Next-to-leading power two-loop soft functions for the Drell-Yan process at threshold

Sebastian Jaskiewicz

Parton Showers and Resummation
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Motivations and focus

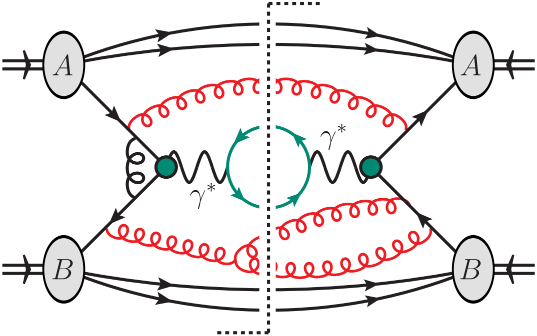
$$A(p_A) + B(p_B) \to \gamma^*(Q^2)[\to \ell(l_1)\bar{\ell}(l_2)] + X(p_X)$$

Threshold limit:

$$z = \frac{Q^2}{\hat{s}} \to 1$$

Define power counting parameter λ:

$$\lambda = \sqrt{1-z}$$



Schematic form for production cross-sections near threshold, $z \to 1$:

$$\hat{\sigma}(z) = \sum_{n=0}^{\infty} \alpha_s^n \left[c_n \delta(1-z) + \sum_{m=0}^{2n-1} \left(c_{nm} \left[\frac{\ln^m(1-z)}{1-z}\right]_+ + \quad d_{nm} \ln^m(1-z) \quad \right) + \ldots \right]$$

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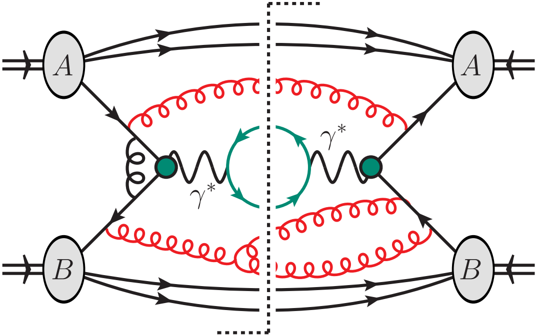

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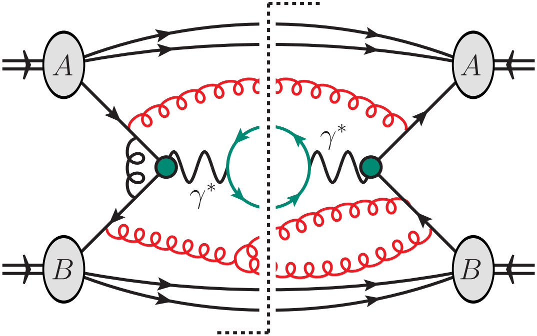

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Recent *next-to-leading power* progress

Subleading power resummed thrust spectrum for $H \to gg$ (LL)

[I. Moult, I. Stewart, G. Vita, H. Zhu, 1804.04665, 1910.14038]

Drell-Yan and Higgs production at threshold (LL)

[M. Beneke, A.Broggio, M. Garny, **SJ**, R. Szafron, L. Vernazza, J.Wang, 1809.10631]

[M. Beneke, A.Broggio, **SJ**, L. Vernazza, 1912.01585] [N. Bahjat-Abbas, D. Bonocore, J. Sinninghe Damsté, E. Laenen, L. Magnea, L. Vernazza, C. White, 1905.13710]

[M. Beneke, M. Garny, **SJ**, R. Szafron, L. Vernazza, J.Wang, 1910.12685](with numerics)

Resummation of rapidity logarithms: the EE correlator in N=4 SYM (LL)

[I. Moult, G. Vita, K. Yan, 1912.02188]

Mass effects and Endpoint Divergences (LL, NLL)

[Z. L. Liu, M.Neubert, 1912.08818] [C. Anastasiou, A. Penin, 2004.03602]

[Z. L. Liu, B. Mecaj, M.Neubert, X.Wang, 2009.04456, 2009.06779]

Refactorization and d-dimensional resummation in DIS(LL)

[M. Beneke, M. Garny, **SJ**, R. Szafron, L. Vernazza, J.Wang, 2008.04943]→see Leonardo's talk

Violation of KSZ theorem in SCET

[M. Beneke, M. Garny, R. Szafron, J.Wang, 1907.05463]

Power-enhanced QED corrections to $B_q \to \mu^+\mu^-$ (LL)

[M. Beneke, C. Bobeth , R. Szafron, 1708.09152, 1908.07011]

Drell-Yan q_T Resummation of Fiducial Power Corrections

[M. Ebert, J. Michel, I. Stewart, F. Tackmann, 2006.11382]

NLP Threshold corrections for colour-singlet cross sections

[M. van Beekveld, E. Laenen, J. Sinninghe Damsté, L. Vernazza, 2101.07270] →see Melissa's talk

Drell Yan: Factorization of the partonic cross-section

First let us compare leading power and *next*-to-leading power cross-sections schematically:

$$\frac{d\sigma_{\rm DY}}{dQ^2} \sim \sum_{a,b} \int_0^1 dx_a dx_b \, f_{a/A}(x_a) f_{b/B}(x_b) \left(\hat{\sigma}_{ab}^{\rm LP}(z) \; + \; \hat{\sigma}_{ab}^{\rm NLP}(z) \; + \ldots \right) + \mathcal{O}\left(\Lambda/Q\right)$$

