

# Lattice QCD for AMBER

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# HadStruc Collaboration

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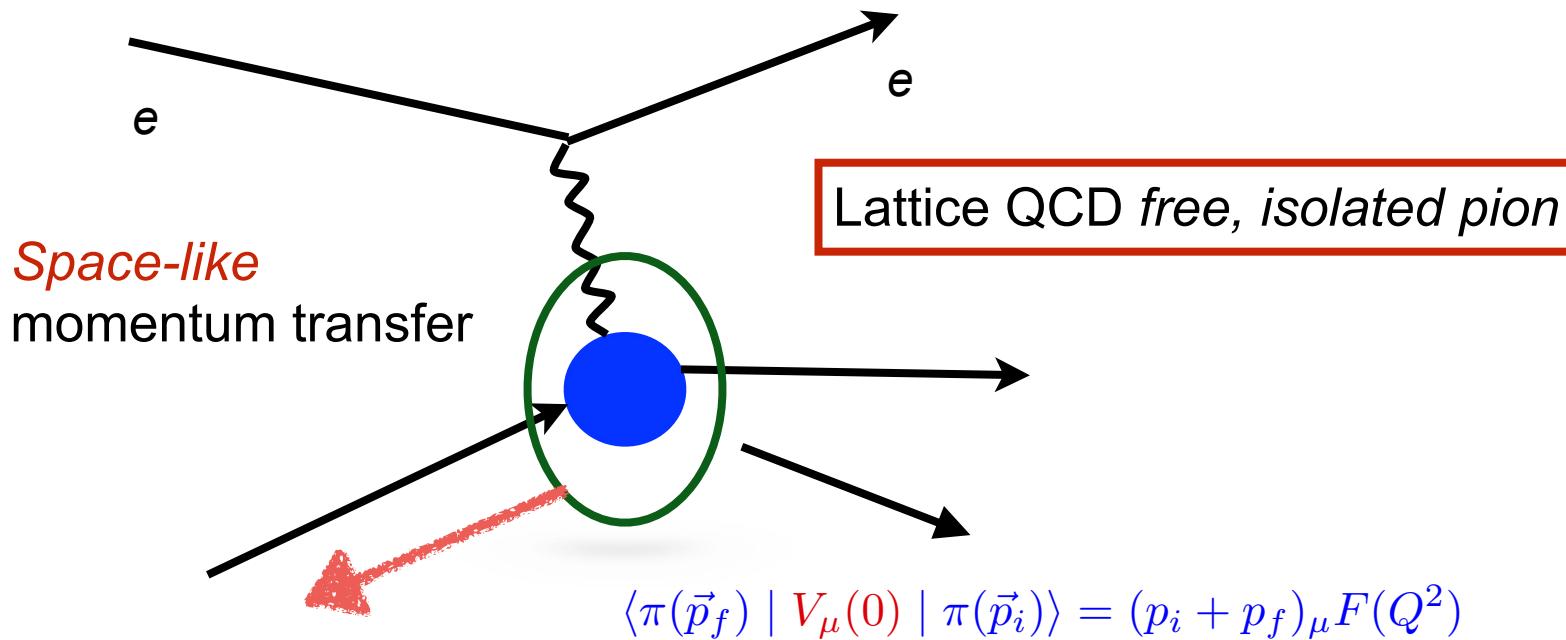
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Peking University, Beijing, China

# Outline

- One-dimensional Pion Structure - Form Factors and PDFs
- GPDs and 3D Images
- Where are the gluons?
- Summary

# Paradigm: Pion EM form factor

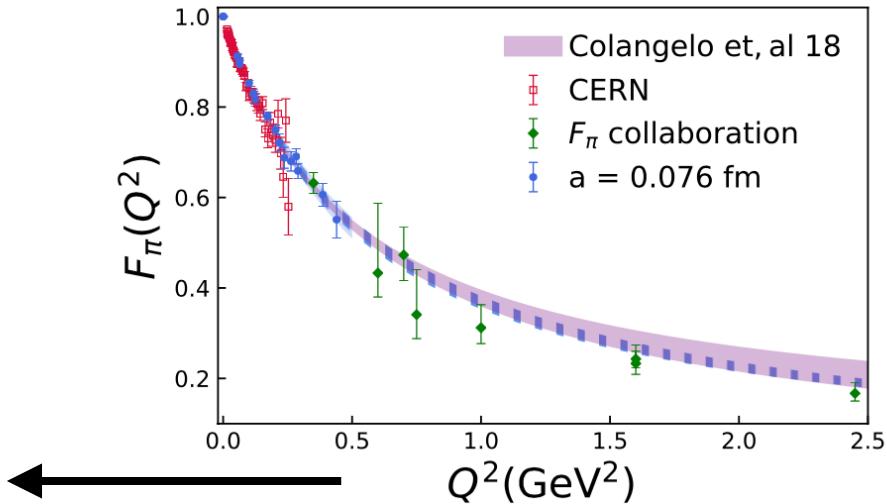


where

$$\begin{aligned} V_\mu &= \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d \\ -Q^2 &= [E_\pi(\vec{p}_f) - E_\pi(\vec{p}_i)]^2 - (\vec{p}_f - \vec{p}_i)^2 \end{aligned}$$

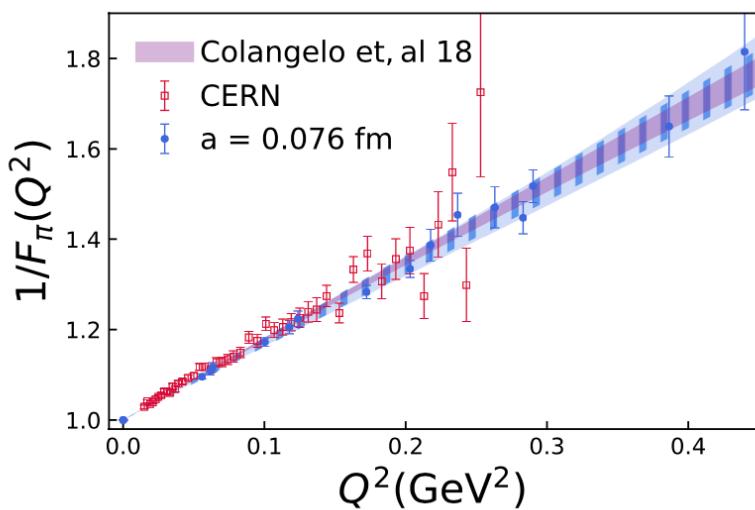
$$\Gamma_{\pi^+ \mu \pi^+}(t_f, t; \vec{p}, \vec{q}) = \sum_{\vec{x}, \vec{y}} \langle 0 | \phi(\vec{x}, t_f) V_\mu(\vec{y}, t) \phi^\dagger(0) | \rangle e^{-i\vec{p} \cdot \vec{x}} e^{-i\vec{q} \cdot \vec{y}}$$

*Charge Radius*



*Partonic degrees of freedom*

See Nikhil - QDA

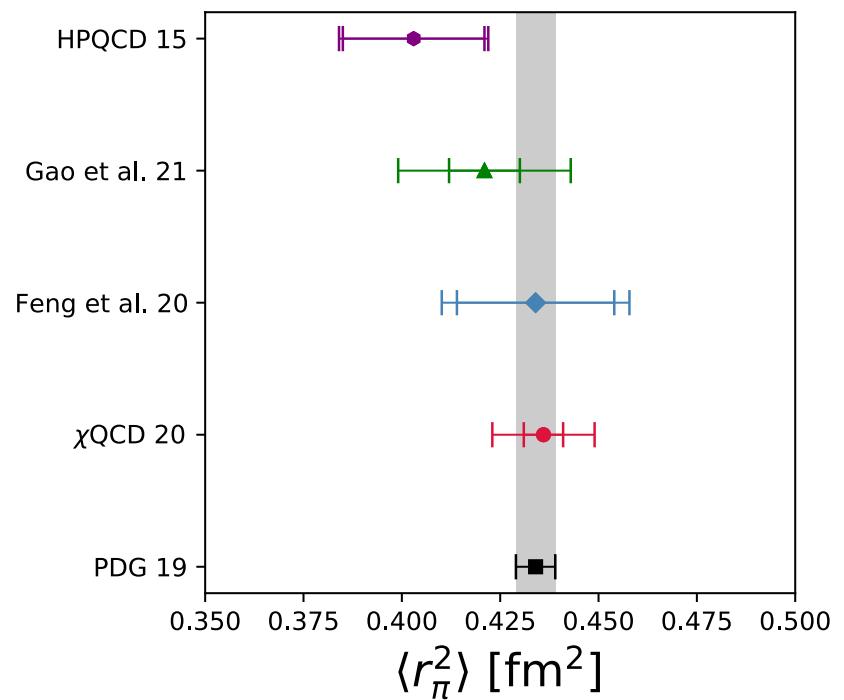
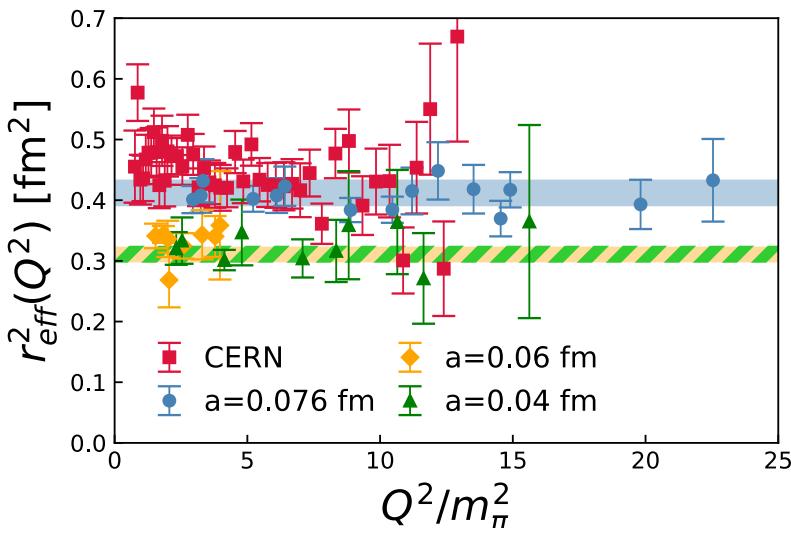


