New exact solutions of dissipative hydrodynamics

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arXiv:2003.08859

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Outline

New, exact solutions of relativistic Navier-Stokes (NS) and Israel-Stewart (IS) theory

→ spherically symmetric Hubble-flow: great amount of freedom of dissipative coefficients

Asymptotically perfect fluid solutions

→ effects of dissipative coefficients in final state measurments?

Applications of the new, relativistic, dissipative solutions

→ indirect description of experimental data

→ producing new, non relativistic solutions

Asymptotic perfect fluid attractors of non relativistic, dissipative solutions

I. New, relativistic, dissipative solutions

Relativistic hydrodynamics (Navier-Stokes)

Local conservation of the four momentum and the particle number:

$$\partial_\mu \left(n u^\mu\right) = 0$$

$$\partial_\mu T^{\mu\nu} = 0$$

The energy-momentum tensor is:

$$T^{\mu\nu} = (\varepsilon + p)\, u^\mu u^\nu - p g^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu} - \Delta^{\mu\nu} \Pi$$

The heat current (with the heat conductivity λ):

$$q^\mu = \lambda \left(g^{\mu\nu} - u^\mu u^\nu\right) \left(\partial_\nu T - T u^\rho \partial_\rho u_\nu\right)$$

The following terms describes the viscous effects:

ζ: bulk viscosity
η: shear viscosity

$$\pi^{\mu\nu} = \eta \left[\Delta^{\mu\rho} \partial_\rho u^\nu + \Delta^{\nu\rho} \partial_\rho u^\mu\right] - \frac{2}{d} \eta \Delta^{\mu\nu} \partial_\rho u^\rho$$

$$\Pi = -\zeta \partial_\rho u^\rho$$

Relativistic hydrodynamics (Israel-Stewart)

Local conservation of the four momentum and the particle number:

$$\partial_\mu \left(n u^\mu \right) = 0$$

$$\partial_\mu T^{\mu\nu} = 0$$

The energy-momentum tensor is:

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$$\Pi = -\zeta \partial_\rho u^\rho - \tau_\Pi u_\rho \partial^\rho \Pi$$

Relativistic hydrodynamics (Israel-Stewart)

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$$\Pi = -\zeta \partial_\rho u^\rho - \tau_\Pi u_\rho \partial^\rho \Pi$$

To close the equation system:

EoS: ε=κp

In this work: *κ=const.*

ζ: bulk viscosity

η: shear viscosity

Hubble-type solutions: scale variable

Hubble-type velocity field: $$u^\mu = \frac{x^\mu}{\tau} = \gamma\left(1, \frac{r_x}{t}, \frac{r_y}{t}, \frac{r_z}{t}\right)$$

Scale equation: $$u^\mu \partial_\mu s = 0$$

Directional scale variables: $$s_x = \frac{r_x}{t},\ s_y = \frac{r_y}{t},\ s_z = \frac{r_z}{t}$$

Satisfy the scale equation separately: $$u^\mu \partial_\mu s_i = \partial_\tau s_i = 0$$

Hubble-type solutions: equations to solve

Navier-Stokes theory

Continuity equation: $\partial_\tau n + \frac{d}{\tau} n = 0$

Energy conservation $\partial_\tau p + \left(1 + \frac{1}{\kappa}\right) \frac{d}{\tau} p = \frac{d^2}{\tau^2} \frac{\zeta}{\kappa}$

Euler-equation: $p\tau - \zeta d = \phi(\tau)$

Entropy equation: $\partial_\tau \sigma + \frac{d}{\tau} \sigma = \frac{d^2}{\tau^2} \frac{\zeta}{T} \geq 0$

Ansatz for bulk viscosity: $\zeta = \zeta_0 \frac{p}{p_0}$

Israel-Stewart theory

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Energy conservation: $\partial_\tau p + \left(1 + \frac{1}{\kappa}\right) \frac{d}{\tau} p = -\frac{d}{\tau} \frac{\Pi}{\kappa}$

Bulk pressure: $\Pi = -\zeta \frac{d}{\tau} - \tau_\Pi \dot{\Pi}$

Euler-equation: $p + \Pi = \Psi(\tau)$

Entropy equation: $\partial_\tau \sigma + \frac{d}{\tau} \sigma = -\frac{d}{\tau} \frac{\Pi}{T} \geq 0$

Ansatz for bulk viscosity: $\zeta = \Pi \frac{\zeta_0}{\Pi_0}$

M. Nagy, M. Csanád, Z. Jiang, T. Csörgő: arXiv:1909.02498

T. Csörgő, G. K.: arXiv:2003.08859

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$\tau_\Pi \rightarrow 0$, or Π is constant

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Heat conduction and shear viscosity cancelled!

Hubble-type solutions: equations to solve

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Analytic solutions of NS equations, with κ=const

The solution of the pressure is: $$p(\tau) = p_0 \left(\frac{p_A}{p_0}\right)^{1-\frac{\tau_0}{\tau}} \left(\frac{\tau_0}{\tau}\right)^{d\left(1+\frac{1}{\kappa}\right)}, \quad \frac{p_A}{p_0} = f_{A,0} = \exp\left[\frac{d^2\zeta_0}{\kappa_0 p_0 \tau_0}\right]$$

The temperature has a generalized form: $$T = T_0 \left(\frac{T_A}{T_0}\right)^{1-\frac{\tau_0}{\tau}} \left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\kappa_0}} \mathcal{T}\left(s_x, s_y, s_z\right)$$

T. Csörgő, G. K.: arXiv:2003.08859

Conserved charge, μ > 0

$$p = nT$$

$$\frac{T_A}{T_0} = \frac{p_A}{p_0} = f_{0,A} = \exp\left(\frac{\zeta_0 d^2}{\kappa_0 p_0 \tau_0}\right)$$

$$n = n_0 \left(\frac{\tau_0}{\tau}\right)^d \mathcal{V}\left(s_x, s_y, s_z\right)$$

No conserved charge, μ = 0

$$p = \frac{T\sigma}{1+\kappa}$$

$$\frac{T_A}{T_0} = \left(\frac{\sigma_A}{\sigma_0}\right)^{\frac{1}{\kappa_0}} = \exp\left(\frac{\zeta_0 d^2}{\kappa_0 p_0 \tau_0} \frac{1}{1+\kappa_0}\right)$$

$$\sigma = \sigma_0 \left(\frac{\sigma_A}{\sigma_0}\right)^{1-\frac{\tau_0}{\tau}} \left(\frac{\tau_0}{\tau}\right)^d \mathcal{V}\left(s_x, s_y, s_z\right), \quad \frac{\sigma_A}{\sigma_0} = f_{0,A}^{\frac{\kappa_0}{1+\kappa_0}} = \exp\left(\frac{\zeta_0 d^2}{p_0 \tau_0} \frac{1}{1+\kappa_0}\right)$$

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$$n = n_0 \left(\frac{\tau_0}{\tau}\right)^d \mathcal{V}\left(s_x, s_y, s_z\right)$$

No conserved charge, μ = 0

$$\mathcal{T}\left(s_x, s_y, s_z\right) = \frac{1}{\mathcal{V}\left(s_x, s_y, s_z\right)}$$

$$p = \frac{T\sigma}{1+\kappa}$$

$$\frac{T_A}{T_0} = \left(\frac{\sigma_A}{\sigma_0}\right)^{\frac{1}{\kappa_0}} = \exp\left(\frac{\zeta_0 d^2}{\kappa_0 p_0 \tau_0} \frac{1}{1+\kappa_0}\right)$$

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Analytic solutions of IS equations, with κ=const

Bulk pressure:

$$\Pi(\tau) = \Pi_0 \left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\tau_\Pi}\frac{\zeta_0}{\Pi_0}} \exp\left(-\frac{\tau-\tau_0}{\tau_\Pi}\right)$$

Pressure:

$$p(\tau) = p_A \left(\frac{\tau_0}{\tau}\right)^{d\left(1+\frac{1}{\kappa}\right)} \left[1 + \frac{p_0 - p_A}{p_A} \cdot \frac{\Gamma\left(B, \frac{\tau}{\tau_\Pi}\right)}{\Gamma\left(B, \frac{\tau_0}{\tau_\Pi}\right)}\right]$$

