Symmetry in Particle Physics, Problem Sheet 9

1. Consider a vector q transforming to the *fundamental* representation of $SU(N_c)$:

$$q_i \to (q_i)' = U_{ij}\, q_j\,, \qquad U = e^{-i\theta^a t^a}\,, \quad a = 1, \dots N_c^2 - 1\,.$$

The vector $q^\dagger$ transforms according to the *conjugate* representation.

(a) Show that the generators of the conjugate representation are $\bar{t}^a = -(t^a)^T$.

(b) Consider the tensor $T_{ij} = q_i q_j^*$, and decompose it as follows

$$T_{ij} = A_{ij} + S_{ij}\,, \qquad A_{ij} \equiv \frac{1}{3}\delta_{ij}(q_k q_k^*) \qquad S_{ij} \equiv \left[q_i q_j^* - \frac{1}{3}\delta_{ij}(q_k q_k^*)\right]\,.$$

Show that A_{ij} is invariant under $SU(N_c)$, while S_{ij} transforms according to the adjoint representation.

(c) Let $N_c > 2$ and construct the tensor

$$T_i = \epsilon_{ijk} q_j q_k\,.$$

Using $\det(U) = 1$, show that T_i transforms according to the conjugate representation.

2. For each representation R of a compact Lie group G, consider the *quadratic Casimir* operator $T^2(R) = T^a(R) T^a(R)$.

(a) Show that T^2 commutes with every generator T^a, and hence $T^2(R) = C_R\, \mathbb{1}_R$, where $\mathbb{1}_R$ is the identity matrix in the vector space spanning representation R.

(b) Given that the generators of each representation are normalised as follows

$$\mathrm{Tr}[T^a(R)\, T^b(R)] = T_R\, \delta^{ab}\,.$$

where T_R depends on the representation, show that T_R and C_R are related by

$$C_R\ \dim(R) = T_R\ \dim(G)\,.$$

where $\dim(G)$ is the dimension of the group.

(c) Given the normalisation $T_F = 1/2$, derive

$$C_F = \frac{N_c^2 - 1}{2N_c} \qquad C_A = T_A\,.$$

3. Consider the Lagrangian

$$\mathcal{L} = i\bar{\psi}_1\gamma^\mu\partial_\mu\psi_1 + i\bar{\psi}_2\gamma^\mu\partial_\mu\psi_2 - m\left(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2\right) ,$$

where $\psi_1(x)$ and $\psi_2(x)$ are two Dirac spinor fields, with $\bar{\psi}_i = \psi_i^\dagger\gamma^0$ $(i = 1, 2)$, and m is a real parameter. We have seen in the lectures that the Lagrangian is invariant under a global $U(1) \times SU(2)$ transformation of the form

$$\psi_i(x) \to e^{-i\alpha}U_{ij}\psi_j(x) , \qquad U = \exp\left[-i\,\alpha_a\,\frac{\sigma_a}{2}\right] ,$$

where $\alpha, \alpha_1, \alpha_2, \alpha_3$ are real constant parameters, and σ_a, $a = 1, 2, 3$ are the three Pauli matrices.

(a) Consider now the following **local** $U(1) \times SU(2)$ transformation

$$\psi_i(x) \to e^{-i\frac{g_1}{2}\alpha(x)}U_{ij}(x)\psi_j(x) \qquad U = \exp\left[-i\,g_2\,\alpha_a(x)\,\frac{\sigma_a}{2}\right] ,$$

where g_1 and g_2 are constants, whereas $\alpha(x)$ and $\alpha_a(x), a = 1, 2, 3$ are arbitrary functions of the space-time point x. Show that the Lagrangian is not invariant any more under such transformation, and compute the corresponding variation $\delta\mathcal{L}$

(b) We can modify the Lagrangian so that it is invariant under a local $U(1) \times SU(2)$ transformation by promoting the ordinary derivative ∂_μ to a covariant derivative D_μ as follows

$$D_\mu = \partial_\mu + i\frac{g_1}{2}B_\mu + i\frac{g_2}{2}(W_a)_\mu\sigma_a ,$$

where B^μ, and W_i^μ are vector gauge fields. How should B^μ, and W_a^μ transform so that $\mathcal{L}$ is still invariant under local $U(1) \times SU(2)$ transformations?

(c) You want this very same Lagrangian to describe electromagnetism, and you know that the particles described by ψ_2 are electrically neutral. How can you accommodate this in the theory?

Hint. Consider a suitable linear combination of gauge fields.

(d) What is the electric charge of the particles described by ψ_1?