# Charge Radius

Challenges mirror those of experimental measurement.

$$F_\pi(Q^2) = \sum_{k=0}^{k_{\max}} a_k z^k \quad z\text{-expansion}$$

$$z(t, t_{\text{cut}}, t_0) = \frac{\sqrt{t_{\text{cut}} - t} - \sqrt{t_{\text{cut}} - t_0}}{\sqrt{t_{\text{cut}} - t} + \sqrt{t_{\text{cut}} - t_0}}$$



# A history of lattice QCD through no-go theorems

- ~~You can't place a chiral gauge theory on a discretized lattice~~

Domain-wall Fermions: *D.Kaplan, Phys.Lett.B 288 (1992) 342*

Overlap Fermions: *R.Narayanan, H.Neuberger, Nucl.Phys.B 443 (1995) 305*

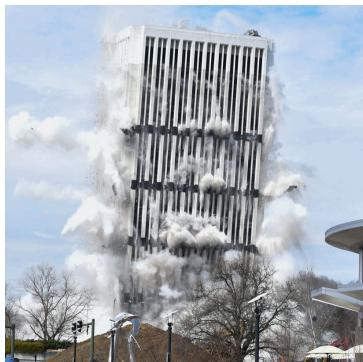
- ~~You can't investigate scattering on a Euclidean lattice~~

“Luscher’s Method”: *M.Luscher, Nucl.Phys.B 354 (1991) 531*

See *David Wilson, Tuesday and many parallel talks*

- ~~You can't compute matrix elements of light-cone operators on a Euclidean lattice~~

LaMET: *X.Ji, Phys.Rev.Lett. 110 (2013) 262002*



Theorems did  
not fall - we  
found way to  
drive around  
them



Transformed our ability to exploit internal structure of hadrons

# Hadron Structure: No-go Theorem?

- **First Challenge:**

- Euclidean lattice precludes calculation of light-cone/time-separated correlation functions

PDFs, GPDs, TMDs

$$q(x, \mu) = \int \frac{d\xi^-}{4\pi} e^{-ix\xi^- P^+} \langle P | \bar{\psi}(\xi^-) \gamma^+ e^{-ig \int_0^{\xi^-} d\eta^- A^+(\eta^-)} \psi(0) | P \rangle$$

So.... ...Use *Operator-Product-Expansion* to formulate in terms of Mellin Moments with respect to Bjorken x.

—————→  $\langle P | \bar{\psi} \gamma_{\mu_1} (\gamma_5) D_{\mu_2} \dots D_{\mu_n} \psi | P \rangle \rightarrow P_{\mu_1} \dots P_{\mu_n} a^{(n)}$

- **Second Challenge:**

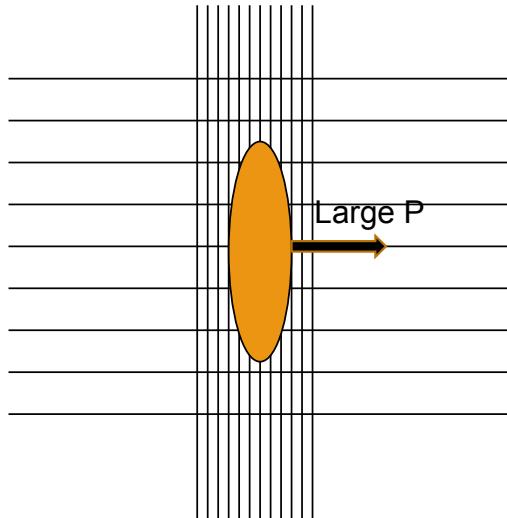
- Discretised lattice: power-divergent mixing for higher moments

Moment Methods

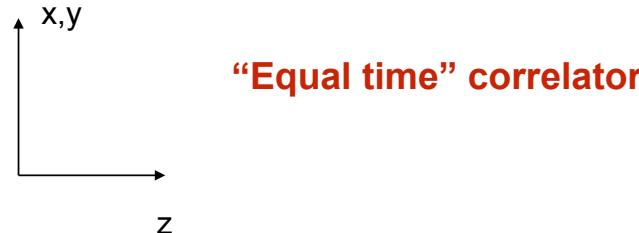
Recent work by ETMC - later

- Extended operators: Z.Davoudi and M. Savage, PRD 86,054505 (2012)
  - Valence heavy quark: W.Detmold and W.Lin, PRD73, 014501 (2006)

# Solution....



Large-Momentum Effective Theory (LaMET)



X. Ji, Phys. Rev. Lett. 110, 262002 (2013).

X. Ji, J. Zhang, and Y. Zhao, Phys. Rev. Lett. 111, 112002 (2013).

J. W. Qiu and Y. Q. Ma, arXiv:1404.686.

$$q(x, \mu^2, P^z) = \int \frac{dz}{4\pi} e^{izk^z} \langle P | \bar{\psi}(z) \gamma^z e^{-ig \int_0^z dz' A^z(z')} \psi(0) | P > \\ + \mathcal{O}((\Lambda^2/(P^z)^2), M^2/(P^z)^2)$$



$$q(x, \mu^2, P^z) = \int_x^1 \frac{dy}{y} Z\left(\frac{x}{y}, \frac{\mu}{P^z}\right) q(y, \mu^2) + \mathcal{O}(\Lambda^2/(P^z)^2, M^2/(P^z)^2)$$

# Pseudo-PDFs

- Pseudo-PDF (pPDF) recognizing generalization of PDFs in terms of *Ioffe Time*.  $\nu = p \cdot z$

B.Ioffe, PL39B, 123 (1969); V.Braun  
et al, PRD51, 6036 (1995)

A.Radyushkin, Phys. Rev. D 96, 034025 (2017)

$$M^\alpha(p, z) = \langle p | \bar{\psi} \gamma^\alpha U(z; 0) \psi(0) | p \rangle$$

$p = (p^+, m^2/2p^+, 0_T)$      $\downarrow$      $z = (0, z_-, 0_T)$                   Ioffe-Time Distribution

$$M^\alpha(z, p) = 2p^\alpha \mathcal{M}(\nu, z^2) + 2z^\alpha \mathcal{N}(\nu, z^2)$$

Ioffe-time pseudo-Distribution (**pseudo-ITD**) generalization to *space-like z*

Lattice “building blocks” that of quasi-PDF approach.

$$\mathcal{M}(\nu, z^2) = \int_{-1}^1 dx e^{i\nu x} \mathcal{P}(x, z^2)$$

$\Downarrow$  *Lorentz covariant*

$$f(x) = \mathcal{P}(x, 0) \underset{z_3^2 \rightarrow 0}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu e^{-i\nu x} \mathcal{M}(\nu, -z_3^2)$$

# “Good Lattice Cross Sections”

$$\sigma_n(\nu, \xi^2, P^2) = \langle P | T\{\mathcal{O}_n(\xi)\} | P \rangle$$

Ma and Qiu, Phys. Rev. Lett. 120 022003

*Expressed in coordinate space*

where

$$\sigma_n(\nu, \xi^2, P^2) = \sum_a \int_{-1}^1 \frac{dx}{x} f_a(x, \mu^2) K_n^a(x\nu, \xi^2, x^2 P^2, \mu^2) + \mathcal{O}(\xi^2 \Lambda_{\text{QCD}}^2)$$

*Calculated in  
LQCD*

Parton Distribution  
function

Short distance scale

*Calculated in perturbation  
theory (“process dependent”)*

$$\mathcal{O}(\xi) = \bar{\psi}(0) \Gamma W(0, 0 + \xi) \psi(\xi)$$

← Encompasses qPDF/pPDF

$$\mathcal{O}_S(\xi) = \xi^4 Z_S^2 [\bar{\psi}_q \psi_q](\xi) [\bar{\psi}_q \psi](0)$$

Gauge-Invariant Currents

$$\mathcal{O}_{V'}(\xi) = \xi^2 Z_{V'}^2 [\bar{\psi}_q \xi \cdot \gamma \psi_{q'}](\xi) [\bar{\psi}_{q'} \xi \cdot \gamma \psi](0)$$

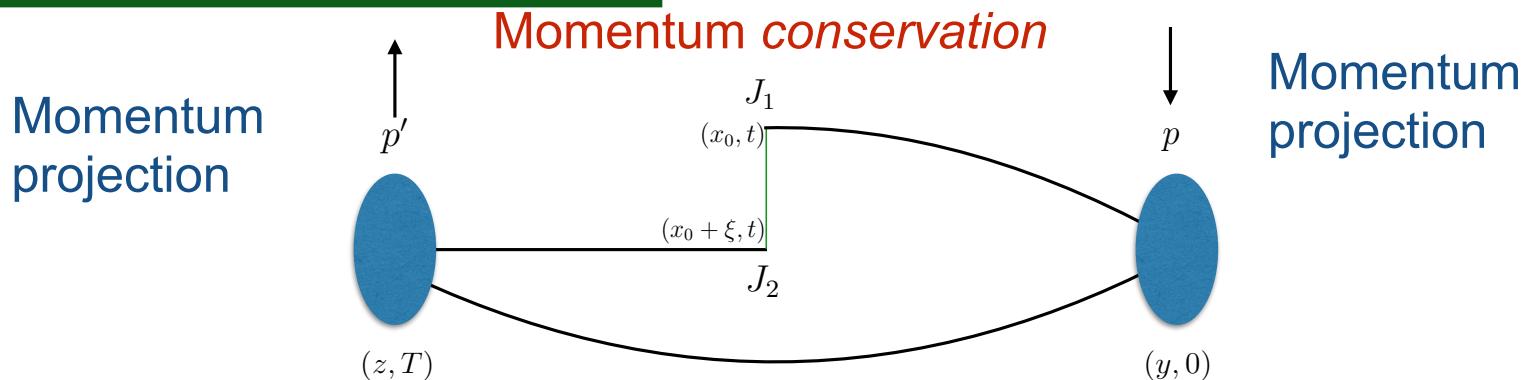
← Flavor-changing

+ analogous gluon operators

# Good Lattice Cross Section

Sufian et al., Phys. Rev. D 99, 074507  
(2019); arXiv:2001.04960

Sequential-Source Approach



Process, i.e. current, dependent

$$\frac{1}{2} [\sigma_{V,A}^{\mu\nu}(\xi, p) + \sigma_{A,V}^{\mu\nu}(\xi, p)]$$

$$\equiv \epsilon^{\mu\nu\alpha\beta} \xi_\alpha p_\beta T_1(\nu, \xi^2) + (p^\mu \xi^\nu - \xi^\mu p^\nu) T_2(\nu, \xi^2)$$