Constants:

$$p_A = p_0 - \frac{\Pi_0 d}{\kappa} \left(\frac{\tau_0}{\tau_\Pi}\right)^{-B} \exp\left(\frac{\tau_0}{\tau_\Pi}\right) \Gamma\left(B, \frac{\tau_0}{\tau_\Pi}\right)$$

$$B = d\left(1 + \frac{1}{\kappa} - \frac{\zeta_0}{\Pi_0}\frac{1}{\tau_\Pi}\right)$$

M. Nagy, M. Csanád, Z. Jiang, T. Csörgő: arXiv:1909.02498

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II. Asymptotically perfect fluid solutions

Asymptotically perfect fluid solutions

In the $\tau \gg \tau_0$ limit, both the NS and IS cases lead to the same asymptotic perfect fluid temperature profile and pressure:

$$T \sim T_A \left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\kappa_0}} \mathcal{T}\left(s_x, s_y, s_z\right) \qquad p \sim p_A \left(\frac{\tau_0}{\tau}\right)^{d\left(1+\frac{1}{\kappa_0}\right)}$$

If μ=0 the entropy density asymptotically equals to a perfect fluid form (and if μ≠0 the particle density is unchanged):

$$\sigma \sim \sigma_A \left(\frac{\tau_0}{\tau}\right)^{d} \mathcal{V}\left(s_x, s_y, s_z\right) \qquad \frac{T_A}{T_0} = \left(\frac{\sigma_A}{\sigma_0}\right)^{\frac{1}{\kappa_0}} = \exp\left(\frac{\zeta_0 d^2}{\kappa_0 p_0 \tau_0} \frac{1}{1+\kappa_0}\right)$$

The bulk viscosity is absorbed to the asymptotic normalization constants!

The effect of bulk viscosity is scaled out!

T. Csörgő, L. P. Csernai, Y. Hama, T. Kodama: arXiv:nucl-th/0306004

Asymptotically perfect fluid solutions

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The bulk viscosity is absorbed to the **asymptotic normalization constants**!

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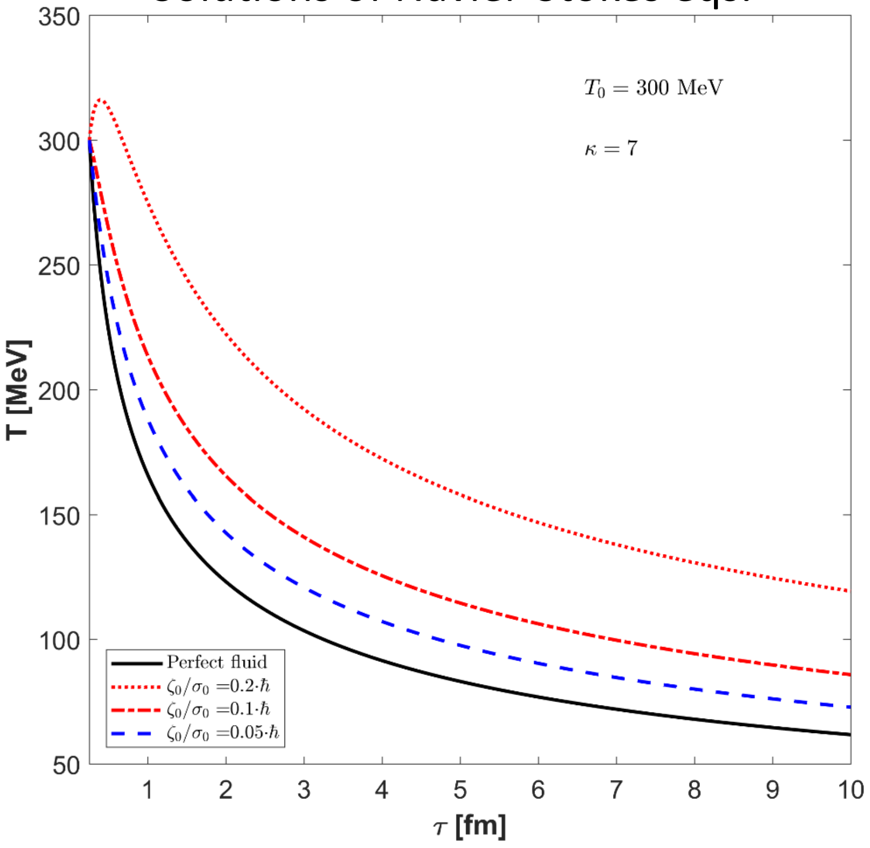
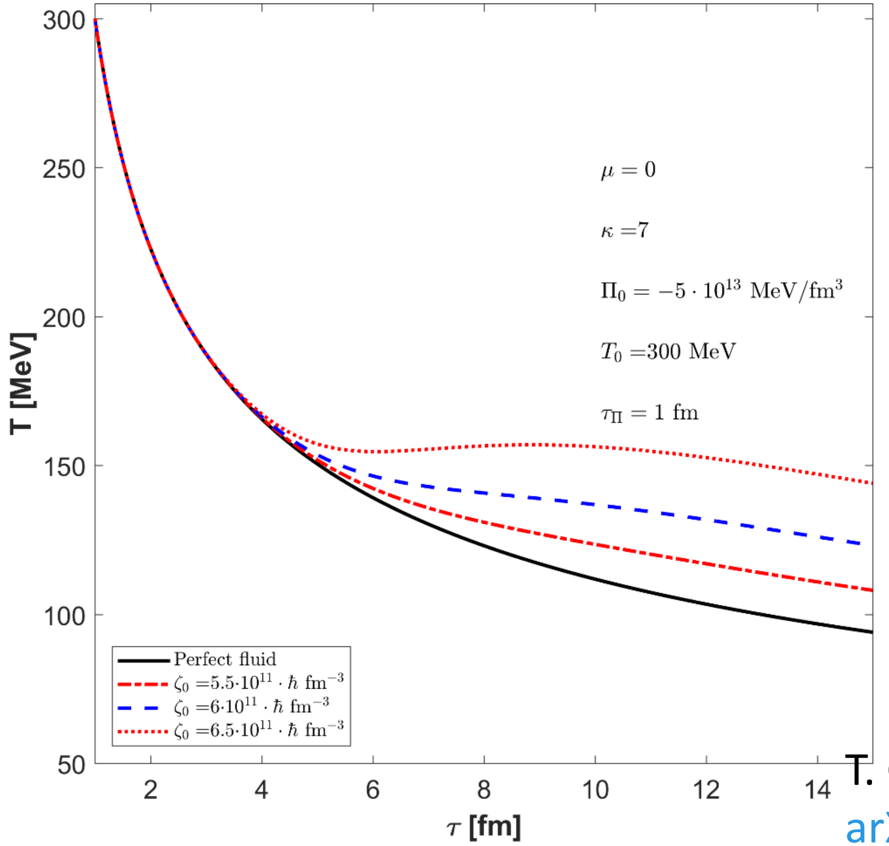
Evolution of the temperature: same initial conditions

Solutions of Navier-Stokes eqs.



Solutions of Isreal-Stewart eqs.