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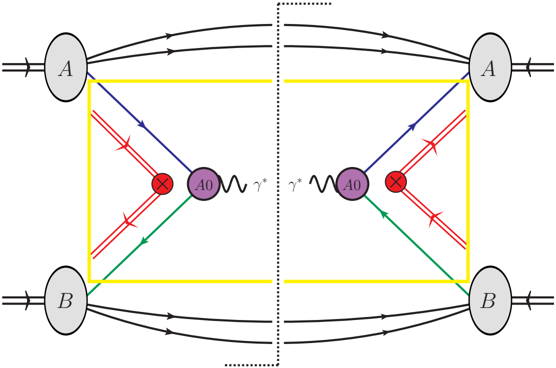

$$\hat{\sigma}^{\rm LP}(z) = Q\, H(Q^2)\, S_{\rm DY}(\Omega)$$

[G. P. Korchemsky G. Marchesini, 1993] [S. Moch, A. Vogt, hep-ph/0508265]

[T. Becher, M. Neubert, G. Xu, 0710.0680]

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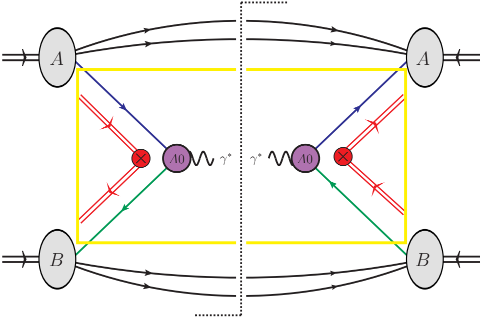

$$S_{\mathrm{DY}}(\Omega) = \int \frac{dx^0}{4\pi} e^{i\Omega\, x^0/2} \frac{1}{N_c} \mathrm{Tr} \langle 0 | \bar{\mathbf{T}} \left[Y_+^\dagger(x) Y_-(x) \right] \mathbf{T} \left[Y_-^\dagger(0) Y_+(0) \right] | 0 \rangle$$

$$Y_\pm(x) \quad = \quad \mathbf{P} \exp \left[i g_s \int_{-\infty}^0 ds\, n_\mp A_s (x + s n_\mp) \right].$$

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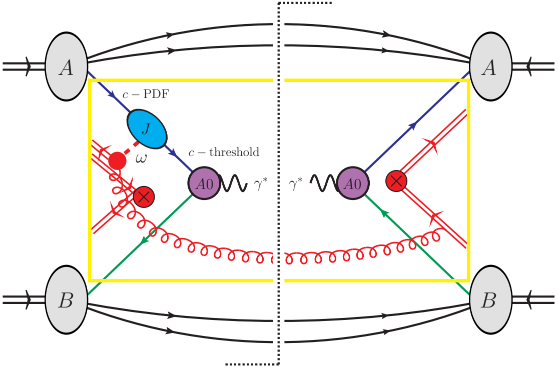

$$\hat{\sigma}^{\rm NLP}(z) = \sum_{\rm terms} [C \otimes J \otimes \bar{J}]^2 \otimes S$$

[M. Beneke, A. Broggio, M. Garny, **SJ**, R. Szafron, L. Vernazza, J. Wang, 1809.10631]

[M. Beneke, A. Broggio, **SJ**, L. Vernazza, 1912.01585]

Brief introduction to NLP SCET

In this talk we employ position-space SCET formalism

[M. Beneke, A. Chapovsky, M. Diehl, Th. Feldmann, hep-ph/0206152]

$$\mathcal{L}_{\text{SCET}} = \sum_{i=1}^{N} \mathcal{L}_{c_i} + \mathcal{L}_{\text{soft}}$$

where each of the Lagrangians belonging to a collinear direction is expanded in powers of the small parameter $\lambda = \sqrt{1-z}$:

$$\mathcal{L}_{c_i} = \underbrace{\mathcal{L}_{c_i}^{(0)}}_{\text{LP}} + \underbrace{\mathcal{L}_{c_i}^{(1)}}_{\mathcal{O}(\lambda^1)} + \underbrace{\mathcal{L}_{c_i}^{(2)}}_{\mathcal{O}(\lambda^2)} + \ldots \qquad \text{E.g.} \qquad \mathcal{L}_{c}^{(1)} = \bar{\chi}_c \, i x_\perp^\mu \left[i n_- \partial \mathcal{B}_\mu^+ \right] \frac{\not{n}_+}{2} \chi_c$$

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Generic N-jet operator has the form:

[M. Beneke, M. Garny, R. Szafron, J. Wang, 1712.04416, 1712.07462, 1808.04742, 1907.05463]

$$J = \int \prod_{i=1}^{N} \prod_{k_i=1}^{n_i} dt_{ik_i} \; C(\{t_{ik_i}\}) \prod_{i=1}^{N} \; J_i(t_{i_1}, t_{i_2}, ..., t_{i_{n_i}})$$

where the Js are constructed by multiplying collinear gauge invariant building blocks in the same direction (up to $\mathcal{O}(\lambda^2)$)

$$\chi_i(t_i n_{i+}) \equiv W_i^\dagger \, \xi_i \qquad\qquad \mathcal{A}_{i\perp}^\mu(t_i n_{i+}) \equiv W_i^\dagger [iD_{\perp i}^\mu \, W_i]$$

by acting on these with derivatives $i\partial_{\perp i}^\mu \sim \lambda$, and insertions of subleading SCET Lagrangian in a time-ordered product with lower power current.