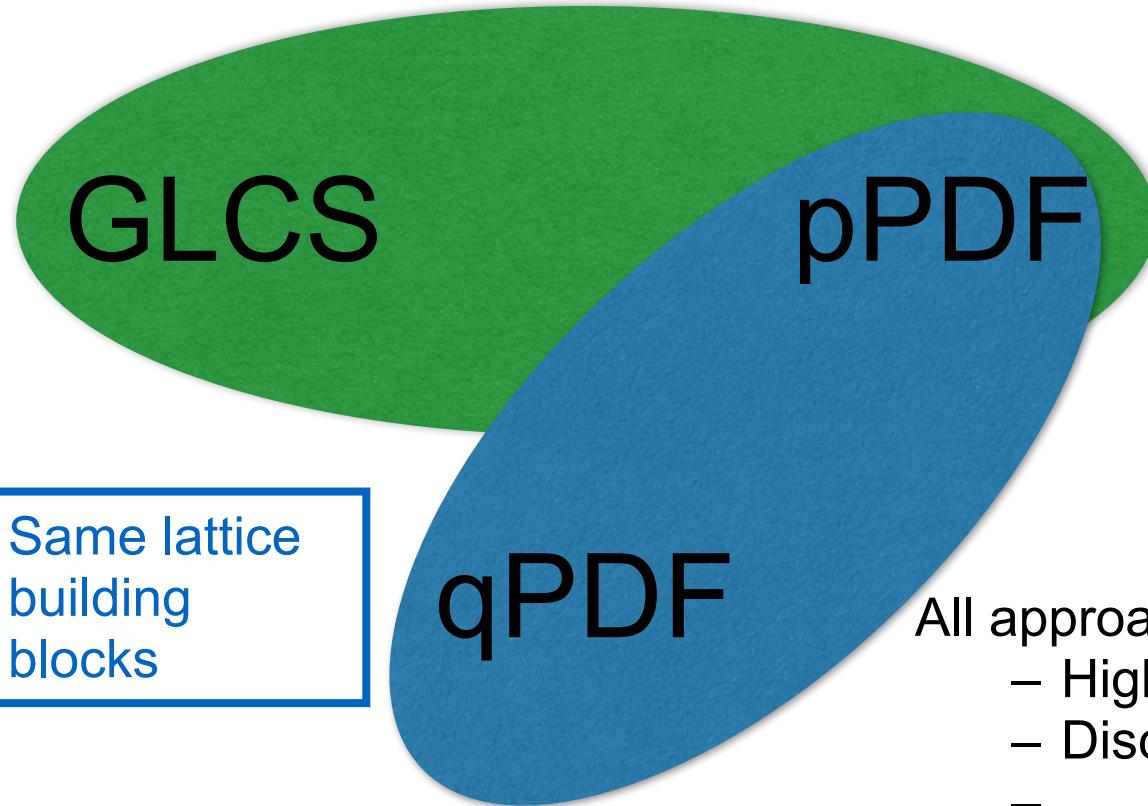
Perturbative kernel:

N.B. We're inconsistent  $\omega \leftrightarrow \nu$ !

$$\tilde{\sigma}_{VA}^{q(1)}(\tilde{\omega}, q^2) = \int_0^1 \frac{dx}{x} \tilde{K}^{(1)}(x\tilde{\omega}, q^2, \mu^2) f_{q_v/q}^{(0)}(x, \mu^2) + \int_0^1 \frac{dx}{x} \tilde{K}^{(0)}(x\tilde{\omega}, q^2, \mu^2) f_{q_v/q}^{(1)}(x, \mu^2).$$

Y-Q Ma

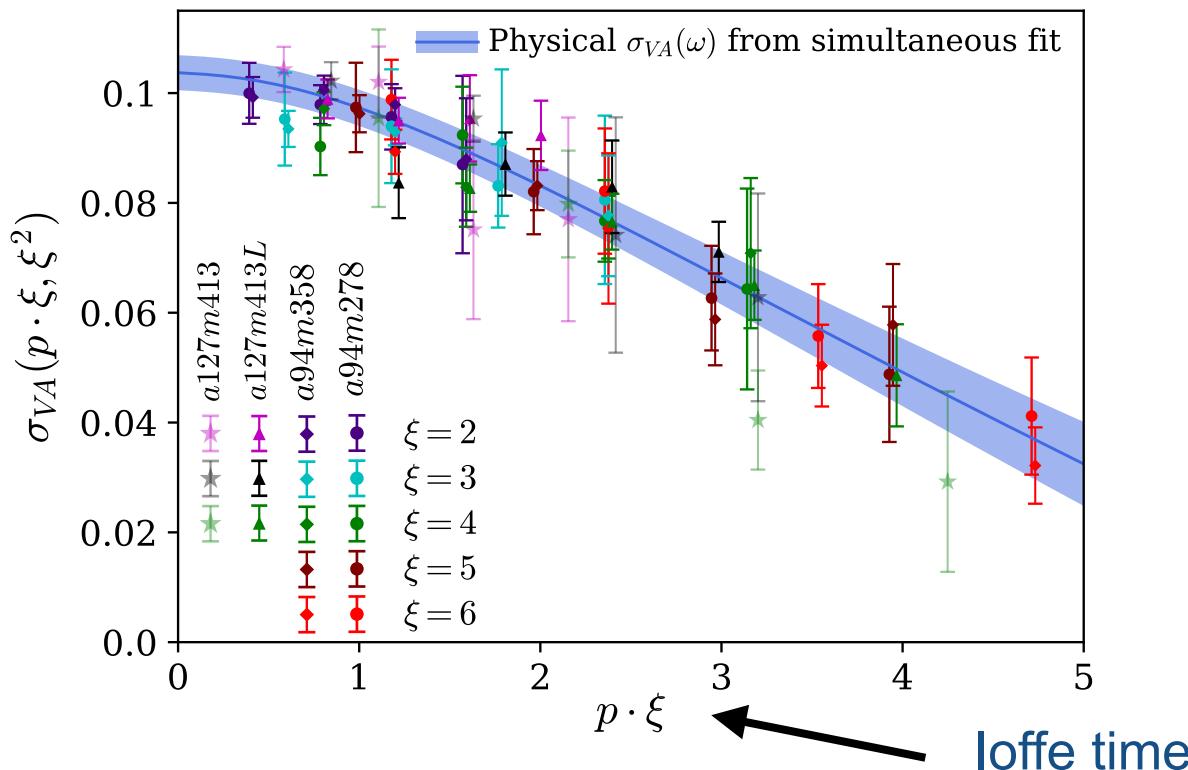
## Analogous matching to light-cone PDFs



- All approaches should give same after:
- Higher Twist
  - Discretization Uncertainties
  - .....

*All methods rely on reaching high momentum.*

# Lattice Cross Sections



$$\sigma_{VA}(\omega, \xi^2) = \sum_{k=0}^{k_{\max}=4} \lambda_k \tau^k + b_1 m_\pi + b_2 a + b_3 \xi^2 + b_4 a^2 p^2 + b_5 e^{-m_\pi(L-\xi)}$$

$$\tau = \frac{\sqrt{\omega_{\text{cut}} + \omega} - \sqrt{\omega_{\text{cut}}}}{\sqrt{\omega_{\text{cut}} + \omega} + \sqrt{\omega_{\text{cut}}}}$$

"Ioffe Time Distribution"

"Z-expansion fit"

# Inverse problem: extract PDF

“Inverse Problem” - ill-posed inverse Fourier transform.

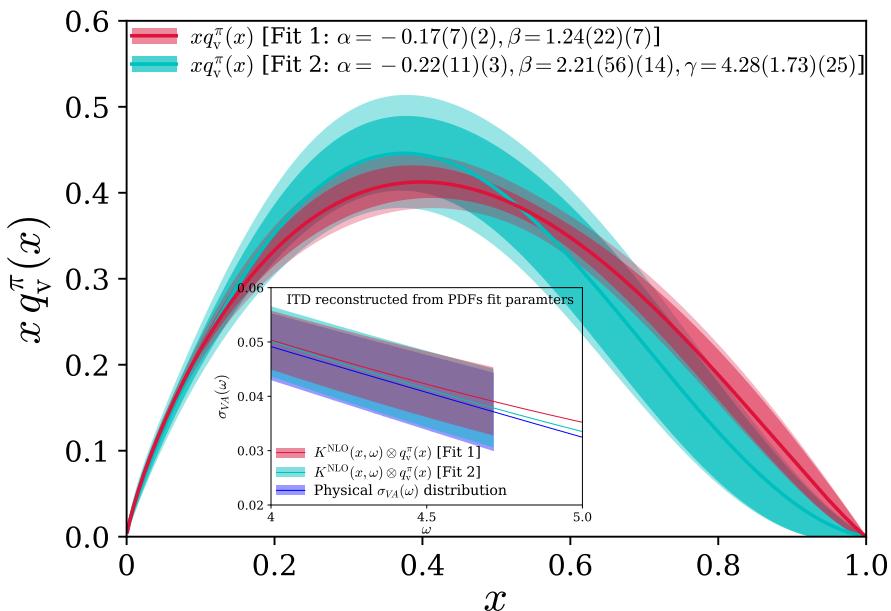
$$\sigma_n(\nu, \xi^2, P^2) = \sum_a \int_{-1}^1 \frac{dx}{x} f_a(x, \mu^2) K_n^a(x\nu, \xi^2, x^2 P^2, \mu^2) + \mathcal{O}(\xi^2 \Lambda_{\text{QCD}}^2)$$

Calculate on Lattice

Extract PDF?