T. Csörgő, G. K.: arXiv:2003.08859

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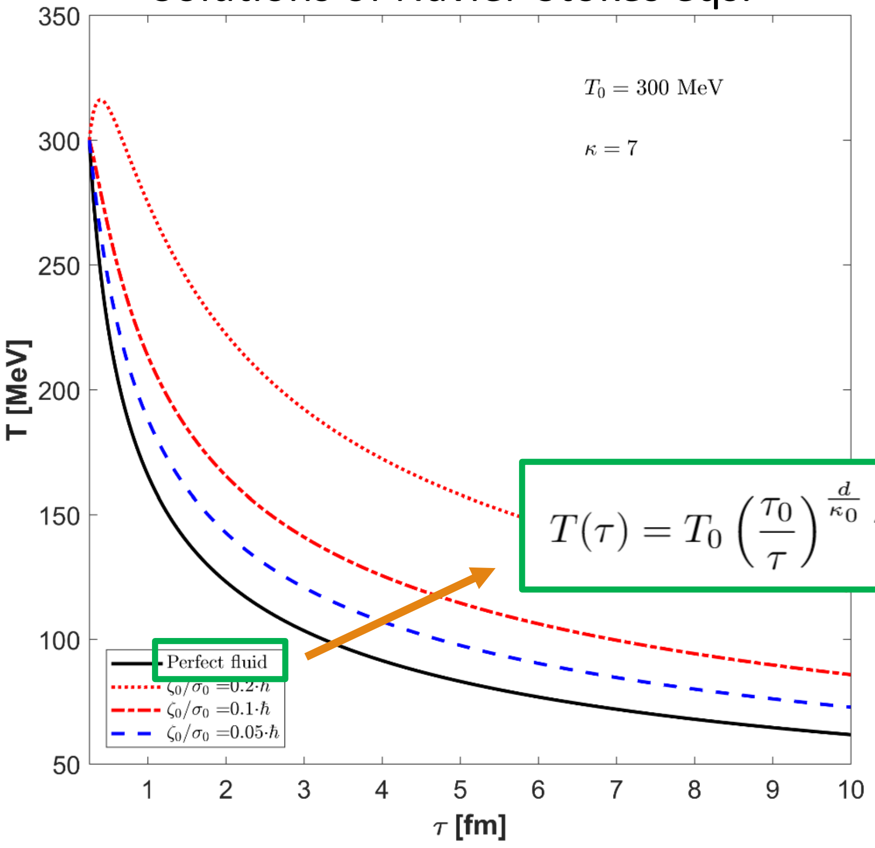

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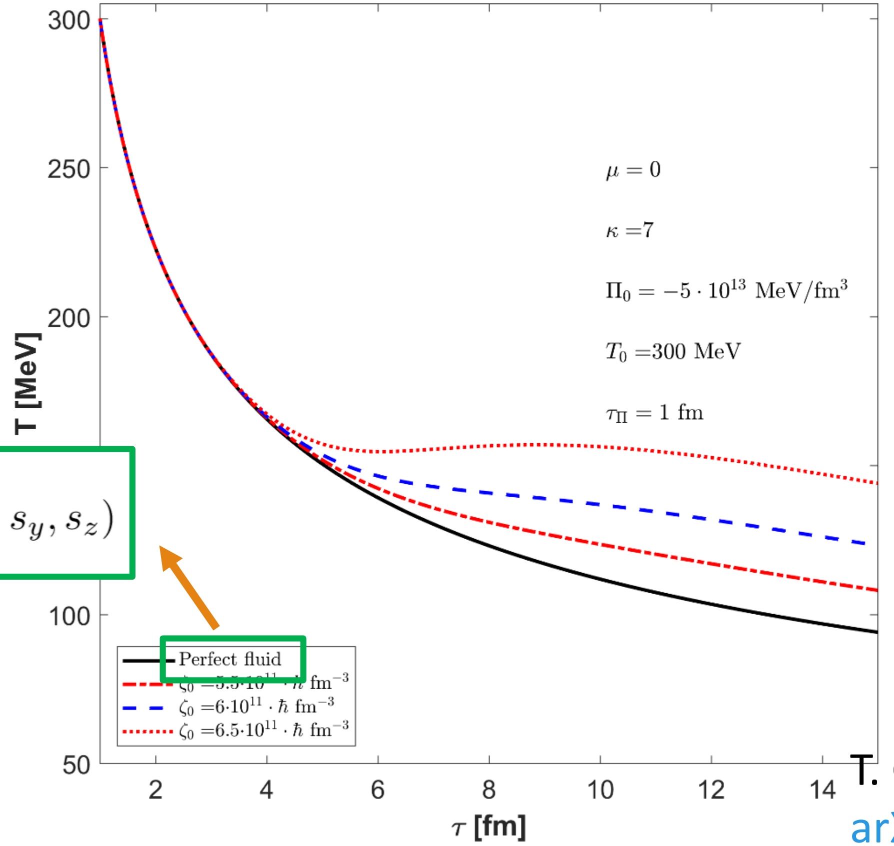

$$T(\tau) = T_0 \left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\kappa_0}} \mathcal{T}\left(s_x, s_y, s_z\right)$$

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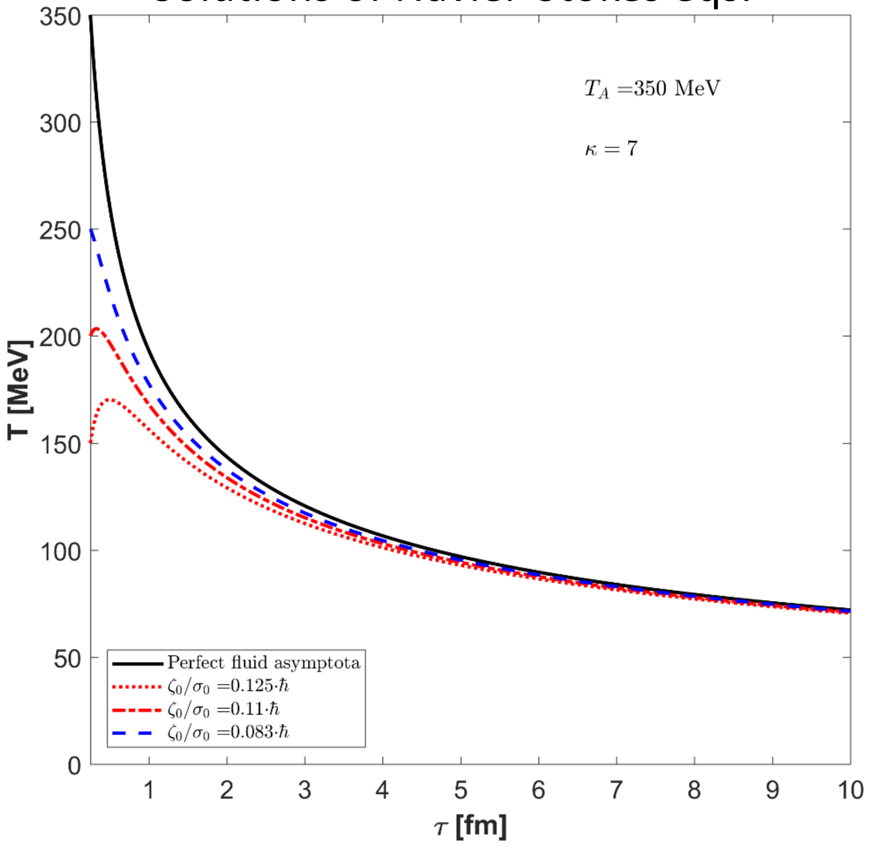
Evolution of the temperature: same attractor

Solutions of Navier-Stokes eqs.



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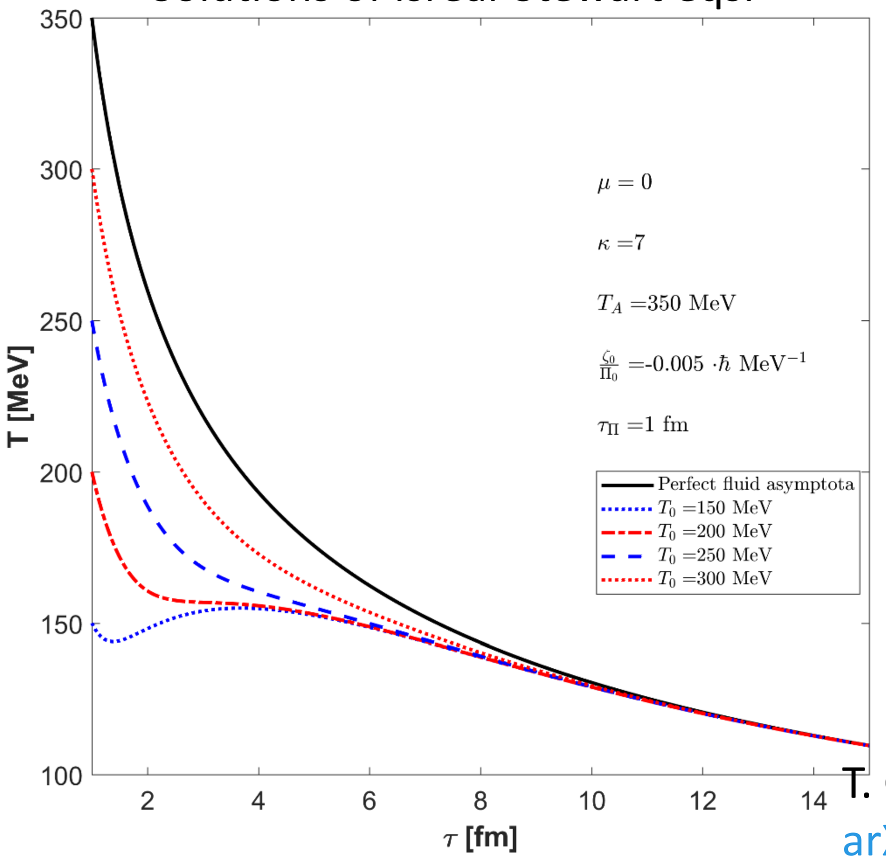