Brief introduction to NLP SCET

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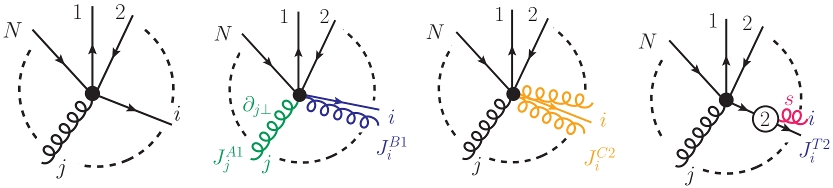
Generic leading power N-jet operator:



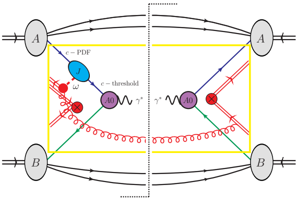
NLP factorization formula for Drell-Yan

The partonic cross-section is

$$\Delta(z) = \frac{1}{(1-\epsilon)} \frac{\hat{\sigma}(z)}{z} \qquad \Delta_{\rm NLP}(z) = \Delta^{dyn}_{\rm NLP}(z) + \Delta^{kin}_{\rm NLP}(z)$$

where [M. Beneke, A.Broggio, **SJ**, L. Vernazza, 1912.01585]

$$\Delta^{dyn}_{\rm NLP}(z) = -\frac{2}{(1-\epsilon)} Q \left[\left(\frac{\not{n}_-}{4} \right) \gamma_{\perp\rho} \left(\frac{\not{n}_+}{4} \right) \gamma^{\rho}_{\perp} \right]_{\beta\gamma}$$
$$\times \int d(n_+p)\, C^{\,A0,A0}\,(n_+p, x_b n_- p_B)\, C^{*\,A0A0}\,(\, x_a n_+ p_A,\, x_b n_- p_B)$$
$$\times \sum_{i=1}^{5} \int \{d\omega_j\} \;\; J_{i,\gamma\beta}\,(n_+p, x_a n_+ p_A; \{\omega_j\}) \quad S_i(\Omega; \{\omega_j\}) \quad + \text{h.c.}$$



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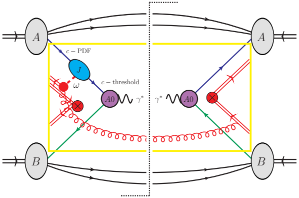

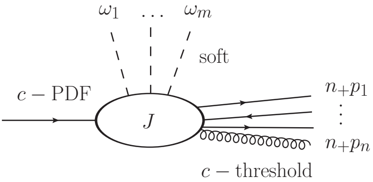

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In this talk, we focus on $\mathcal{O}(\alpha_s^2)$ calculation of:

$$S_i(\Omega; \{\omega_j\}) = \int \frac{dx^0}{4\pi}\, e^{i\Omega\, x^0/2} \int \left\{ \frac{dz_{j-}}{2\pi} \right\} e^{-i\omega_j z_{j-}}\, S_i(x_0; \{z_{j-}\})$$

Generalized soft functions

We make use of the soft building blocks

$$\mathcal{B}^{\mu}_{\pm} = Y^{\dagger}_{\pm}\,[i\,D^{\mu}_{s}\,Y_{\pm}]\;, \quad q^{\pm} = Y^{\dagger}_{\pm}\,q_s$$

The relevant soft functions are

$$S_1(x^0; z_-) \;=\; \frac{1}{N_c}\,\mathrm{Tr}\langle 0|\bar{\mathbf{T}}\left[Y^{\dagger}_{+}(x^0)Y_{-}(x^0)\right]\mathbf{T}\left(\left[Y^{\dagger}_{-}(0)Y_{+}(0)\right]\frac{i\partial^{\nu}_{\perp}}{in_-\partial}\mathcal{B}^{+}_{\nu_\perp}(z_-)\right)|0\rangle$$

$$S_3(x^0; z_-) \;=\; \frac{1}{N_c}\,\mathrm{Tr}\,\langle 0|\bar{\mathbf{T}}\left[Y^{\dagger}_{+}(x^0)Y_{-}(x^0)\right]$$
$$\times\,\mathbf{T}\left(\left[Y^{\dagger}_{-}(0)Y_{+}(0)\right]\frac{1}{(in_-\partial)^2}\left[\mathcal{B}^{+\,\mu_\perp}(z_-),\left[in_-\partial\mathcal{B}^{+}_{\mu_\perp}(z_-)\right]\right]\right)|0\rangle$$

$$S^{AB}_{4;\mu\nu,bf}(x^0; z_{1-}, z_{2-}) \;=\; \frac{1}{N_c}\,\mathrm{Tr}\,\langle 0|\bar{\mathbf{T}}\left[Y^{\dagger}_{+}(x^0)Y_{-}(x^0)\right]_{ba}$$
$$\times\,\mathbf{T}\left(\left[Y^{\dagger}_{-}(0)Y_{+}(0)\right]_{af}\mathcal{B}^{+A}_{\mu_\perp}(z_{1-})\,\mathcal{B}^{+B}_{\nu_\perp}(z_{2-})\right)|0\rangle$$

$$S_{5;bfgh,\sigma\lambda}(x^0; z_{1-}, z_{2-}) \;=\; \frac{1}{N_c}\,\langle 0|\bar{\mathbf{T}}\left[Y^{\dagger}_{+}(x^0)Y_{-}(x^0)\right]_{ba}$$
$$\times\,\mathbf{T}\left(\left[Y^{\dagger}_{-}(0)Y_{+}(0)\right]_{af}\frac{g_s^2}{(in_-\partial_{z_1})(in_-\partial_{z_2})}\,q_{+\sigma g}(z_{1-})\,\bar{q}_{+\lambda h}(z_{2-})\right)|0\rangle$$

Not functions of Wilson lines only!

