Spin tensor and pseudo-gauges in relativistic nuclear collisions

Nora Weickgenannt

NW, E. Speranza, X.-l. Sheng, Q. Wang. D. H. Rischke, arXiv:2005.01506 (accepted for PRL)
E. Speranza, NW, EPJA 57 (2021) 5, 155

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- **Spin hydrodynamics is based on conserved quantities:**

energy-momentum tensor

$$\partial_\mu T^{\mu\nu} = 0$$

+ **total** angular-momentum tensor

$$J^{\lambda,\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu} + \hbar S^{\lambda,\mu\nu}$$

orbital part — spin tensor

$$\partial_\lambda J^{\lambda,\mu\nu} = 0 \quad \Longrightarrow \quad \hbar \partial_\lambda S^{\lambda,\mu\nu} = T^{[\nu\mu]}$$

$$a^{[\mu} b^{\nu]} \equiv a^\mu b^\nu - a^\nu b^\mu$$

- Given a Lagrangian density $\mathcal{L}$, how to calculate $T^{\mu\nu}$ and $S^{\lambda,\mu\nu}$?

Textbook: general relativity

- Physical objects determined by space-time geometry

- Energy-momentum tensor: variation with respect to metric tensor

$$T^{\mu\nu} = -2\frac{\delta \mathcal{L}}{\delta g_{\mu\nu}}$$

- Spin tensor: ?

- Conserved currents obtained from Noether's theorem, consider spinor fields
- Energy-momentum tensor: invariance under space-time translations

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \partial^\nu \psi + \partial^\nu \bar{\psi} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} - g^{\mu\nu} \mathcal{L}$$

- Spin tensor: invariance under rotations

$$S^{\lambda,\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \psi)} f^{\mu\nu} \psi - \bar{\psi} f^{\mu\nu} \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \bar{\psi})}$$

generators of rotation: $f^{\mu\nu} = -\frac{i}{2}\sigma^{\mu\nu}$

- Apply Noether's theorem to Dirac Lagrangian:

$$\mathcal{L}_D(x) = \frac{i\hbar}{2}\,\bar{\psi}(x)\gamma\cdot\overleftrightarrow{\partial}\,\psi(x) - m\,\bar{\psi}(x)\psi(x)$$

- Spin tensor:

$$S_C^{\lambda,\mu\nu} = -\frac{1}{2}\epsilon^{\lambda\mu\nu\alpha}\bar{\psi}\gamma_\alpha\gamma_5\psi$$

- Energy-momentum tensor:

$$T_C^{\mu\nu} = \frac{i\hbar}{2}\bar{\psi}\gamma^\mu\overleftrightarrow{\partial}^\nu\psi$$

$\Longrightarrow$ not symmetric in μ and ν!

- Why is this inconsistent with general relativity?

- Definition of $T^{\mu\nu}$ and $S^{\lambda,\mu\nu}$ depends on choice of pseudo-gauge.
  F. W. Hehl, Rept. Math. Phys. 9, 55 (1976)
  F. Becattini, W. Florkowski, and E. Speranza, PLB789, 419 (2019)
  E. Speranza, NW, EPJA 57 (2021) 5, 155
  L. Tinti, W. Florkowski, arXiv:2007.04029
  S. Li, M. Stephanov, H.-U. Yee, arXiv: 2011.12318
- Pseudo-gauge transformation:

$$T'^{\mu\nu} = T_C^{\mu\nu} + \frac{\hbar}{2}\partial_\lambda(\Phi^{\lambda,\mu\nu} + \Phi^{\nu,\mu\lambda} + \Phi^{\mu,\nu\lambda}),$$
$$S'^{\lambda,\mu\nu} = S_C^{\lambda,\mu\nu} - \Phi^{\lambda,\mu\nu} + \hbar\,\partial_\rho Z^{\mu\nu,\lambda\rho}$$