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Evolution of the temperature: same attractor

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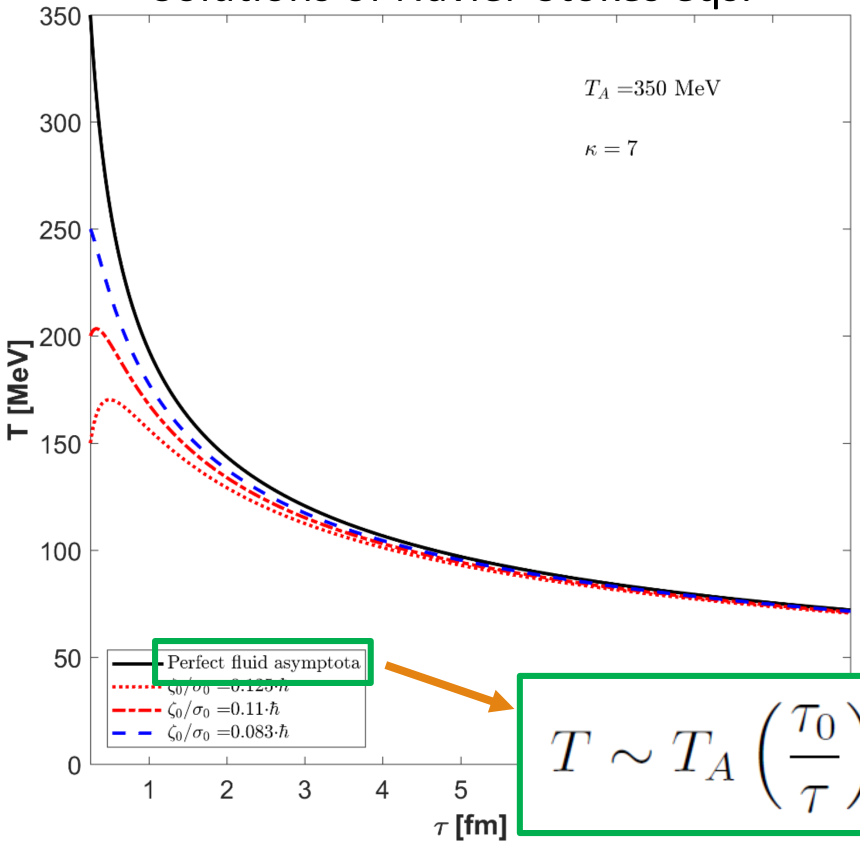

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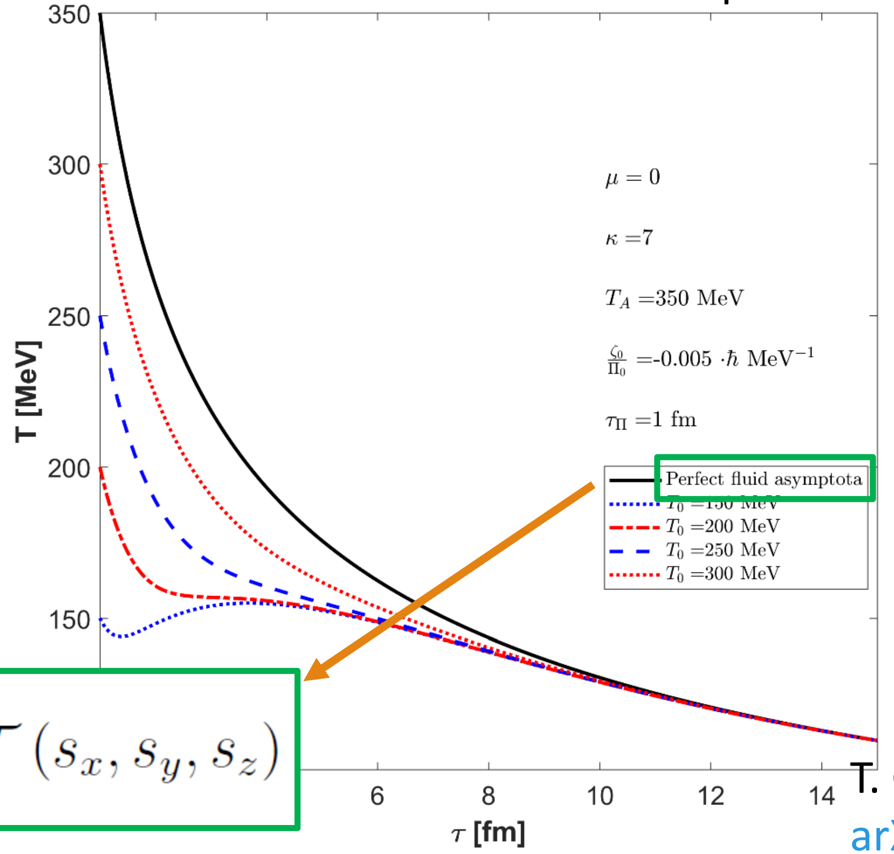