Generalized soft functions: matrix elements

Single emission:

$$\langle g^A(k)|\mathbf{T}\left(Y_-^\dagger(0)Y_+(0)\ \frac{i\partial_\perp^\nu}{in_-\partial}\mathcal{B}^+_{\nu\perp}(z_-)\right)|0\rangle = \mathbf{T}^A\,\frac{g_s}{(n_-k)}\left[k_\perp^\eta - \frac{k_\perp^2}{(n_-k)}n_-^\eta\right]\epsilon^*_\eta(k)\ e^{iz_-k}$$

Double emission:

$$\langle g^{K_1}(k_1)g^{K_2}(k_2)|\mathbf{T}\left[\ Y_-^\dagger(0)Y_+(0)\quad \frac{i\partial_\perp^\mu}{in_-\partial}\mathcal{B}^+_{\mu\perp}(z_-)\ \right]|0\rangle \quad =$$

$$g_s^2\mathbf{T}^{K_2}\,\mathbf{T}^{K_1}\,\frac{1}{(n_-k_1)}\frac{n_-^{\eta_2}}{(n_-k_2)}\left[k_{1\perp}^{\eta_1} - \frac{k_{1\perp}^2}{(n_-k_1)}n_-^{\eta_1}\right]\epsilon^*_{\eta_1}(k_1)\epsilon^*_{\eta_2}(k_2)\ e^{iz_-k_1}$$

$$+\ldots$$

$$+g_s^2\,if^{K_1K_2K}\mathbf{T}^K\frac{1}{n_-(k_1+k_2)}\Bigg(-\frac{(k_{1\perp}^{\eta_2}+k_{2\perp}^{\eta_2})\,n_-^{\eta_1}}{(n_-k_1)} + \frac{(k_{1\perp}^{\eta_1}+k_{2\perp}^{\eta_1})\ n_-^{\eta_2}}{(n_-k_2)}$$

$$-\frac{n_-^{\eta_1}n_-^{\eta_2}}{n_-(k_1+k_2)(n_-k_1)(n_-k_2)}\Big[(n_-k_1)\ \left(k_{1\perp}^2 + k_{1\perp}\cdot k_{2\perp}\right)$$

$$-(n_-k_2)\left(k_{2\perp}\cdot k_{1\perp} + k_{2\perp}^2\right)\Big]\Bigg)\epsilon^*_{\eta_1}(k_1)\epsilon^*_{\eta_2}(k_2)\ e^{iz_-(k_1+k_2)}$$

$$+\ldots$$

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$$+\ldots$$

$$+g_s^2\,if^{K_1K_2K}\mathbf{T}^K\frac{1}{n_-(k_1+k_2)}\left(-\frac{\left(k_{1\perp}^{\eta_2}+k_{2\perp}^{\eta_2}\right)n_-^{\eta_1}}{(n_-k_1)} + \frac{\left(k_{1\perp}^{\eta_1}+k_{2\perp}^{\eta_1}\right)\ n_-^{\eta_2}}{(n_-k_2)}\right.$$

$$-\frac{n_-^{\eta_1}n_-^{\eta_2}}{n_-(k_1+k_2)(n_-k_1)(n_-k_2)}\Big[(n_-k_1)\ \left(k_{1\perp}^2 + k_{1\perp}\cdot k_{2\perp}\right)$$

$$\left.-(n_-k_2)\left(k_{2\perp}\cdot k_{1\perp}+k_{2\perp}^2\right)\Big]\right)\epsilon^*_{\eta_1}(k_1)\epsilon^*_{\eta_2}(k_2)\ e^{iz_-(k_1+k_2)}$$

$$+\ldots$$

→ $\delta(\omega - n_-k_1)$ constraint in the integrand.

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$$\langle g^A(k)|\mathbf{T}\left(Y_-^\dagger(0)Y_+(0)\ \frac{i\partial_\perp^\nu}{in_-\partial}\mathcal{B}^+_{\nu_\perp}(z_-)\right)|0\rangle = \mathbf{T}^A\frac{g_s}{(n_-k)}\left[k_\perp^\eta - \frac{k_\perp^2}{(n_-k)}n_-^\eta\right]\epsilon^*_\eta(k)\ \ e^{iz_-k}$$

Double emission:

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$$g_s^2\mathbf{T}^{K_2}\ \mathbf{T}^{K_1}\ \frac{1}{(n_-k_1)}\frac{n_-^{\eta_2}}{(n_-k_2)}\left[k_{1\perp}^{\eta_1} - \frac{k_{1\perp}^2}{(n_-k_1)}n_-^{\eta_1}\right]\epsilon^*_{\eta_1}(k_1)\epsilon^*_{\eta_2}(k_2)\ \ e^{iz_-k_1}$$

$$+\ldots$$

$$+g_s^2\, if^{K_1K_2K}\mathbf{T}^K\frac{1}{n_-(k_1+k_2)}\left(-\frac{\left(k_{1\perp}^{\eta_2}+k_{2\perp}^{\eta_2}\right)n_-^{\eta_1}}{(n_-k_1)} + \frac{\left(k_{1\perp}^{\eta_1}+k_{2\perp}^{\eta_1}\right)\ n_-^{\eta_2}}{(n_-k_2)}\right.$$

$$-\frac{n_-^{\eta_1}n_-^{\eta_2}}{n_-(k_1+k_2)(n_-k_1)(n_-k_2)}\Big[(n_-k_1)\ \left(k_{1\perp}^2+k_{1\perp}\cdot k_{2\perp}\right)$$

$$\left.-(n_-k_2)\left(k_{2\perp}\cdot k_{1\perp}+k_{2\perp}^2\right)\Big]\right)\epsilon^*_{\eta_1}(k_1)\epsilon^*_{\eta_2}(k_2)\ e^{iz_-(k_1+k_2)}$$

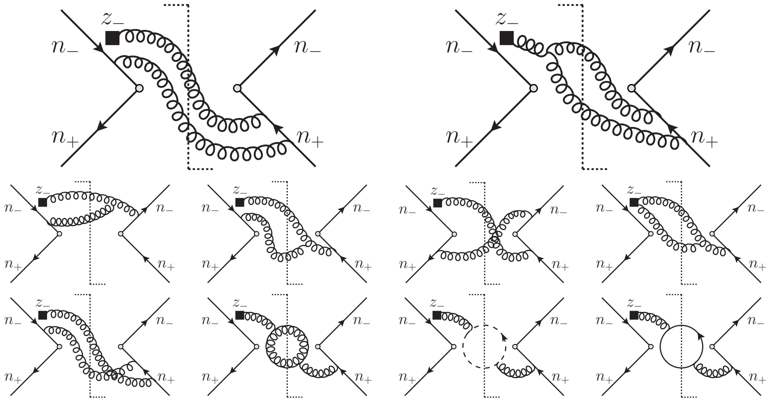
$$+\ldots$$

$\rightarrow\quad \delta(\omega - n_-k_1 - n_-k_2)$ constraint in the integrand.

Some sample diagrams

For the S_1 soft function:

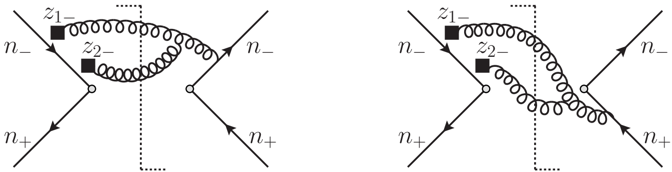
$$S_1(x^0; z_-) = \frac{1}{N_c} \mathrm{Tr}\langle 0|\bar{\mathbf{T}}\left[Y_+^\dagger(x^0)Y_-(x^0)\right]\mathbf{T}\left(\left[Y_-^\dagger(0)Y_+(0)\right]\frac{i\partial_\perp^\nu}{in_-\partial}\mathcal{B}^+_{\nu_\perp}(z_-)\right)|0\rangle$$



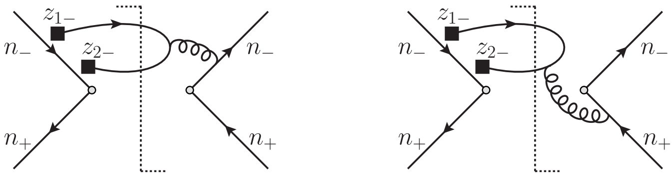
...

Some sample diagrams

For the S_4 soft function:



For the S_5 soft function:



The calculation

Methods developed for calculations of two-loop soft functions at *leading power.*

[Y. Li, S. Mantry, F. Petriello, 1105.5171] [T. Becher, G. Bell, S. Marti, 1201.5572]

[A. Ferroglia, B. Pecjak, L.L. Yang, 1207.4798]

First, we find the relevant topologies for, and perform, the reduction. For example:

$$P_1 = (k_1 + k_2)^2, \quad P_2 = n_+ k_2, \quad P_3 = n_-(k_1 + k_2),$$

$$P_4 = k_1^2 \ , \quad P_5 = k_2^2 \ , \quad P_6 = \big(\Omega - n_- k_1 - n_- k_2 - n_+ k_1 - n_+ k_2\big) \ , \quad P_7 = \big(\omega - n_- k_1\big)$$

$$\hat{I}_{\mathfrak{T}}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = (4\pi)^4 \left(\frac{e^{\gamma_E} \mu^2}{4\pi} \right)^{2\epsilon} \int \frac{d^d k_1}{(2\pi)^{d-1}} \frac{d^d k_2}{(2\pi)^{d-1}} \prod_{i=1}^{7} \frac{1}{P_i^{\alpha_i}}$$

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$$\delta(k_1^2) = \frac{1}{2\pi i}\left[\frac{1}{k_1^2 + i0^+} - \frac{1}{k_1^2 - i0^+}\right]$$

[C. Anastasiou, K. Melnikov, hep-ph/0207004]