$\Longrightarrow$ equations of motion invariant

$$\partial_\mu T'^{\mu\nu} = 0 \qquad \partial_\lambda J'^{\lambda,\mu\nu} = 0$$

and global charges invariant

$$P^\nu \equiv \int_\Sigma d\Sigma_\mu T^{\mu\nu} = \int_\Sigma d\Sigma_\mu T'^{\mu\nu}$$
$$J^{\mu\nu} \equiv \int_\Sigma d\Sigma_\lambda J^{\lambda,\mu\nu} = \int_\Sigma d\Sigma_\lambda J'^{\lambda,\mu\nu}$$

Pseudo-gauge transformations change global spin

Different splitting into spin and orbital angular momentum

Canonical vs. Belinfante pseudo-gauge

- Currents derived from Noether's theorem:
  Canonical energy-momentum tensor $T_C^{\mu\nu}$ and spin tensor $S_C^{\lambda,\mu\nu}$

- Apply pseudo-gauge transformation with

$$\Phi^{\lambda,\mu\nu} = S_C^{\lambda,\mu\nu}, \qquad Z^{\mu\nu,\lambda\rho} = 0$$

  $\Longrightarrow$ Belinfante currents

$$T_B^{\mu\nu} = \frac{i\hbar}{4}\left(\bar{\psi}\gamma^\mu \overleftrightarrow{\partial}^\nu \psi + \bar{\psi}\gamma^\nu \overleftrightarrow{\partial}^\mu \psi\right)$$

$$S_B^{\lambda,\mu\nu} = 0$$

- Spin tensor vanishes, full angular momentum contained in orbital part

- Belinfante energy-momentum tensor couples to gravity in conventional general relativity
  $\Longrightarrow$ Traditionally considered as "physical" energy-momentum tensor
  $\Longrightarrow$ "Physical" pseudo-gauge in quantum theory?

- "Physical" currents $\rightarrow$ enter observables

- Observable for polarization in heavy-ion collisions:

Pauli-Lubanski vector for particles with momentum p

F. Becattini, V. Chandra, L. Del Zanna, E. Grossi, AP338, 32 (2013)
F. Becattini, arXiv:2004.04050
E. Speranza, NW, EPJA 57 (2021) 5, 155
L. Tinti, W. Florkowski, arXiv:2007.04029

$$\Pi_\mu = -\frac{1}{2m}\epsilon_{\mu\nu\alpha\beta}p^\nu \int_\Sigma d\Sigma_\lambda J^{\lambda,\alpha\beta}$$

total angular momentum

$\Longrightarrow$ Form of Pauli-Lubanski vector independent of pseudo-gauge

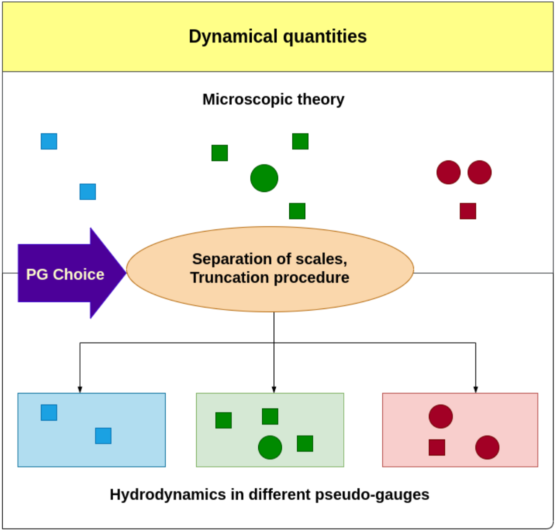
Are all pseudo-gauges equally "physical"?