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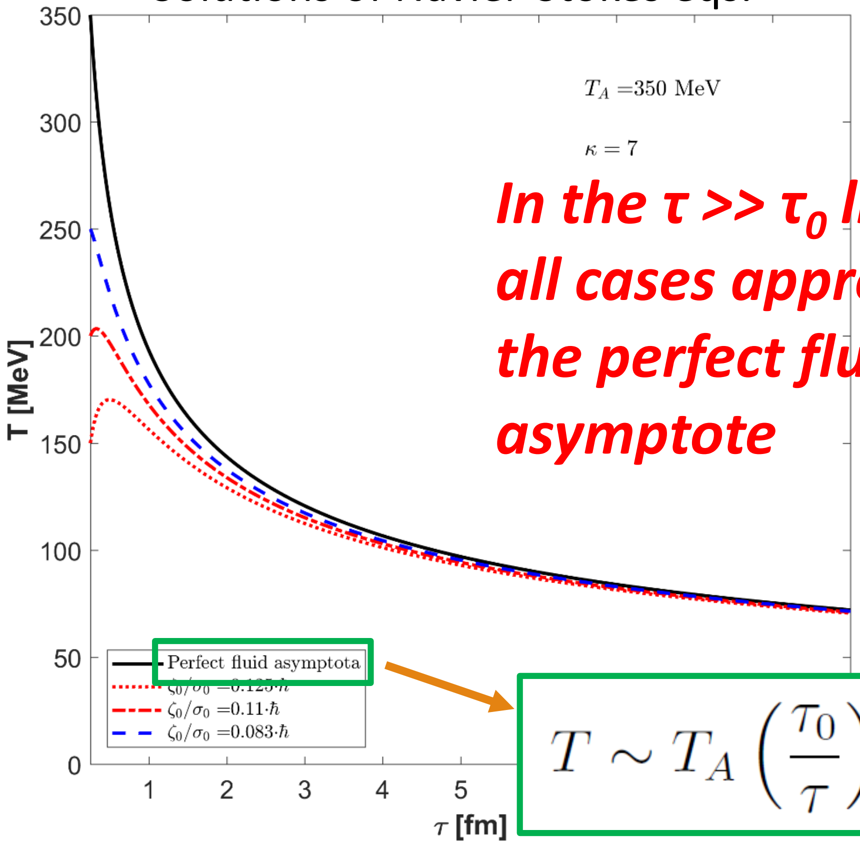
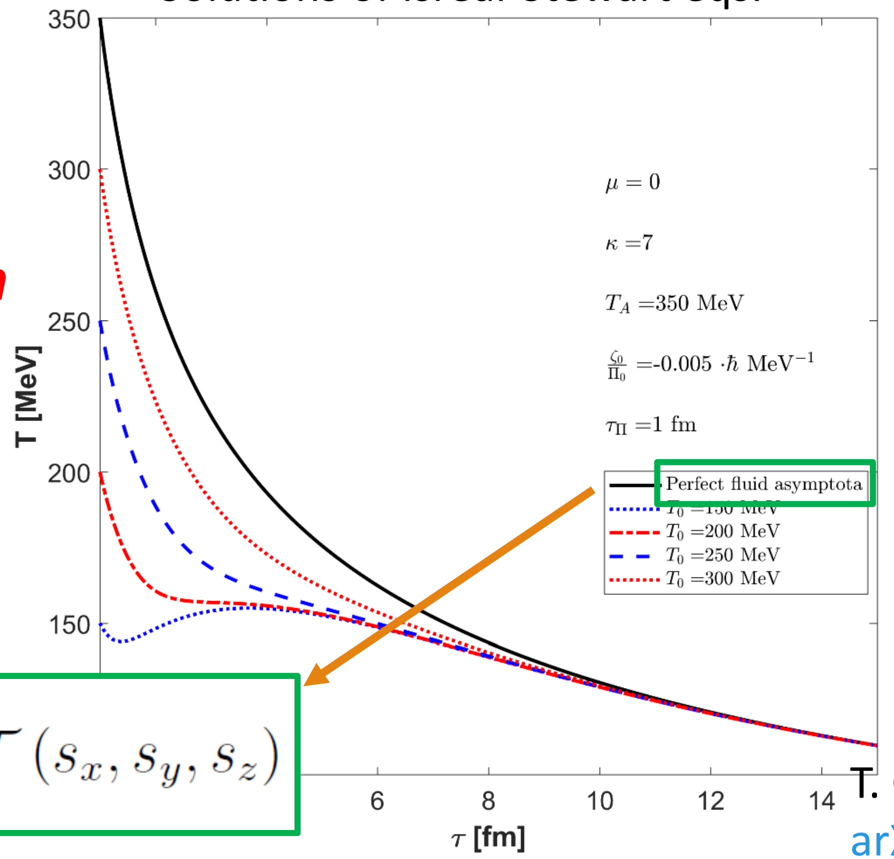
T. Csörgő, G. K.: arXiv:2003.08859

Evolution of the temperature: same attractor

Solutions of Navier-Stokes eqs.



Solutions of Isreal-Stewart eqs.



In the $\tau >> \tau_0$ limit, all cases approach the perfect fluid asymptote

$$T \sim T_A \left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\kappa_0}} \mathcal{T}\left(s_x, s_y, s_z\right)$$

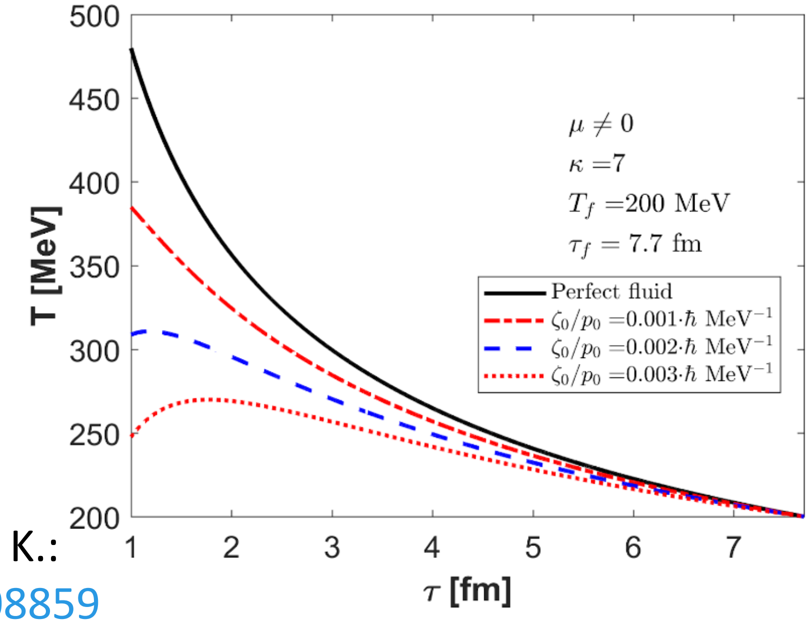
T. Csörgő, G. K.: arXiv:2003.08859

III. Applications

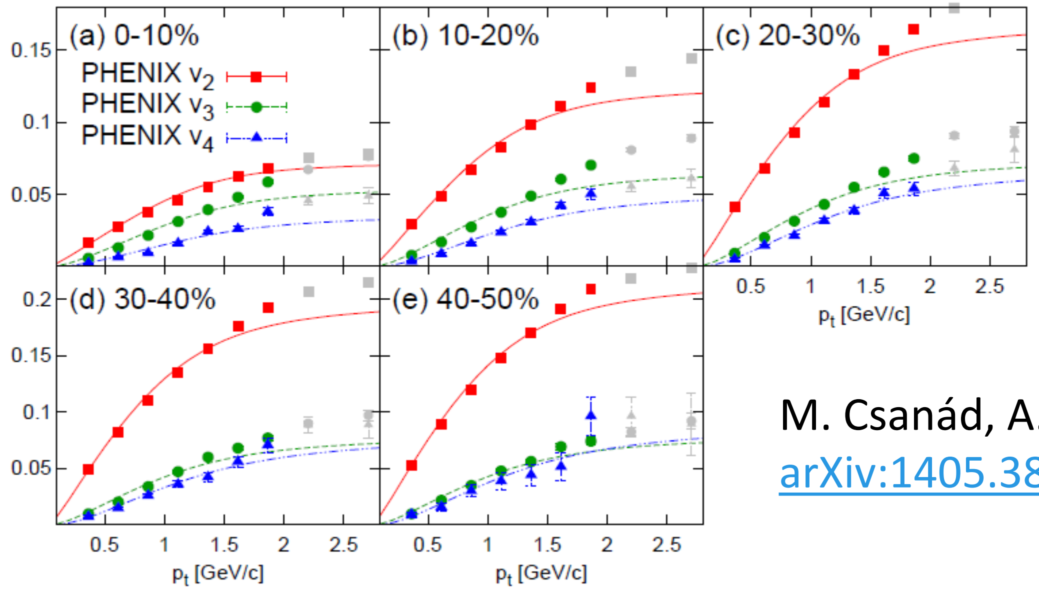
1st application of the solutions of NS eqs.

In arXiv:1405.3877: v_2, v_3 and v_4 were reproduced for $s_{NN}^{1/2}$ = 200 GeV Au+Au collisions with τ_f=7.7 fm/c and T_f=200 MeV final state parameters

We co-varied the initial conditions so that exactly the same freeze-out parameters are obtained



T. Csörgő, G. K.: arXiv:2003.08859



M. Csanád, A. Szabó: arXiv:1405.3877

Further applications of the solutions of NS eqs.