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- The reduction is implemented in `LiteRed`
- 9 topologies are needed to reduce the soft functions
- We have 8 Master Integrals (MIs)
  - 5 MIs implementing the $\delta(\omega - n_- k_1)$ constraint: $\hat{I}_1$ – $\hat{I}_5$
  - 2 MIs with $\delta(\omega - n_- k_1 - n_- k_2)$: $\hat{I}_6$ and $\hat{I}_7$
  - 1 MI with $\delta(\omega_1 - n_- k_1)\,\delta(\omega_2 - n_- k_2)$: $\hat{I}_8$

Reduced expressions

For completeness:

$$
\begin{aligned}
S_1^{(2)2r0v}(\Omega,\omega) \quad = \quad & \frac{\alpha_s^2}{(4\pi)^2} C_F^2 \frac{8\left(2-9\epsilon+9\epsilon^2\right)}{\epsilon^2\,\omega\,(\Omega-\omega)^2}\,\hat{I}_1 \\
& + \frac{\alpha_s^2}{(4\pi)^2} C_F C_A \Bigg[\frac{(2-3\epsilon)\left(-4\Omega+\epsilon\,(\omega+19\Omega)+4\epsilon^2\,(\omega-7\Omega)-16\epsilon^3(\omega-\Omega)\right)}{\epsilon^2(1-2\epsilon)\,\omega\,\Omega\,(\Omega-\omega)^2}\,\hat{I}_1 \\
& - \frac{(1-4\epsilon^2)}{\epsilon\,\omega\,\Omega}\hat{I}_2 + \frac{(3\Omega-10\epsilon\,\Omega+16\epsilon^2(\omega+\Omega))}{2(1-2\epsilon)\,\omega\,\Omega}\hat{I}_3 + \frac{(\Omega-3\omega)}{2\omega}\,\hat{I}_4 \\
& + \Omega\,\hat{I}_5 + \frac{(9-20\epsilon+12\epsilon^2-2\epsilon^3)}{\epsilon^2\,(3-2\epsilon)\omega^2(\Omega-\omega)}\hat{I}_6 + (\Omega-\omega)\hat{I}_7 \Bigg] \\
& - \frac{\alpha_s^2}{(4\pi)^2} C_F\, n_f\, \frac{4(1-\epsilon)^2}{\epsilon\,(3-2\epsilon)\omega^2(\Omega-\omega)}\,\hat{I}_6
\end{aligned}
$$

For the remaining soft functions:

$$
S_3^{(2)}(\Omega,\omega) = \frac{\alpha_s^2}{(4\pi)^2} C_F C_A \frac{2(1-\epsilon)}{(3-2\epsilon)\omega^2(\Omega-\omega)}\hat{I}_6
$$

$$
S_4^{(2)}(\Omega,\omega_1,\omega_2) = -\frac{\alpha_s^2}{(4\pi)^2} C_F C_A \frac{2(1-\epsilon)\omega_2(\omega_1-\omega_2)}{(\omega_1+\omega_2)^4(\Omega-\omega_1-\omega_2)}\hat{I}_8
$$

$$
S_5^{(2)}(\Omega,\omega_1,\omega_2) = \frac{\alpha_s^2}{(4\pi)^2}\left(C_F^2 - \frac{1}{2}C_F C_A\right)\frac{8(-1+\epsilon)\omega_2}{(\omega_1+\omega_2)^3(\Omega-\omega_1-\omega_2)}\hat{I}_8
$$

DE method for MIs

Convenient to change to dimensionless variable $\omega \to r\,\Omega$

$$I_1'(r) = \frac{1}{\Omega^2}\left(\frac{\Omega}{\mu}\right)^{4\epsilon}\hat{I}_1(\Omega, r), \qquad I_2'(r) = \frac{1}{\Omega}\left(\frac{\Omega}{\mu}\right)^{4\epsilon}\hat{I}_2(\Omega, r),$$

$$I_3'(r) = \left(\frac{\Omega}{\mu}\right)^{4\epsilon}\hat{I}_3(\Omega, r), \qquad I_4'(r) = \Omega\left(\frac{\Omega}{\mu}\right)^{4\epsilon}\hat{I}_4(\Omega, r),$$

$$I_5'(r) = \Omega^2\left(\frac{\Omega}{\mu}\right)^{4\epsilon}\hat{I}_5(\Omega, r)$$

System of DEs can be put into canonical form [J. Henn, 1304.1806]

$$\frac{d\vec{I}(r)}{dr} = \epsilon\, A(r)\cdot\vec{I}(r)$$

$$I_1'(r) = \frac{2(1-r)^2}{2-9\epsilon+9\epsilon^2} I_1(r),$$

$$I_3'(r) = \frac{1}{\epsilon^2} I_3(r),$$

$$I_4'(r) = -\frac{1}{\epsilon^2(1-r)} I_4(r),$$

$$I_5'(r) = -\frac{1+r}{2\epsilon^2(1-r)r} I_4(r) + \frac{1}{\epsilon^2 r} I_5(r)$$

$$A(r) = \begin{bmatrix} -\frac{1}{r}+\frac{3}{1-r} & 0 & 0 & 0 \\ \frac{2}{r} & -\frac{2}{r} & 0 & 0 \\ \frac{2}{r} & \frac{2}{r} & \frac{4}{1-r} & 0 \\ \frac{1}{r} & \frac{1}{r} & \frac{1}{r} & -\frac{2}{r} \end{bmatrix}$$