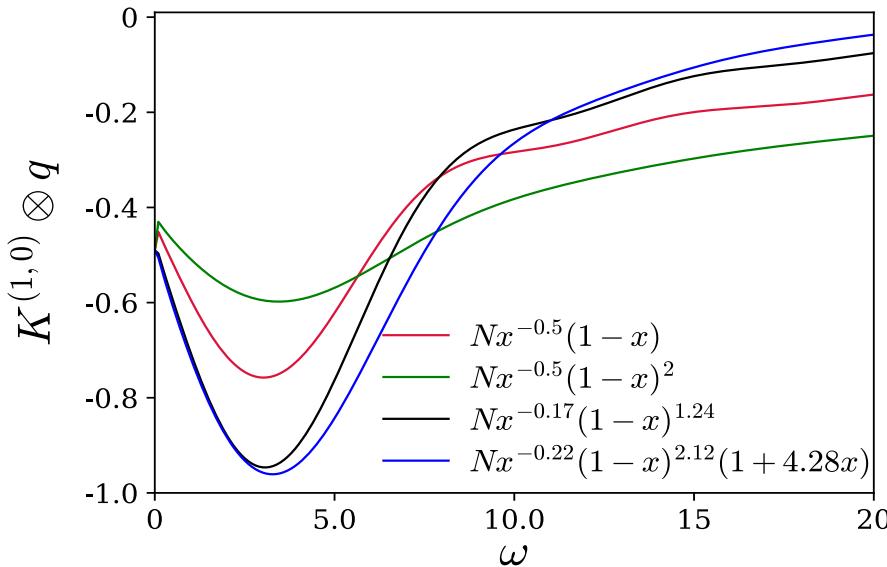
Calculate in PQCD

Similar challenge to global fitting community!



$$q_v^\pi(x) = \frac{x^\alpha (1-x)^\beta (1+\gamma x)}{B(\alpha+1, \beta+1) + \gamma B(\alpha+2, \beta+1)}$$

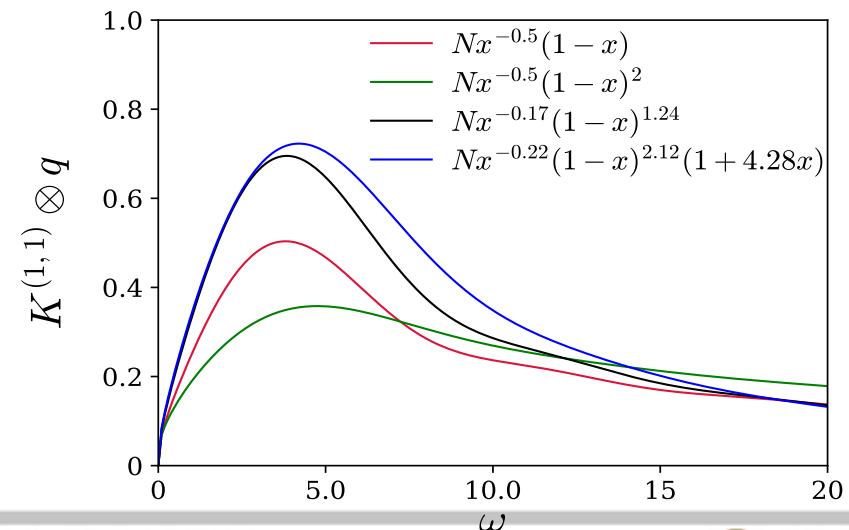
# NLO term well-controlled

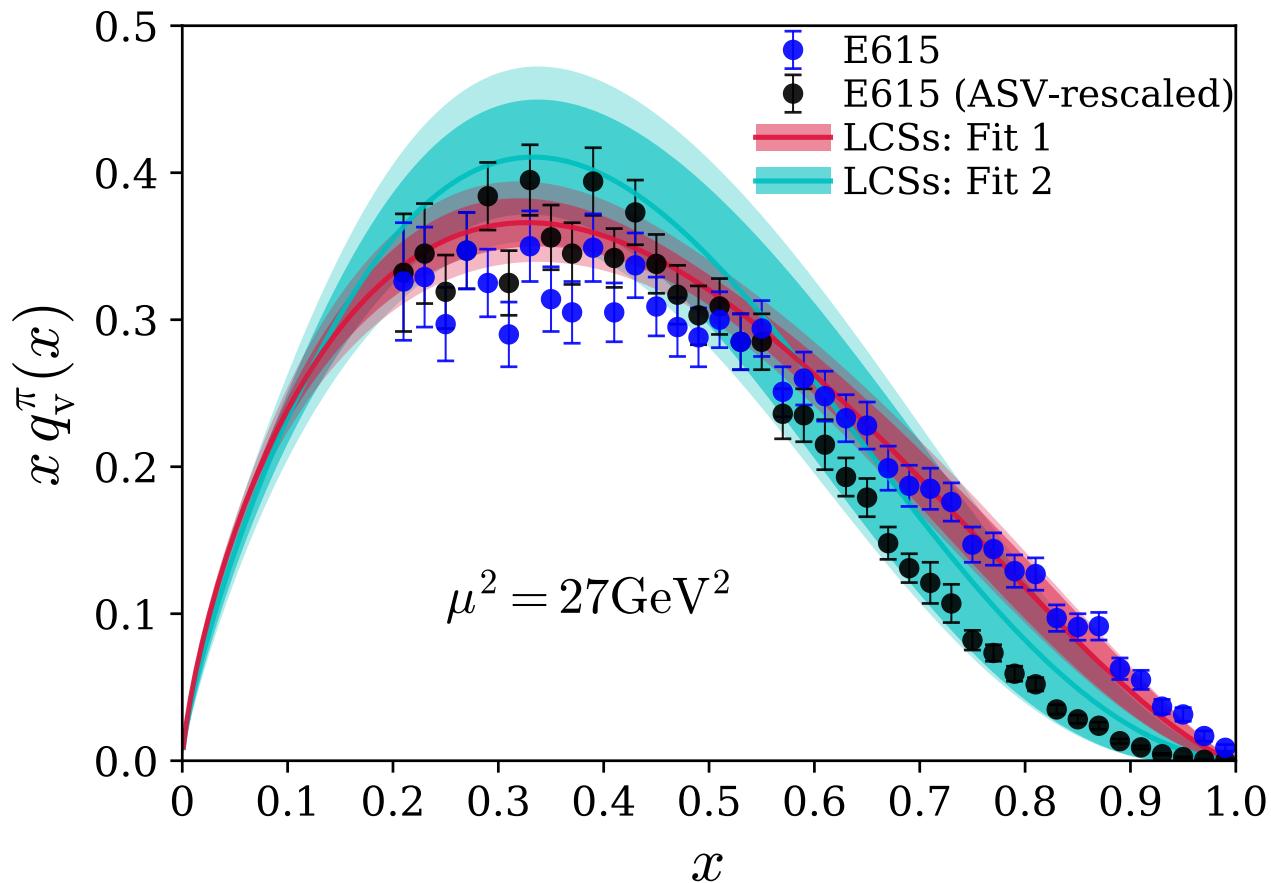


## Pion Valence Quark Distribution at Large $x$ from Lattice QCD

Raza Sabbir Sufian,<sup>1</sup> Colin Egerer,<sup>2</sup> Joseph Karpie,<sup>3</sup> Robert G. Edwards,<sup>1</sup> Balint Joo,<sup>1</sup> Yan-Qing Ma,<sup>4,5,6</sup> Kostas Orginos,<sup>1,2</sup> Jian-Wei Qiu,<sup>1</sup> and David G. Richards<sup>1</sup>

Sufian et al., Phys. Rev. D102, 05408 (2020)

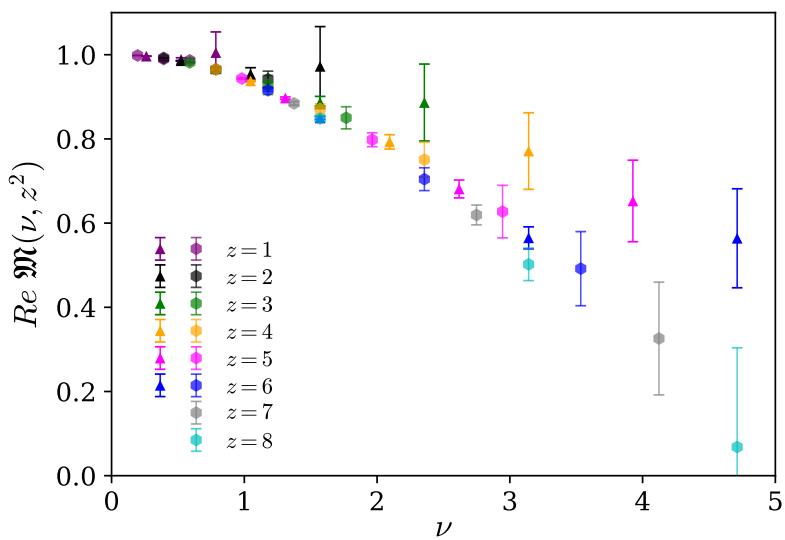




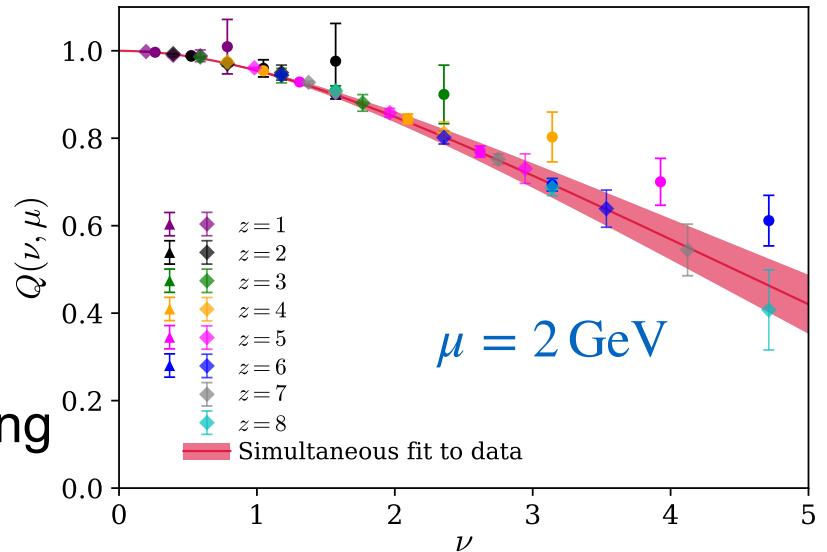
Determine large- $x$  behavior  $\rightarrow$  need for finer resolution and reach in Ioffe time.