Producing new, dissipative solutions of non relativistic hydro

1st step: Non relativistic limit of the relativistic solution → ***Spherically symmetric solution of non relativistic, dissipative hydro*** (manuscript is in preparation)

2nd step: Ellipsoidal generalization

3rd step: Add rotation to the velocity field → $v = v_{Hubble} + v_{rot}$

Result: ***Ellipsoidally symmetric, rotating, dissipative fireball solution of non relativistic hydro***

T. Csörgő, G. K.:
arXiv:2003.08859

IV. New, non relativistic, dissipative solutions

- with perfect fluid attractors -

Spherically symmetric, dissipative fireball solution

Velocity field and self similarity:

$$\vec{v} = \frac{\dot{R}}{R}(r_x, r_y, r_z) \longrightarrow (\partial_t + \vec{v}\nabla)s = 0, \quad s = \frac{r^2}{R^2}$$

Particle density, temperature and ideal gas approach:

$$n(\vec{r},t) = n_0 \left(\frac{R_0}{R}\right)^d \mathcal{V}(s)$$

$$T(\vec{r},t) = T_0 f_T(t) \mathcal{T}(s)$$

$$p(\vec{r},t) = p_0 f_T(t) \left(\frac{R_0}{R}\right)^d \mathcal{V}(s)\mathcal{T}(s)$$

$$\mathcal{V}(s) = \frac{1}{\mathcal{T}(s)} \exp\left(-\frac{C_E}{2}\int_0^s \frac{du}{\mathcal{T}(u)}\right)$$

Energy and momentum conservation:

$$\kappa \partial_t \ln(f_T) + d\frac{\dot{R}}{R} = \frac{\zeta d^2}{p}\left(\frac{\dot{R}}{R}\right)^2 \qquad R\ddot{R} = C_E \frac{T_0}{m} f_T(t)$$

Two possible solution of the energy conservation:

1. with homogeneous pressure: $\nu(s)\tau(s)=1$, $C_E=0$, $\zeta=\zeta(p)$
2. with inhomogeneous pressure: $\zeta \sim p$

Spherically symmetric, dissipative fireball solution
- with homogeneous pressure -

If the pressure is homogeneous, then *v(s)τ(s)=1*, C_E*=0* so the Euler equation and ζ are:

$$\ddot{R} = 0 \longrightarrow \dot{R} = \text{const.} \longrightarrow R = \dot{R}t + R_0 \sim \dot{R}t$$

$$\zeta \equiv \zeta(p(t))$$

With that, the energy conservation becomes:

$$\kappa \partial_t \ln(f_T) + \frac{d}{t} = \frac{\zeta(p(t))}{p(t)} \frac{d^2}{t^2}$$

If the bulk viscosity is linear in pressure*:* $\zeta(p(t)) = \zeta_0 \frac{p(t)}{p_0}$

$$p(t) = p_0 \left(\frac{t_0}{t}\right)^{d\left(1+\frac{1}{\kappa}\right)} \exp\left(\frac{d^2\zeta_0}{\kappa p_0 t_0}\left[1 - \frac{t_0}{t}\right]\right)$$

$$T(t,s) = T_0 \left(\frac{t_0}{t}\right)^{\frac{d}{\kappa}} \exp\left(\frac{d^2\zeta_0}{\kappa p_0 t_0}\left[1 - \frac{t_0}{t}\right]\right)\mathcal{T}(s)$$

$\tau_0 << \tau$ →

Late time approximation:

perfect fluid asymptote

$$T(t) \sim T_A \left(\frac{t_0}{t}\right)^{\frac{d}{\kappa}} \mathcal{T}(s)$$

$$p(t) \sim p_A \left(\frac{t_0}{t}\right)^{d\left(1+\frac{1}{\kappa}\right)}$$

$$T_A = T_0 \exp\left(\frac{d^2\zeta_0}{\kappa p_0 t_0}\right)$$

$$p_A = p_0 \exp\left(\frac{d^2\zeta_0}{\kappa p_0 t_0}\right)$$

Spherically symmetric, dissipative fireball solution - with inhomogeneous pressure -

If the pressure is inhomogeneous, then ζ has to be linear in pressure: $\zeta(t,s) = \zeta_0 \frac{p(t,s)}{p_0}$

Assumption:

$$f_T(t) = g_T(t) \left(\frac{R_0}{R} \right)^{\frac{d}{\kappa}}$$

With that, the energy conservation becomes:

$$\frac{\dot{g_T}}{g_T} = \frac{\zeta_0 d^2}{\kappa p_0} \left(\frac{\dot{R}}{R} \right)^2$$

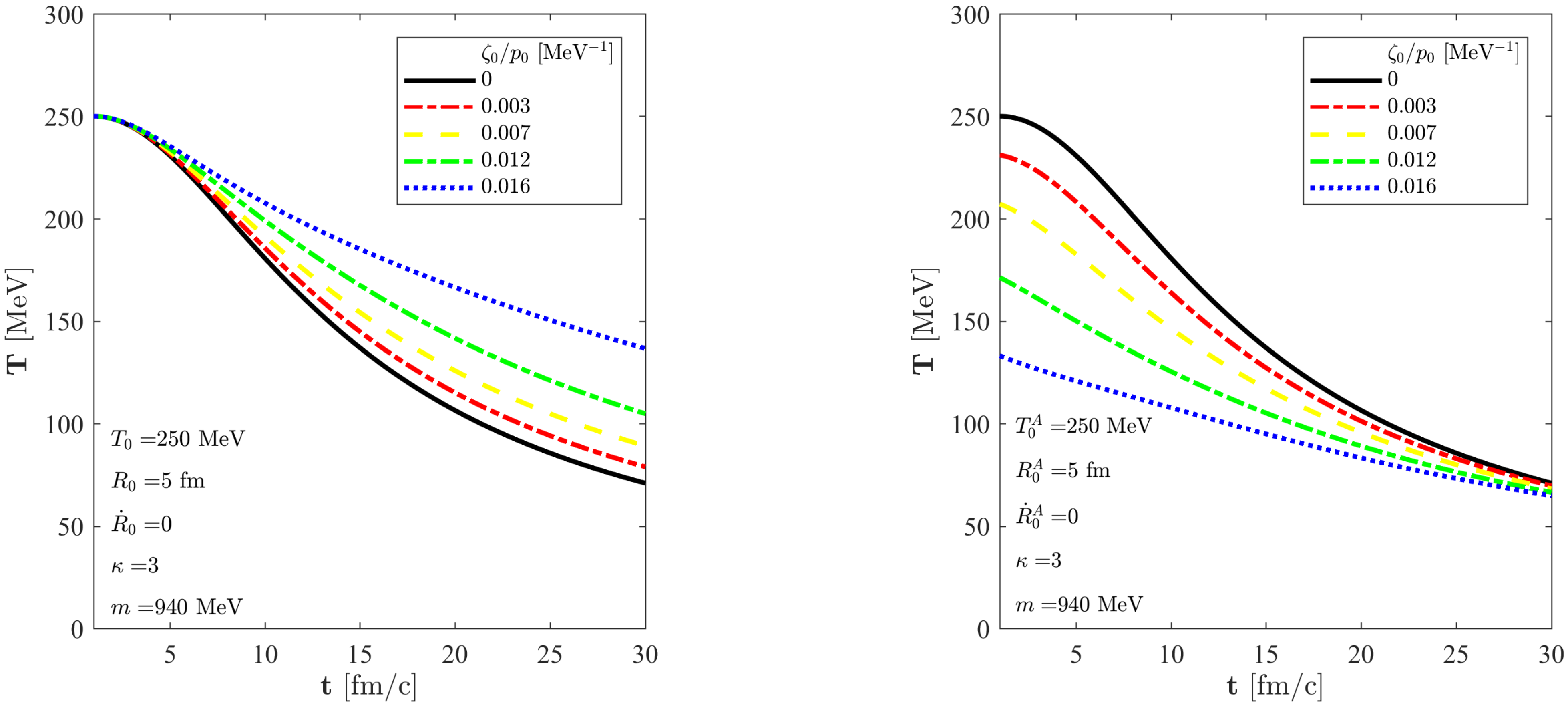
The Euler equation is:

$$R\ddot{R} = C_E \frac{T_0}{m} \left(\frac{R_0}{R} \right)^{\frac{d}{\kappa}} g_T(t)$$

This set of differential equations is solved numerically (next slides)

The asymptotic attractor is a perfect fluid solution again!