MIs results

- We solve the system of differential equations iteratively.
- $I_3(r)$ and $I_4(r)$ can be obtained exactly in d-dimensions.
- Integrand of I_5 includes a ${}_3F_2$ hypergeometric function, hence we resort to keeping exact d-dimensional dependence at endpoints 0 and 1, and expanding elsewhere.

$$\frac{dI_5(r)}{dr} = \epsilon\left[\frac{1}{r}I_1(r) + \frac{1}{r}I_3(r) + \frac{1}{r}I_4(r) - \frac{2}{r}I_5(r)\right]$$

and the solution has the following structure

$$I_5(r) = r^{-2\epsilon}\left[C_5(\epsilon) + \int_1^r dr'\, f_{I_5}(r',\epsilon)\right]\theta(r)\theta(1-r)$$

$$\begin{aligned}
I_5'(r) = \Bigg[& -\frac{\delta(1-r)+\delta(r)}{2\epsilon^3} + \frac{1}{\epsilon^2}\left(2\left[\frac{1}{1-r}\right]_+ + \left[\frac{1}{r}\right]_+\right) + \frac{1}{12\epsilon}\Bigg(5\pi^2\delta(1-r) - \pi^2\delta(r) \\
& - 96\left[\frac{\ln(1-r)}{1-r}\right]_+ - 24\left[\frac{\ln r}{r}\right]_+ - \frac{48\ln(1-r)}{r} - \frac{12\ln r}{1-r}\Bigg) \\
& + \frac{\zeta_3}{3}\big(28\delta(1-r) - 5\delta(r)\big) - \frac{5\pi^2}{3}\left[\frac{1}{1-r}\right]_+ + \frac{\pi^2}{6}\left[\frac{1}{r}\right]_+ + 16\left[\frac{\ln^2(1-r)}{1-r}\right]_+ \\
& + 2\left[\frac{\ln^2 r}{r}\right]_+ + 8\frac{\ln^2(1-r)}{r} + \frac{2(1+r)}{r(1-r)}\ln(1-r)\ln(r) + \frac{\ln^2 r}{2(r-1)} - \frac{7\pi^2}{6} \\
& + \frac{(6-7r)}{(r-1)r}\left(\mathrm{Li}_2(r) - \frac{\pi^2 r}{6}\right)\Bigg]\theta(r)\theta(1-r) + \mathcal{O}(\epsilon)
\end{aligned}$$

Fixed-order checks

The S_1 soft function is the only one with C_F^2 contributions. Combine our results for two-loop soft with tree-level collinear according to factorization theorem

$$\Delta^{dyn\,(2)2r0v}_{\mathrm{NLP-soft},S_{1,C_F^2}}(z) = 4\,Q\,\Omega\,H^{(0)}(Q^2)\int dr\,J^{(0)}_{1,1}\left(x_a(n_+p_A);\omega\right)S^{(2)2r0v}_{1,C_F^2}(\Omega,r)$$

$$\begin{aligned}\Delta^{dyn\,(2)2r0v}_{\mathrm{NLP-soft},S_{1,C_F^2}}(z) \quad = \quad & \frac{\alpha_s^2}{(4\pi)^2}\,C_F^2\biggl(\frac{32}{\epsilon^3}-\frac{128}{\epsilon^2}\ln(1-z)+\frac{256}{\epsilon}\ln^2(1-z)-\frac{112\pi^2}{3\epsilon}\\ & +\frac{32}{3}\left(-32\ln^3(1-z)+14\pi^2\ln(1-z)-62\zeta(3)\right)+\mathcal{O}\left(\epsilon\right)\biggr)\end{aligned}$$

The S_1 1r1v gives leading logs proportional to C_FC_A, which are cancelled exactly by the leading poles of 2r0v contribution to S_1

$$\begin{aligned}\Delta^{dyn\,(2)2r0v}_{\mathrm{NLP-soft},S_{1,C_FC_A}}(z) \quad = \quad & \frac{\alpha_s^2}{(4\pi)^2}\,C_FC_A\biggl(\frac{8}{\epsilon^3}-\frac{4}{3\epsilon^2}(24\ln(1-z)-11)\\ & -\frac{16}{9\epsilon}\left(-36\ln^2(1-z)+33\ln(1-z)+6\pi^2-16\right)\\ & -\frac{256}{3}\ln^3(1-z)+\frac{352}{3}\ln^2(1-z)+\frac{128}{3}\pi^2\ln(1-z)\\ & -\frac{1024}{9}\ln(1-z)-\frac{616\zeta(3)}{3}-\frac{154\pi^2}{9}+\frac{1484}{27}+\mathcal{O}(\epsilon)\biggr)\end{aligned}$$

Further S_1 checks

Interestingly, we can partially validate our results at N^{3}LO by comparing to [N. Bahjat-Abbas, J. Sinninghe Damsté, L Vernazza, C White, 1807.09246]

$$\Delta^{dyn\,(3)}_{\mathrm{NLP-coll},\,C_F^3}(z) \quad = \quad 4Q \int d\omega\, J^{(1)}_{1,1}(x_a\, n_+ p_A;\omega)\, S^{(2)}_{1,C_F^2}(\Omega;\omega)$$

