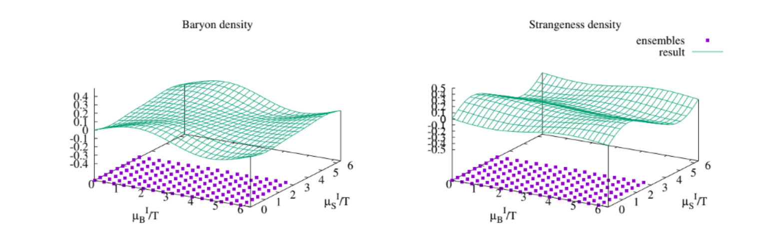
with the fugacity factor $z_i = \exp\left(B_i\hat{\mu}_B + Q_i\hat{\mu}_Q + S_i\hat{\mu}_S\right)$ and $\hat{\mu}_B = \mu_B/T$ etc.
Taylor expand the logdet:

$$\log \mathcal{Z}^{M/B} = \frac{VT^3}{2\pi^2} d_i \frac{m_i^2}{T^2}\sum_{k=1}^{\infty}(\pm)^{k+1}\frac{z_i^k}{k^2}K_2\left(km_i/T\right)$$

For $m_i \gg T$ (everything except πs) the $k = 1$ term dominates (Boltzmann approx.):

$$\log \mathcal{Z}^{M/B} + \log \mathcal{Z}^{\bar{M}/\bar{B}} \approx 2\frac{VT^3}{2\pi^2} d_i \frac{m_i^2}{T^2} K_2\left(m_i/T\right)\cosh\left(B_i\hat{\mu}_B + Q_i\hat{\mu}_Q + S_i\hat{\mu}_S\right)$$

A new dataset: 2102.06625 [hep-lat]

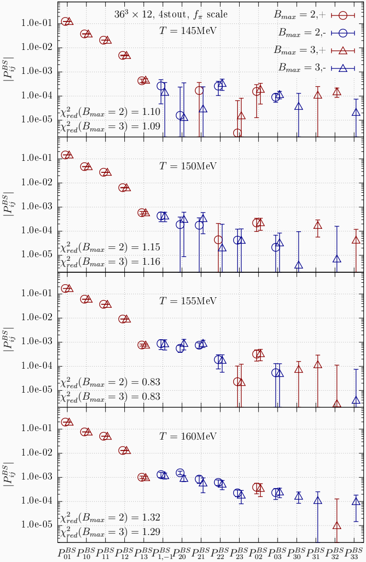


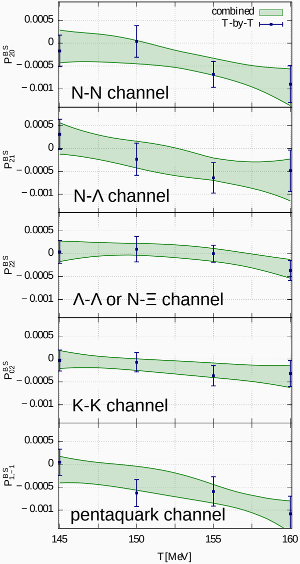
A surface is fitted on the imaginary baryon and strangeness densities, as well as their susceptibilities (**fugacity expansion**):

$$P(T, \mu_B^I, \mu_S^I) = \sum_{j,k} P_{jk}^{BS}(T) cos(j\mu_B^I/T - k\mu_S^I/T)$$

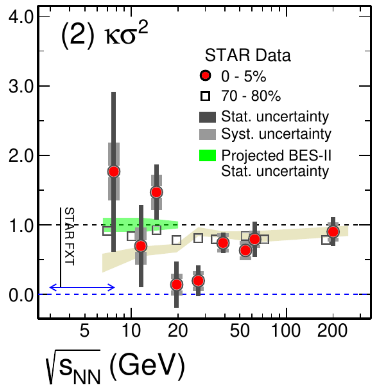
Non-trivial μ_B dependence comes from terms beyond the ideal HRG:
P_{20}^{BS} gets contributions from $N - N$ scattering
P_{21}^{BS} gets contributions from $N - \Lambda$ scattering
P_{22}^{BS} gets contributions from $N - \Xi$ or $\Lambda - \Lambda$ scattering, etc.

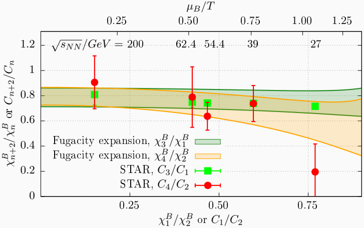
The fugacity expansion coefficients





Extrapolating baryon number fluctuations





Extrapolate with $< S >= 0$, on the crossover line.

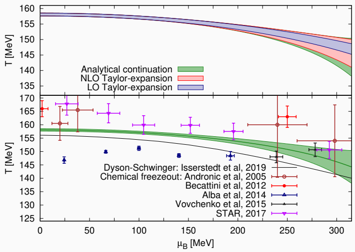
Errors blow up before the interesting physics happens.

STAR: 2001.02852 [nucl-ex]

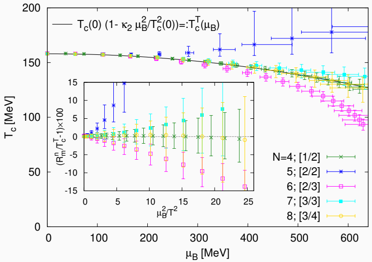
net-proton kurtosis-to-variance

WB: 2102.06625 [hep-lat]

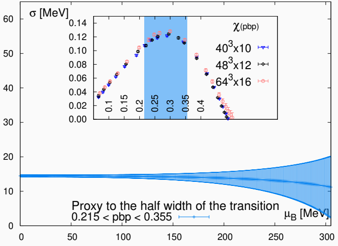
Summary



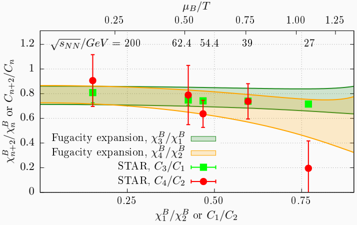
T [MeV]
Analytical continuation
NLO Taylor-expansion
LO Taylor-expansion
Dyson-Schwinger: Isserstedt et al, 2019
Chemical freezeout: Andronic et al, 2005
Becattini et al, 2012
Alba et al, 2014
Vovchenko et al, 2015
STAR, 2017
μB [MeV]



Tc(0) (1- κ2 μB2/Tc2(0))=:TcT(μB)
Tc [MeV]
N=4; [1/2]
5; [2/2]
6; [2/3]
7; [3/3]
8; [3/4]
μB2/T2
μB [MeV]



σ [MeV]
χ(pbp)
40^3x10
48^3x12
64^3x16
Proxy to the half width of the transition
0.215 < pbp < 0.355
μB [MeV]



μB/T
√sNN/GeV = 200
62.4
54.4
39
27
χBn+2/χBn or Cn+2/Cn
Fugacity expansion, χB3/χB1
Fugacity expansion, χB4/χB2
STAR, C3/C1
STAR, C4/C2
χB1/χB2 or C1/C2