Spherically symmetric, dissipative fireball solution
- with inhomogeneous pressure -



Spherically symmetric, dissipative fireball solution
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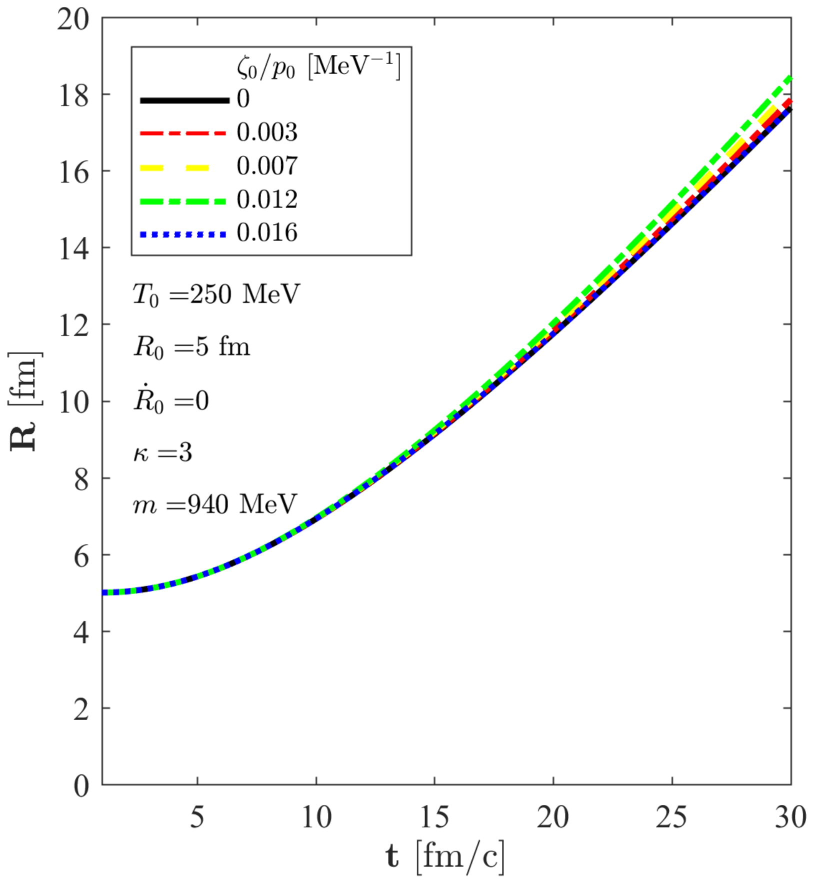
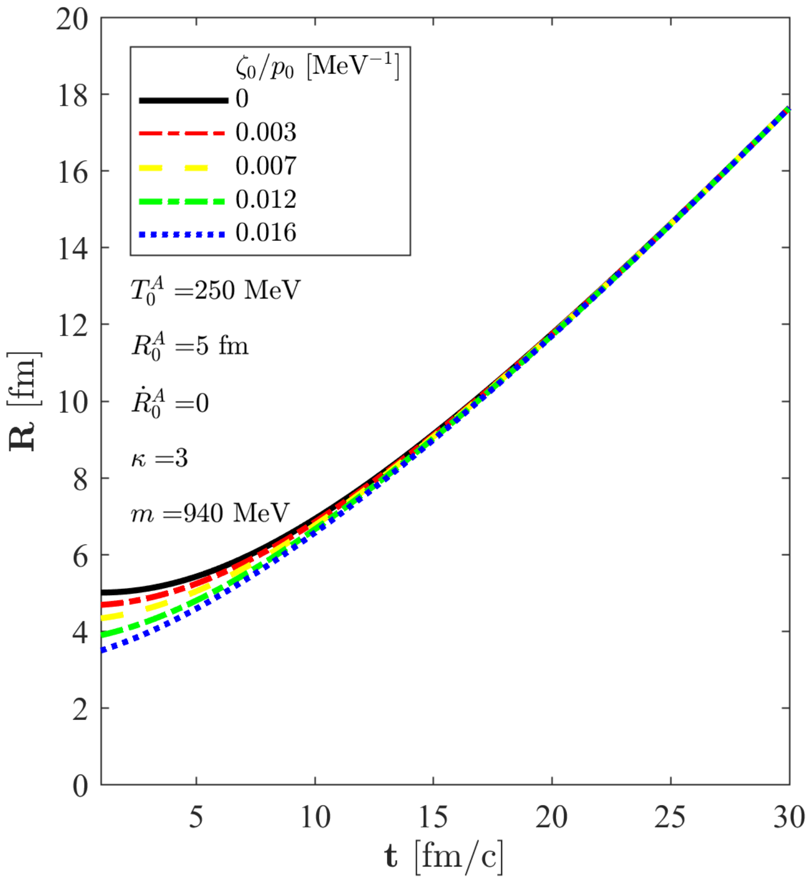
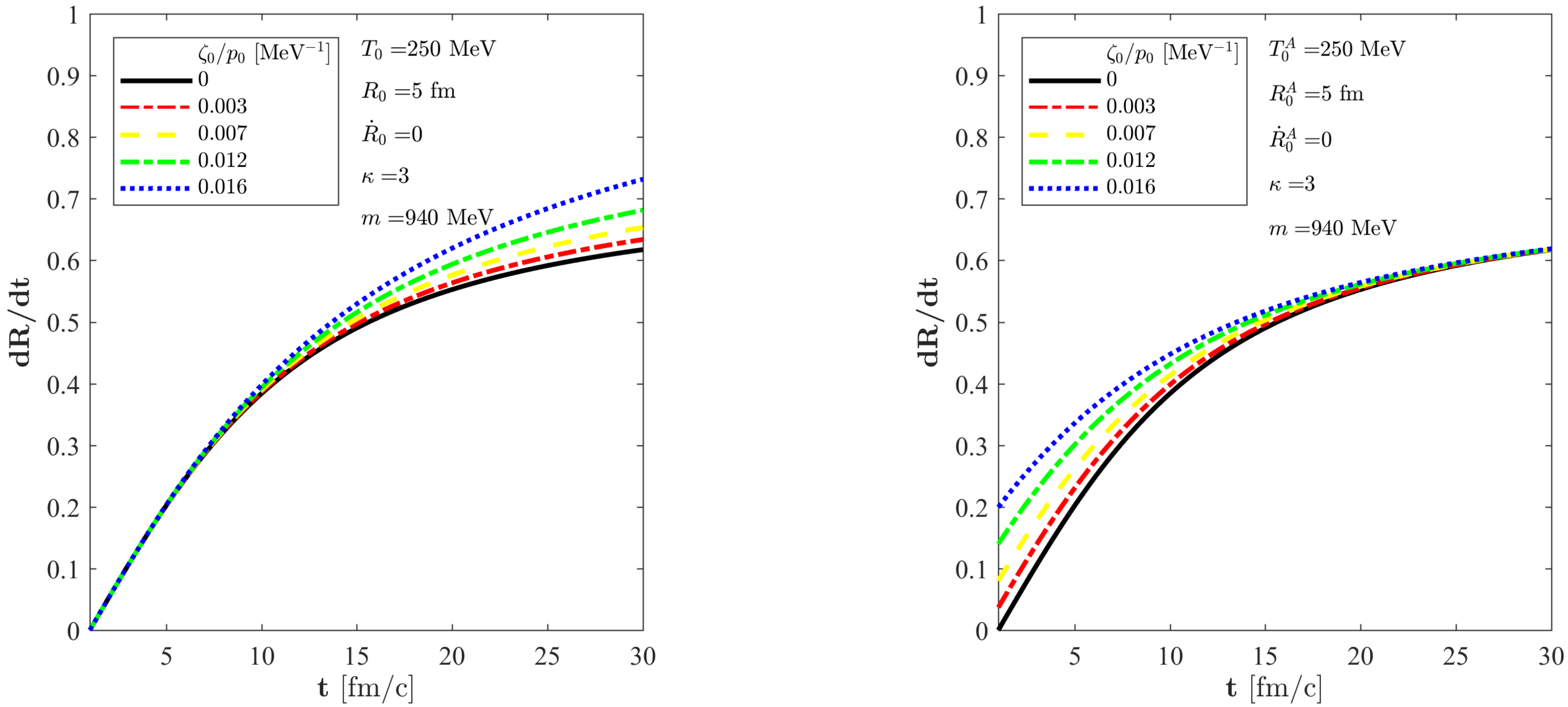