Using one-loop collinear function from [M.Beneke, A.Broggio, **SJ**, L.Vernazza, 1912.01585]

$$S^{(2)2r0v}_{1,C_F^2}(\Omega,\omega) = 8\,\frac{\alpha_s^2}{(4\pi)^2}\,C_F^2\,\left(\frac{\omega\,(\Omega-\omega)^3}{\mu^4}\right)^{-\epsilon}\,\frac{1}{\omega}\,\frac{1}{\epsilon^2}\,\frac{e^{2\epsilon\gamma_E}\,\Gamma[1-\epsilon]}{\Gamma[1-3\epsilon]}\theta(\Omega-\omega)\theta(\omega)$$

We find a closed d-dimensional result, expanding

$$\begin{aligned}\Delta^{dyn\,(3)}_{\mathrm{NLP-coll},\,C_F^3}(z) \quad = \quad & \frac{\alpha_s^3}{(4\pi)^3}C_F^3\Bigg(-\frac{64}{\epsilon^4}+\frac{80(4\ln(1-z)-1)}{\epsilon^3}+\frac{16}{\epsilon^2}\Big(-50\ln^2(1-z)\\ & +25\ln(1-z)+7\pi^2-6\Big)+\frac{1}{\epsilon}\Big(\frac{4000}{3}\ln^3(1-z)-1000\ln^2(1-z)\\ & -560\pi^2\ln(1-z)+480\ln(1-z)+2624\zeta(3)+140\pi^2-128\Big)\\ & -\frac{5000}{3}\ln^4(1-z)+\frac{5000}{3}\ln^3(1-z)+1400\pi^2\ln^2(1-z)\\ & -1200\ln^2(1-z)-700\pi^2\ln(1-z)+640\ln(1-z)\\ & +\zeta(3)(3280-13120\ln(1-z))+\frac{62\pi^4}{5}+168\pi^2-192\Bigg)\end{aligned}$$

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The leading poles of the ω dependent S_1 used for LL resummation in [M. Beneke, A.Broggio, M. Garny, **SJ**, R. Szafron, L. Vernazza, J.Wang, 1809.10631]

$$S^{(2)}_{2\xi} - \frac{1}{4}Z^{(1)}_{2\xi\,x_0}\left(3Z^{(1)}_{2\xi\,2\xi} + Z^{(1)}_{x_0\,x_0}\right)S^{(0)}_{x_0} = \mathcal{O}\left(\frac{1}{\epsilon^2}\right)$$

Expanding our result and keeping only leading poles, we find agreement.
Still to explore: consistent renormalization and resummation beyond LL.

Comments on S_3, S_4 and S_5

By performing the convolution with corresponding collienar functions, we fnd their contribution to the cross-section. Importantly, no leading poles! As assumed in

[M. Beneke, A.Broggio, M. Garny, **SJ**, R. Szafron, L. Vernazza, J.Wang, 1809.10631]

$$\Delta^{dyn\,(2)2r0v}_{\mathrm{NLP-soft},S_3}(z) = 4\,\frac{\alpha_s^2}{(4\pi)^2}\,C_F C_A\,\left(\frac{\Omega^4}{\mu^4}\right)^{-\epsilon}\,\frac{1}{\epsilon}\,\frac{(1-\epsilon)}{(1-2\epsilon)^2(3-2\epsilon)}\,\frac{e^{2\epsilon\gamma_E}\,\Gamma[1-\epsilon]^2}{\Gamma[1-4\epsilon]}\,.$$

$$\Delta^{dyn\,(2)2r0v}_{\mathrm{NLP-soft},S_3}(z) \quad = \quad \frac{\alpha_s^2}{(4\pi)^2}\,C_F C_A\left(\frac{4}{3\epsilon} - \frac{4}{9}\,(12\ln(1-z) - 11) + \mathcal{O}\left(\epsilon\right)\right)$$

And S_4

$$\Delta^{dyn\,(2)2r0v}_{\mathrm{NLP-soft},S_4}(z) = -4\,\frac{\alpha_s^2}{(4\pi)^2}\,C_F C_A\,\left(\frac{\Omega^4}{\mu^4}\right)^{-\epsilon}\,\frac{1}{\epsilon}\,\frac{(1-\epsilon)}{(1-2\epsilon)^2(3-2\epsilon)}\,\frac{e^{2\epsilon\gamma_E}\,\Gamma[1-\epsilon]^2}{\Gamma[1-4\epsilon]}$$

S_3 and S_4 actually cancel!
S_5 does contribute to the cross-section

$$\Delta^{dyn\,(2)2r0v}_{\mathrm{NLP-soft},S_5}(z) = 8\,\frac{\alpha_s^2}{(4\pi)^2}\,\left(C_F^2 - \frac{1}{2}C_F C_A\right)\,\left(\frac{\Omega^4}{\mu^4}\right)^{-\epsilon}\,\frac{(1-\epsilon)}{\epsilon(1-2\epsilon)^2}\,\frac{e^{2\epsilon\gamma_E}\,\Gamma[1-\epsilon]^2}{\Gamma[1-4\epsilon]}$$

Summary

- Next-to-leading power investigations is a thriving field of study with plenty left to explore and the increase precision of resummation.
- Conceptual challenges such as divergent convolution integrals and general renormalization properties remain to be solved $\to$ see the progress in Leonardo's talk next.
- Our two-loop soft function calculation validates the bare factorization theorem for NLP DY up to NNLO, with some checks at N^3LO!
- We provide higher perturbative order data for the objects appearing in the factorization formula to facilitate further investigations into the open conceptual issues such as renormalization properties of NLP objects.

Thank you