# Pseudo-PDF Approach

As for GLCS, this is a short-distance expansion - have to map to scale  $\mu$



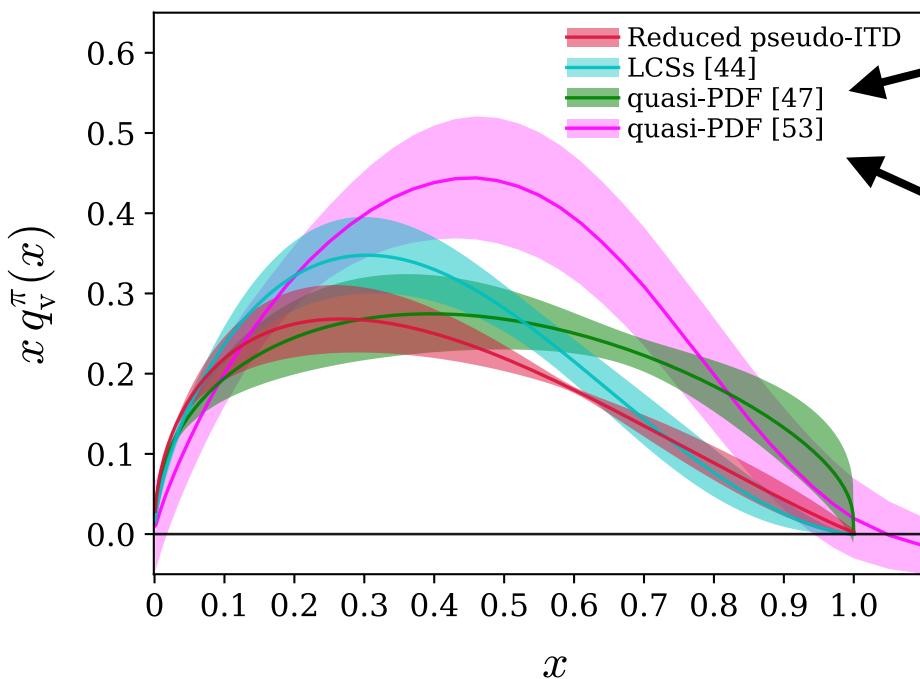
After  
Matching



B.Jóó et al., Phys. Rev. D 100, 114512 (2019).

# Pion PDF

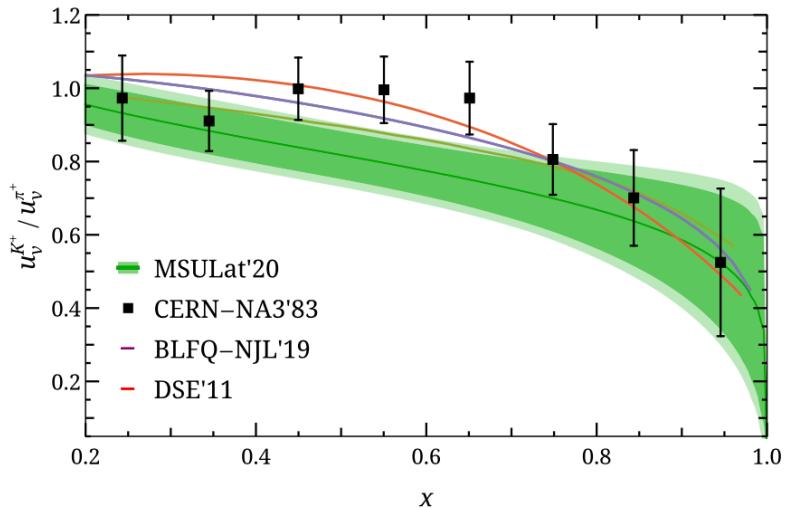
Crucial take-away - *should aim for consistency after systematic uncertainties under control*



T.Izubuchi *et al.*, Phys. Rev. D  
100, 034516

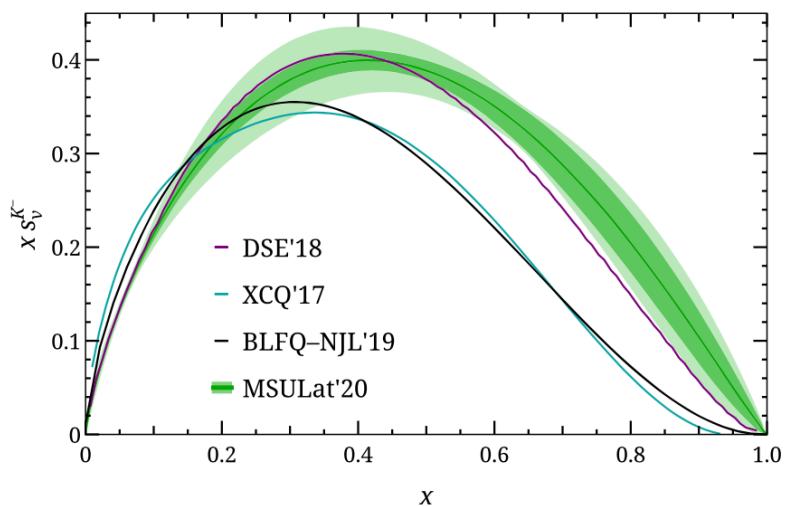
J-H Zhang *et al.*, Phys. Rev. D  
100, 034505

# What about the Kaon?



The hope is that some systematics cancel in ratio

H.W. Lin *et al.*, PHYSICAL REVIEW D 103,  
014516 (2021)



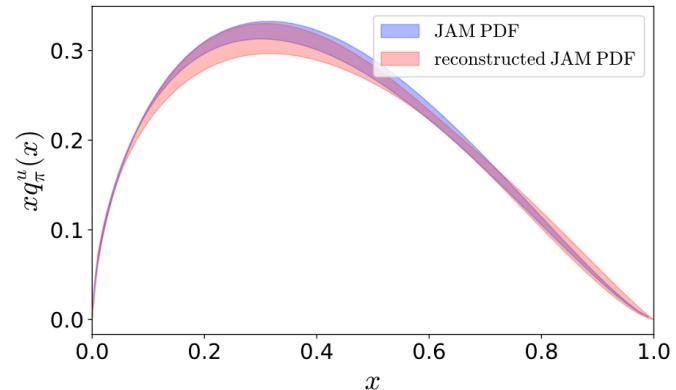
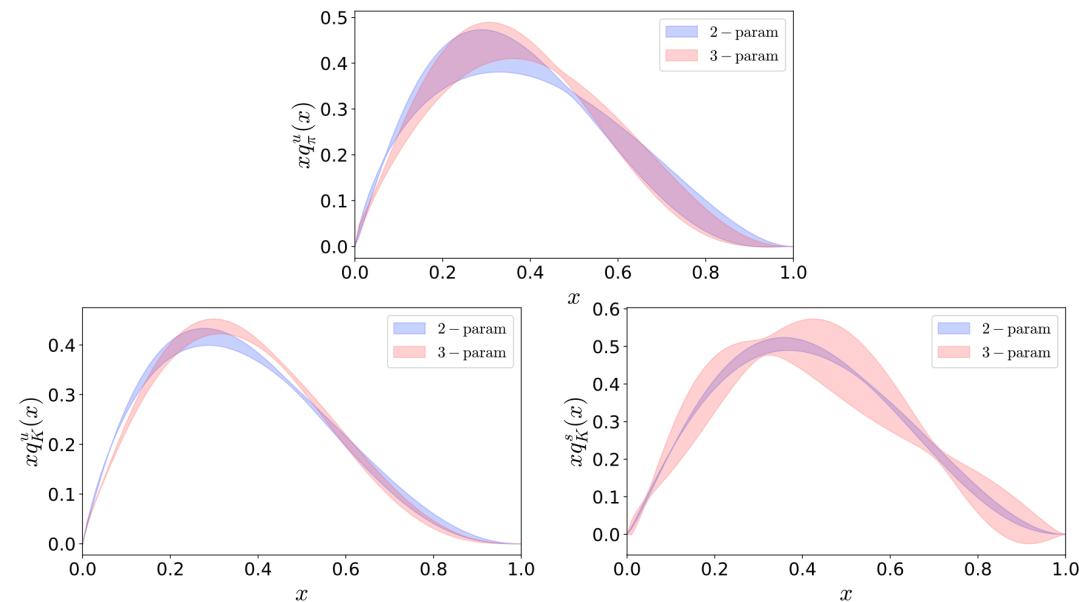
Need reach and resolution in Ioffe time