Spherically symmetric, dissipative fireball solution - with inhomogeneous pressure -



Ellipsoidally symmetric, rotating, dissipative fireball solution

Velocity field: $\vec{v} = \vec{v}_H + \vec{v}_{rot}$ Scales of the fireball: *X(t), Y(t), Z(t)*

$$v_H(\vec{r},t) = \begin{pmatrix} \left(\frac{\dot{X}}{X}\cos^2\vartheta + \frac{\dot{Z}}{Z}\sin^2\vartheta\right) r_x \\ \frac{\dot{Y}}{Y} r_y \\ \left(\frac{\dot{X}}{X}\sin^2\vartheta + \frac{\dot{Z}}{Z}\cos^2\vartheta\right) r_z \end{pmatrix} + \left(\frac{\dot{Z}}{Z} - \frac{\dot{X}}{X}\right)\frac{\sin(2\vartheta)}{2}\begin{pmatrix} r_z \\ 0 \\ r_x \end{pmatrix}$$

→ Hubble-flow

$$v_{rot}(\vec{r},t) = \dot{\vartheta}\begin{pmatrix} r_z \\ 0 \\ -r_x \end{pmatrix} + \dot{\vartheta}\begin{pmatrix} \left(\frac{X}{Z}\cos^2\vartheta + \frac{Z}{X}\sin^2\vartheta\right) r_z \\ 0 \\ -\left(\frac{X}{Z}\sin^2\vartheta + \frac{Z}{X}\cos^2\vartheta\right) r_x \end{pmatrix} + \dot{\vartheta}\left(\frac{X}{Z} - \frac{Z}{X}\right)\frac{\sin(2\vartheta)}{2}\begin{pmatrix} r_x \\ 0 \\ -r_z \end{pmatrix}$$

→ Rotational term of velocity

$$\dot{\vartheta} = \frac{\omega(t)}{2} = \frac{\omega_0}{2}\frac{R_0^2}{R(t)^2}$$

→ Angular velocity

$$R = \frac{X+Z}{2}$$

M. I. Nagy, T. Csörgő: arXiv:1309.4390
M. I. Nagy, T. Csörgő: arXiv:1606.09160
T. Csörgő, M. I. Nagy, I. F. Barna: arXiv:1511.02593

Ellipsoidally symmetric, rotating, dissipative fireball solution

Velocity field: $\vec{v} = \vec{v}_H + \vec{v}_{rot}$

Particle density and scale variable:

$$n(\vec{r},t) = n_0 \frac{X_0 Y_0 Z_0}{XYZ} \mathcal{V}(s) = n_0 \frac{V_0}{V} \mathcal{V}(s) \qquad s = \frac{r_x^2}{X^2} + \frac{r_y^2}{Y^2} + \frac{r_z^2}{Z^2} + \left(\frac{1}{Z^2} - \frac{1}{X^2}\right)\left[\left(r_x^2 - r_z^2\right)\sin^2\vartheta + r_x r_z \sin 2\vartheta\right]$$

Temperature and energy conservation:

$$T = T_0 \left(\frac{X_0 Y_0 Z_0}{XYZ}\right)^{\frac{1}{\kappa}} g_T(t) \mathcal{T}(s), \qquad \mathcal{V}(s) = \frac{1}{\mathcal{T}(s)} \exp\left(-\frac{C_E}{2} \int_0^s \frac{du}{\mathcal{T}(u)}\right)$$

$$\kappa \partial_t [\ln(g_T)] = \frac{\zeta_0}{p_0}\left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z}\right)^2 + \frac{2\eta_0}{p_0}\left[\frac{\dot{X}^2}{X^2} + \frac{\dot{Y}^2}{Y^2} + \frac{\dot{Z}^2}{Z^2} - \frac{1}{3}\left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z}\right)^2\right] + \frac{\eta_0 {\omega_0}^2}{4p_0} \frac{(X_0 + Z_0)^4}{(X+Z)^4} \left(\frac{X}{Z} - \frac{Z}{X}\right)^2$$

Euler equation and angular velocity:

$$X\left(\ddot{X} - R\omega^2\right) = Y\ddot{Y} = Z\left(\ddot{Z} - R\omega^2\right) = C_E \frac{T}{m}$$

$$R = \frac{X+Z}{2}, \qquad \omega = \omega_0 \frac{R_0^2}{R^2}$$

Further applications:
Ellipsoidally symmetric, rotating, dissipative fireball solution

Velocity field: $\vec{v} = \vec{v}_H + \vec{v}_{rot}$

Particle density and scale variable:

$$n(\vec{r},t) = n_0 \frac{X_0 Y_0 Z_0}{XYZ} \mathcal{V}(s) = n_0 \frac{V_0}{V} \mathcal{V}(s) \qquad s = \frac{r_x^2}{X^2} + \frac{r_y^2}{Y^2} + \frac{r_z^2}{Z^2} + \left(\frac{1}{Z^2} - \frac{1}{X^2}\right)\left[\left(r_x^2 - r_z^2\right)\sin^2\vartheta + r_x r_z \sin 2\vartheta\right]$$

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effect of bulk

effect of shear without rotation

effect of shear with rotation

Euler equation and angular velocity:

$$X\left(\ddot{X} - R\omega^2\right) = Y\ddot{Y} = Z\left(\ddot{Z} - R\omega^2\right) = C_E \frac{T}{m}$$

$$R = \frac{X+Z}{2}, \qquad \omega = \omega_0 \frac{R_0^2}{R^2}$$

Summary

New, analytic, exact solutions of relativistic Navier-Stokes and Israel-Stewart equations with spherically symmetric Hubble-flow

The effect of shear viscosity cancel because of the velocity field

The solutions are causal and asymptotically perfect (the effect of bulk viscosity cancels for late times), both for a finite and vanishing μ

These exact solutions tend to the Csörgő-Csernai-Hama-Kodama perfect fluid solution

Cannot decide from final state measurements that the medium evolved as a perfect fluid with higher initial temperature (T_A) or as a viscous fluid with lower initial temperature (T_0)

We were able to reproduce the experimental data in $s_{NN}^{1/2}$ = 200 GeV Au+Au collisions on v_2, v_3 and v_4

Non relativistic limit: new solutions of non relativistic Navier-Stokes theory

Thank you for your attention!

Further application: Asymptotic behaviour (for C_E=0)

If C_E=0, analytic solutions of X, Y and Z scales can be given, and their asymptotic limit are simple:

$$X \propto X_a + \dot{X}_a t, \quad Y \propto Y_a + \dot{Y}_a t, \quad Z \propto Z_a + \dot{Z}_a t, \quad \omega \propto \omega_0 \left[\frac{X_0 + Z_0}{X_a + Z_a + \left(\dot{X}_a + \dot{Z}_a\right) t} \right]^2$$

where

$$\dot{X}_a = \frac{1}{2}\left[\dot{X}_0 - \dot{Z}_0 + \sqrt{(\dot{X}_0 + \dot{Z}_0)^2 + (X_0 + Z_0)^2 \omega_0^2}\right]$$
$$\dot{Y}_a = \dot{Y}_0$$
$$\dot{Z}_a = \frac{1}{2}\left[\dot{Z}_0 - \dot{X}_0 + \sqrt{(\dot{X}_0 + \dot{Z}_0)^2 + (X_0 + Z_0)^2 \omega_0^2}\right]$$

At very late times the constant offsets becomes negligible: $X \propto \dot{X}_a t$, $Y \propto \dot{Y}_a t$, $Z \propto \dot{Z}_a t$

In this asymptotic limit, the rotating and dissipative, ellipsoidal fireball tends to a known, perfect fluid relativistic solution: Csörgő, Csernai, Hama, Kodama: arXiv:nucl-th/0306004

A spherical and irrotational Hubble flow is an asymptotic attractor for rotating and ellipsoidal, finite fireballs

The effects of rotation, shear and bulk viscosity are scaled out from asymptotia