# Moments revisited.....

C. Alexandrou et al, arXiv:2104.02247

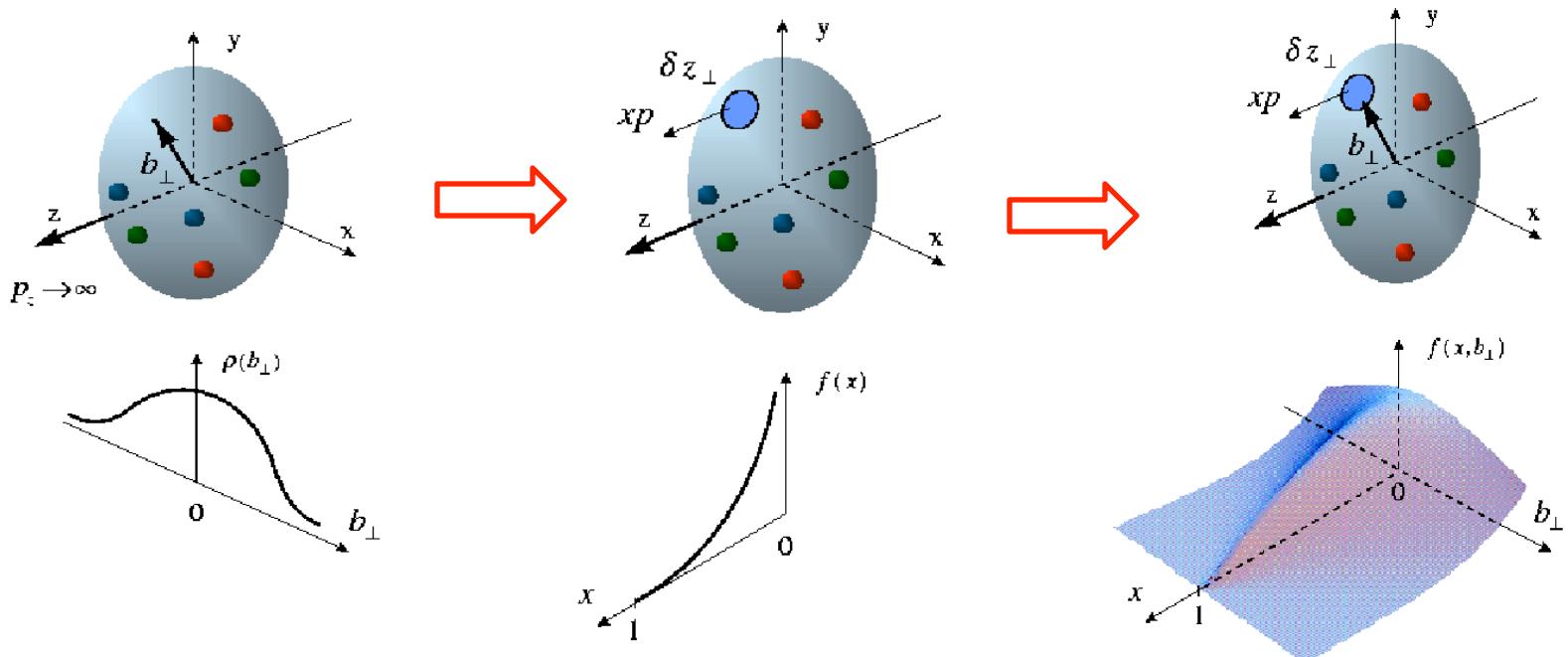
- Recall another approach - use *moments* to recover distributions.  
What can we say with  $\langle x \rangle$ ,  $\langle x^2 \rangle$  and  $\langle x^3 \rangle$ .

A rediscovered message - yes, providing you assume some (well-motivated) parametrization



Inverse problem - *requires additional input*  
LQCD + Expt - *similar challenges*

# Hadron Tomography

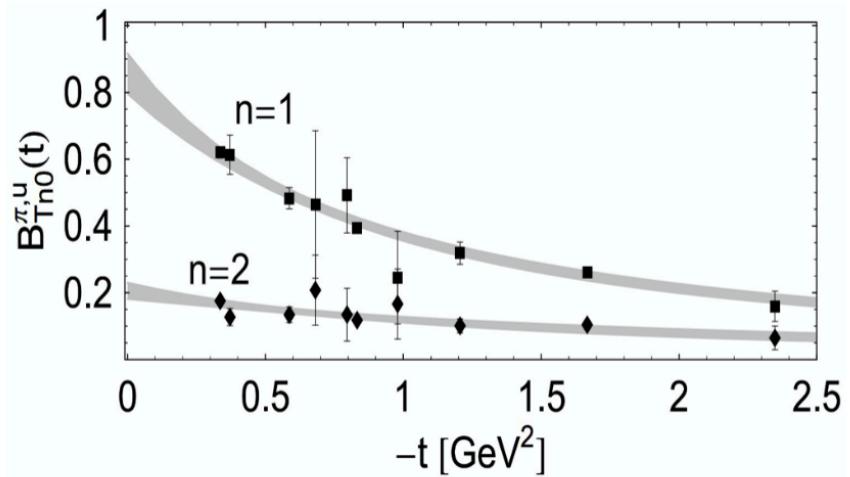
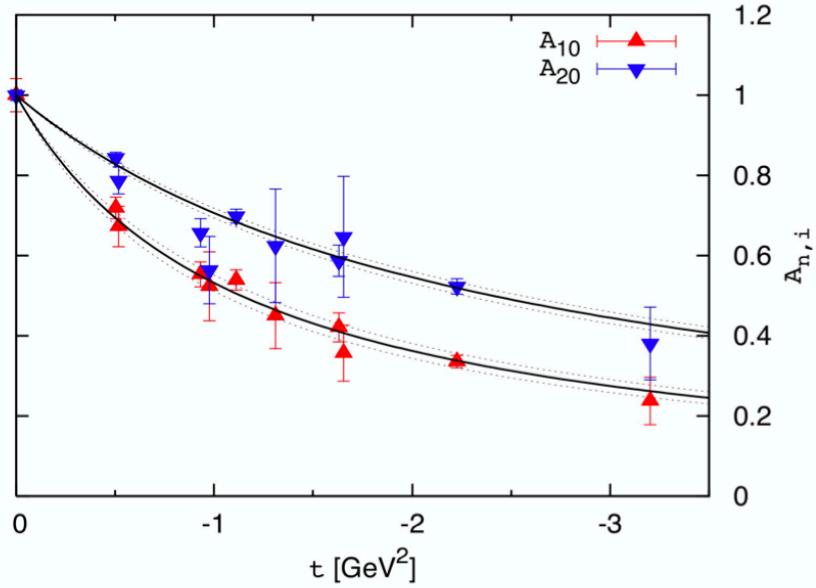


**Form Factors**  
transverse quark  
distribution in  
Coordinate space

**Structure Functions**  
longitudinal  
quark distribution  
in momentum space

**GPDs**  
Fully-correlated  
quark distribution in  
both coordinate and  
momentum space

# Generalized Form Factors



$$\begin{aligned}
 & \langle \pi^+(P + \Delta/2) | \bar{u}(0) \gamma^{\{\mu} i D^{\mu_1} i D^{\mu_2} \dots i D^{\mu_n\}} u(0) | \pi^+(P - \Delta/2) \rangle \\
 &= 2 P^{\{\mu} P^{\mu_1} \dots P^{\mu_n\}} A_{n+1,0}(\Delta^2) + 2 \sum_{i=1, \text{odd}}^n \Delta^{\{\mu} \Delta^{\mu_1} \dots \Delta^{\mu_i} P^{\mu_{i+1}} \dots P^{\mu_n} A_{n+1,i+1}(\Delta^2), \tag{1}
 \end{aligned}$$

GFFs can be related to distributions of mass and OAM!

Brommel et al., PoS LAT 2005, 360; Phys. Rev. Lett. 101, 122001 (2008)

# Hadron Tomography

- Three-dimensional imaging of hadrons

Generalized Parton Distributions

Transverse Momentum Dependent dist

Thanks, Colin Egerer

A.Radyushkin, PRD100, 116011 (2019)

pseudo-GITD

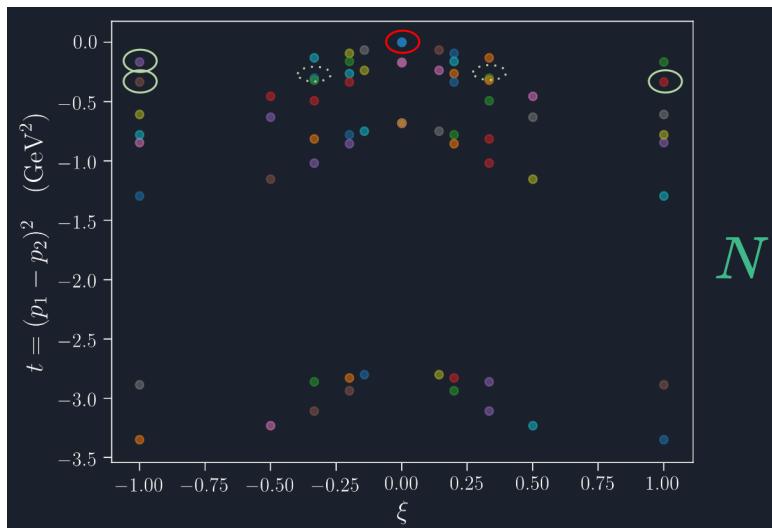
pseudo-GPD

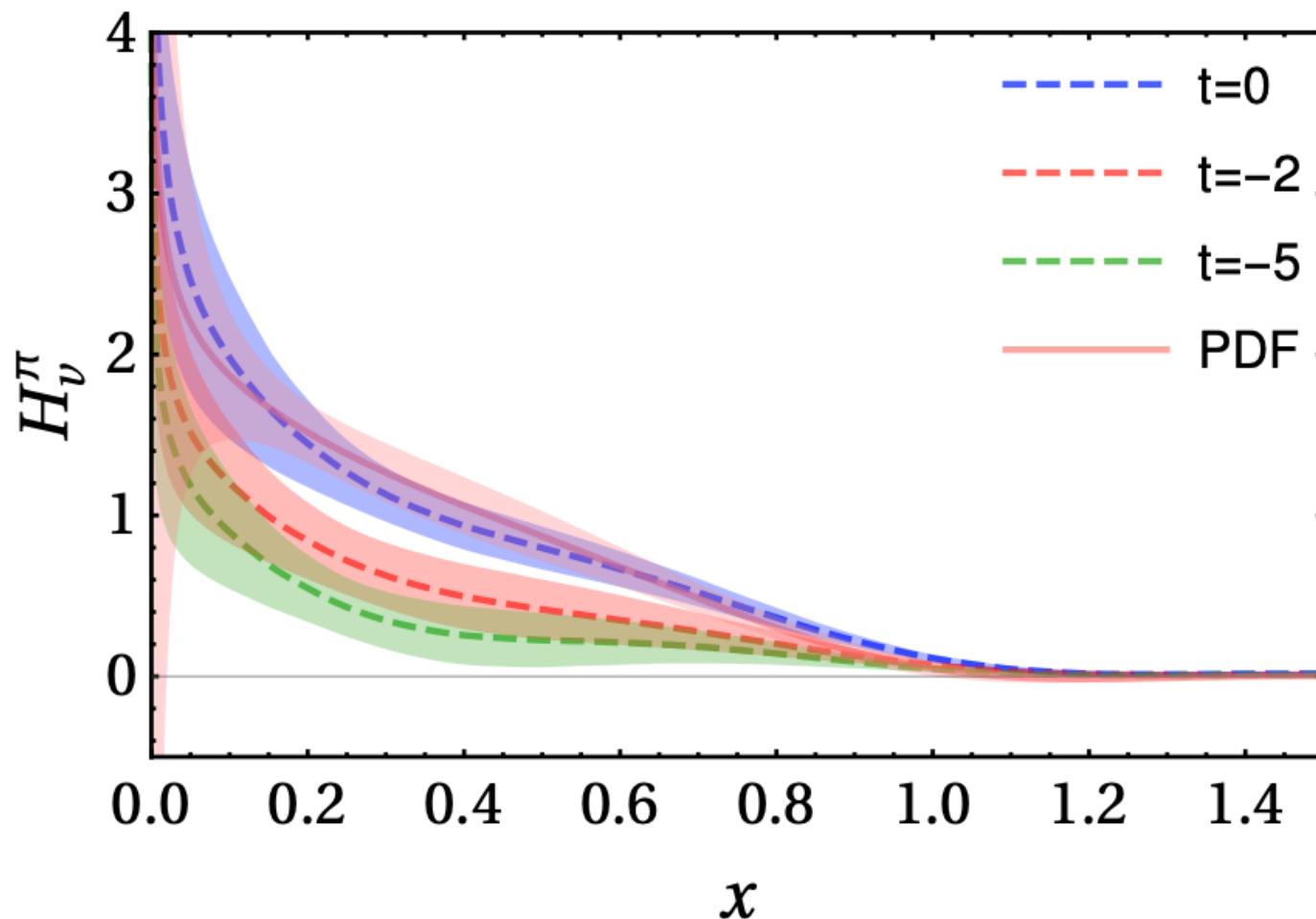
$$\mathcal{M}(\nu, \xi, t; z^2) = e^{i\xi\nu} \int_{-1}^1 dx e^{ix\nu} \mathcal{H}(x, \xi, t, z^2)$$

$$\nu = (\nu_1 + \nu_2)/2$$

$$\xi = \frac{\nu_1 - \nu_2}{\nu_1 + \nu_2}$$

Lattice is complementary  
to experiment and  
essential!

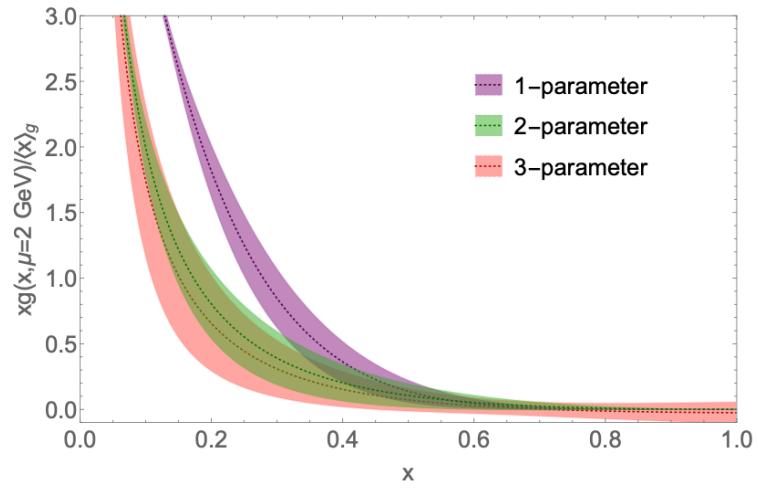
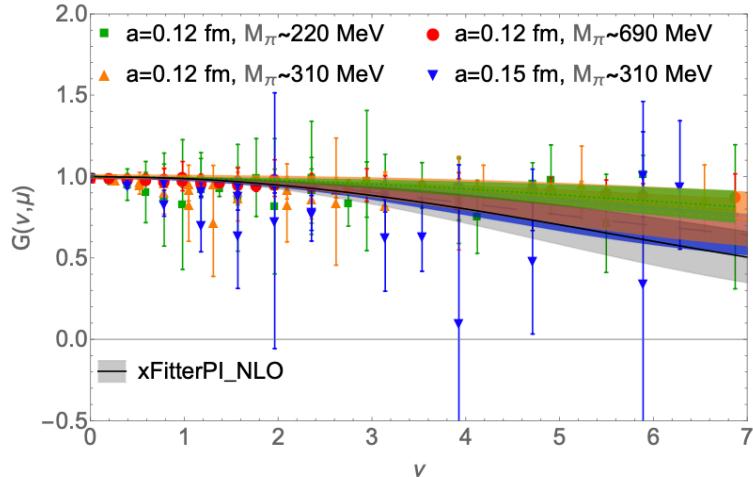




J.-W. Chen, H.-W. Lin and J.-H. Zhang, Nucl. Phys. B 952, 114940

# Flavor-singlet distributions

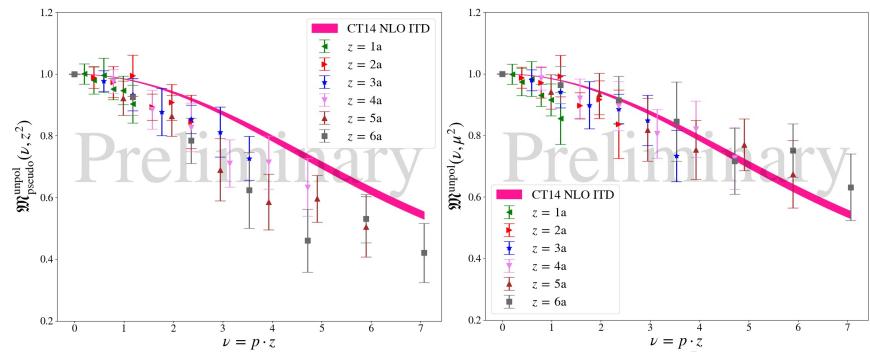
- Much more challenging - severe noise-to-signal ratio



Ioffe-Time Distribution

Preliminary results for Nucleon...

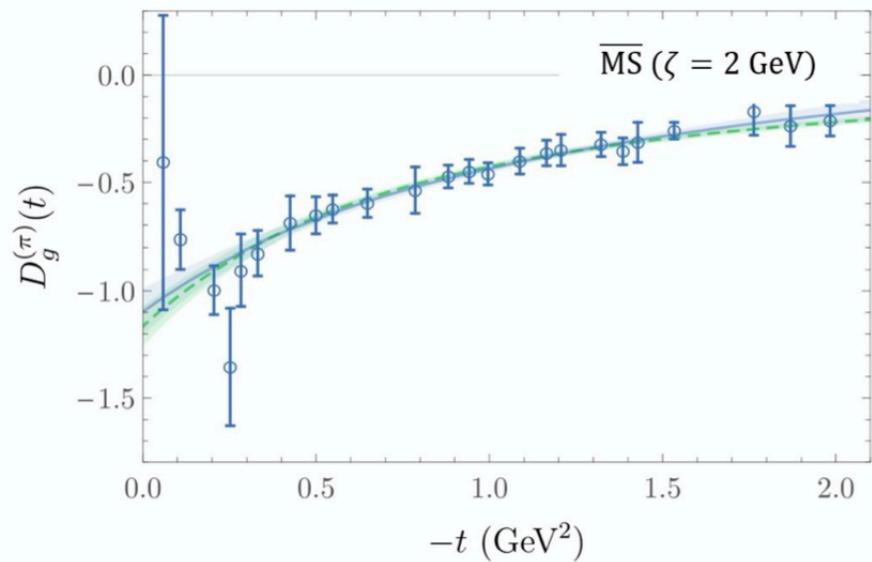
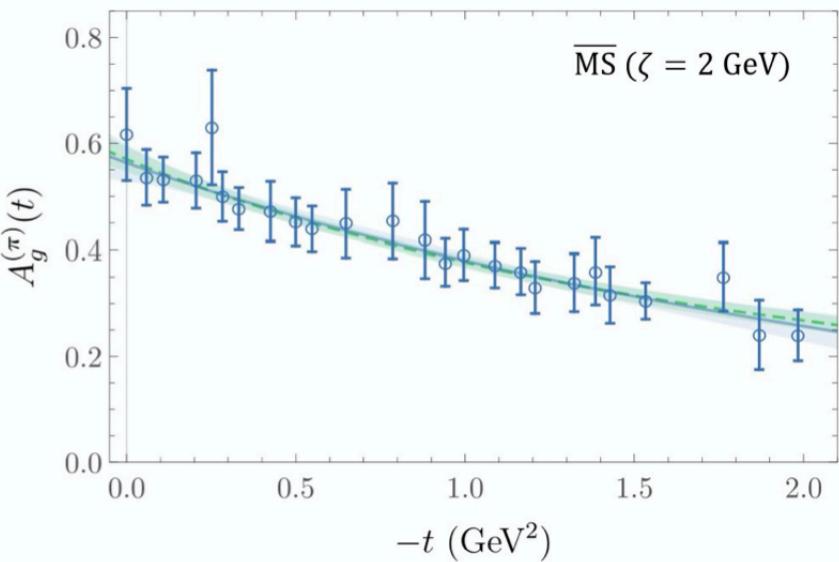
Thanks, Tanjib Khan



Tanjib Khan

13/14

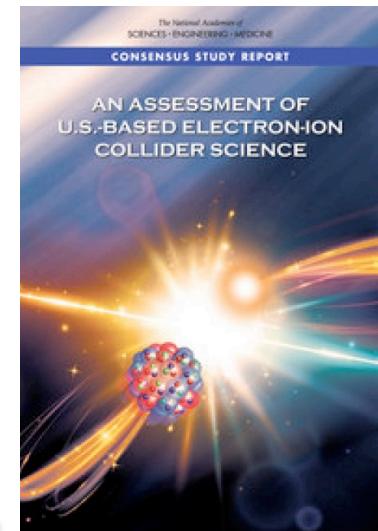
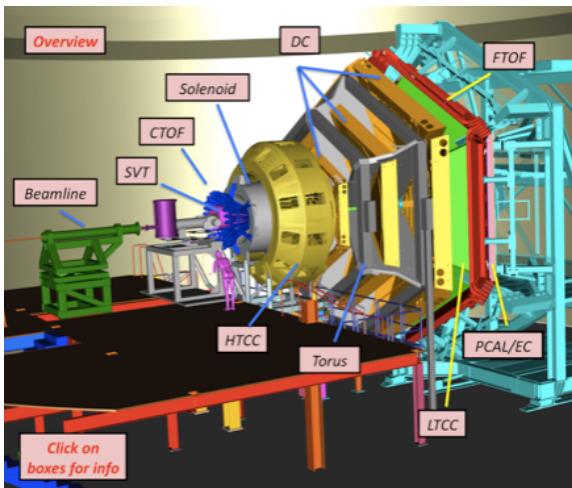
P. Shanahan and W. Detmold, Phys. Rev. D 99, 014511 (2019)



$$\langle \pi(P + \Delta/2) | G_{a\alpha}^{\{\mu} G_a^{\nu\alpha\}} | \pi(P - \Delta/2) \rangle = 2P^\mu P^\nu A_g(\Delta^2) + \frac{1}{2} \Delta^\mu \Delta^\nu D_g(\Delta^2).$$

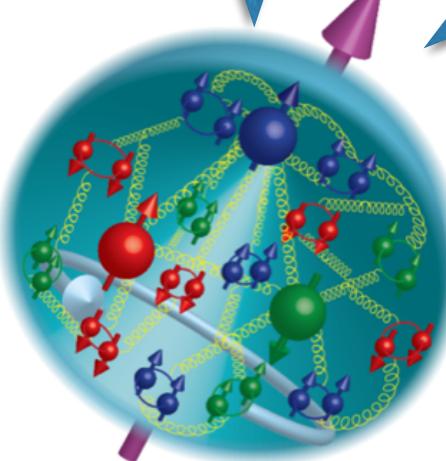
Gives rise to gluon gravitational form factor!

# A New Opportunity in Hadron Structure



Xiangdong Ji

*Lattice QCD*



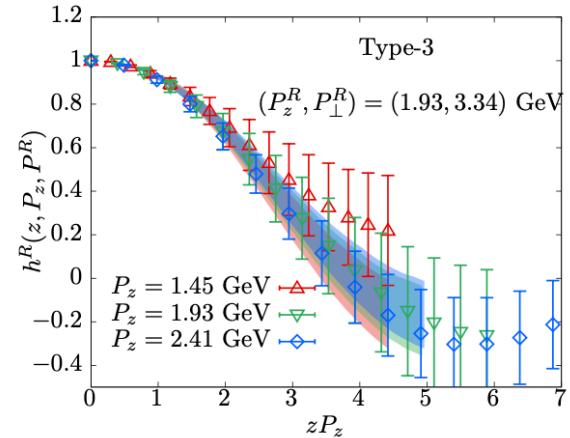
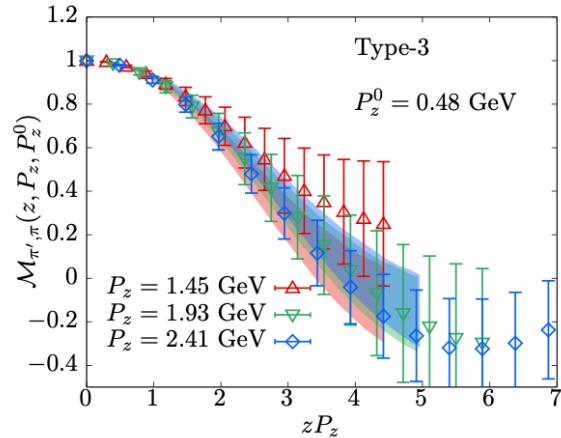
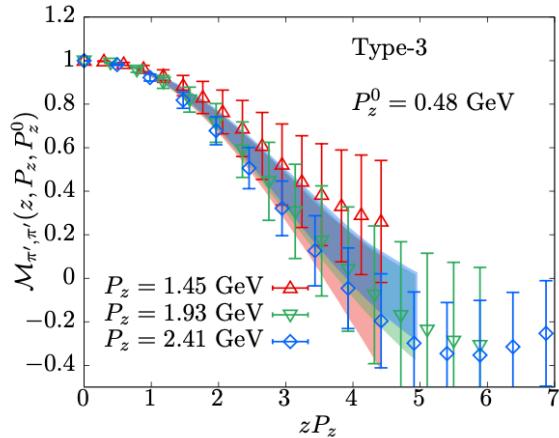
*Future Electron-Ion Collider*

*3D Image of nucleon and nuclei at the femtoscale*

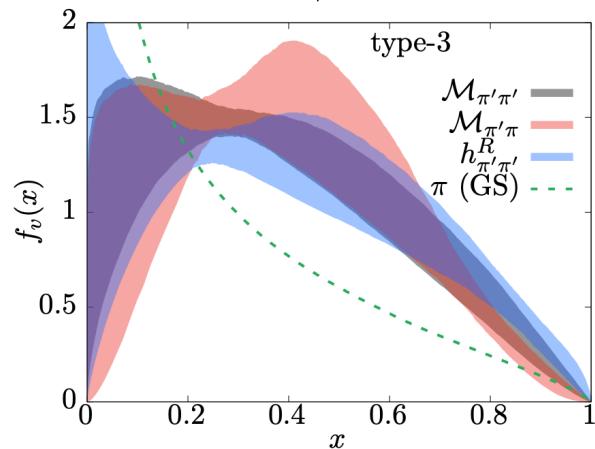
# Summary

- Revolution in the study of x-dependent measures of hadron structure
  - Impact global fitting community?
  - First-Principles calculation
- Solution of inverse problem: common to all attempts to extract PDFs.  
Appeal to global fitting community. *Important activity within CNF*
- To control systematics
  - **fine lattices** - to ensure in perturbative regime
  - **large momenta** - to provide range in Ioffe time
  - **Combined LaMET/pPDF/LCS** - further control over systematics
- New areas:
  - Flavor-singlet PDFs
  - GPD's in pseudo-PDF approach: A.Radyushkin, Phys. Rev. D 100, 116011 (2019).
  - Structure of Excited States

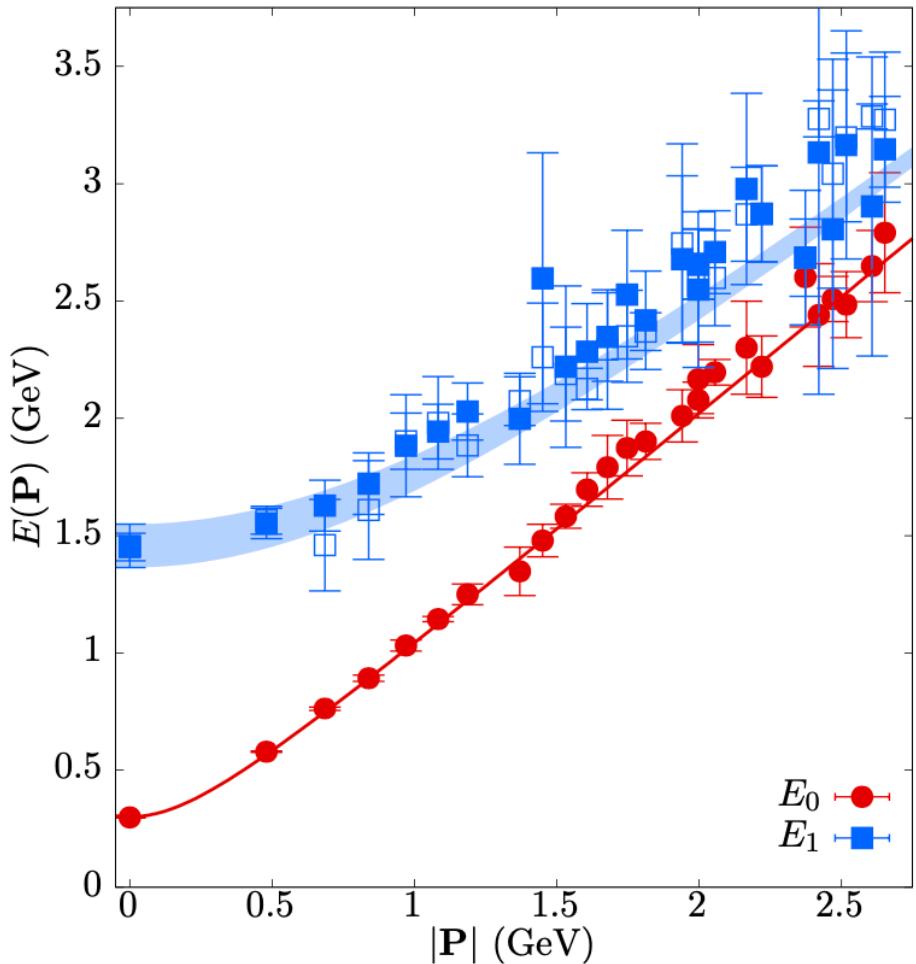
X.Gao *et al.*, arXiv:2101.11632



$$f(x) = Nx^\alpha(1-x)^\beta$$



# Reaching high momentum...



X.Gao *et al.*, arXiv:2101.11632