- Π^μ independent of pseudo-gauge
- However: need to calculate ensemble average of operator

$$\langle \Pi^\mu \rangle \equiv \mathrm{Tr}(\rho\, \Pi^\mu)$$

- Density matrix ρ not known
- Method to (approximately) calculate ensemble average can give different results for different pseudo-gauges
  F. Becattini, W. Florkowski, and E. Speranza, PLB789, 419 (2019)
  F. Becattini, NPA 1005 (2021) 121833
  E. Speranza, NW, EPJA 57 (2021) 5, 155
  K. Fukushima, S. Pu, PLB 817 (2021) 136346
  A. Das, W. Florkowski, R. Ryblewski, R. Singh, PRD 103 (2021) 9, L091502
- $\langle \Pi^\mu \rangle$ dependent on pseudo-gauge

- Hydrodynamics: effective theory, never contains all microscopic information



- Obtain dynamics of densities from equations of motion
  $\Longrightarrow$ Different dynamical quantities for different pseudo-gauges
  $\Longrightarrow$ Dynamics depends on pseudo-gauge

- Main idea of spin hydrodynamics:
  Promote spin tensor to additional dynamical variable

  W. Florkowski, B. Friman, A. Jaiswal, and E. Speranza, PRC 97, no. 4, 041901 (2018)
  W. Florkowski, B. Friman, A. Jaiswal, R. Ryblewski, and E. Speranza, PRD 97, no. 11, 116017 (2018)
  W. Florkowski, F. Becattini, and E. Speranza, APB 49, 1409 (2018)

- Obviously not possible for Belinfante spin tensor
  $\Longrightarrow$ Spin dynamics cannot be described

- $T_C^{\mu\nu}$ not symmetric for free fields
  $\Longrightarrow$ canonical spin tensor not conserved

✗ Physical picture: spin density changed only by interactions

✗ Canonical global spin

$$S_C^{\mu\nu} \equiv \int_\Sigma d\Sigma_\lambda S_C^{\lambda,\mu\nu}$$

no Lorentz tensor

- **In general:** Hypersurface-integrated quantities transform as tensors under Lorentz transformations only if integrand conserved
  For proof see e.g.
  E. Speranza, NW, EPJA 57 (2021) 5, 155

Hilgevoord-Wouthuysen (HW) pseudo-gauge

- Idea for free fields:
  Apply Noether's theorem to Klein-Gordon Lagrangian for spinors
  J. Hilgevoord and S. Wouthuysen, Nuclear Physics 40, 1 (1963)

$$\mathcal{L}_{KG} = \frac{1}{2m}(\hbar^2 \partial_\mu \bar{\psi} \partial^\mu \psi - m^2 \bar{\psi}\psi)$$

- Result:

$$T^{\mu\nu}_{HW} = \frac{\hbar^2}{2m}\left(\partial^\mu \bar{\psi} \partial^\nu \psi + \partial^\nu \bar{\psi} \partial^\mu \psi\right) - g^{\mu\nu}\mathcal{L}_{KG}$$

$$S^{\lambda,\mu\nu}_{HW} = \frac{i\hbar^2}{4m} \bar{\psi}\sigma^{\mu\nu} \overleftrightarrow{\partial}^\lambda \psi$$

- Energy-momentum tensor symmetric for free fields.
- Conserved (nonzero) spin tensor.
- There exists pseudo-gauge transformation from canonical to HW tensors
- Physical interpretation?

Pseudo-gauge and frame choice I

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- Nonrelativistic spin operator given by Pauli matrices: $\frac{1}{2}\boldsymbol{\sigma}$
- How to generalize to relativistic theory?
- Spin vector $\boldsymbol{S}$ connected to global spin by

$$S^{ij} = \epsilon^{ijk} S^k .$$

  Obviously no Lorentz tensor.
- Make this covariant:

$$S_n^{\mu\nu} = -\epsilon^{\mu\nu\alpha\beta} n_\alpha S_\beta$$

  Spin defined in the frame moving with four-velocity $n^\mu \Longleftrightarrow n_\mu S_n^{\mu\nu} = 0$.
- Different choices of pseudo-gauge: different choices of frame vector.
  M. H. L. Pryce, Proc. Roy. Soc. Lond., A195:62–81, 1948
  C. Lorcé, Eur. Phys. J. C (2018) 78:785
  E. Speranza, NW, EPJA 57 (2021) 5, 155
- One preferred reference frame for massive particles: rest frame.

Global spin from spin tensor:

$$S^{\mu\nu} \equiv \int d^3x \, S^{0,\mu\nu}$$

- Canonical choice:
  $\Longrightarrow$ Spin tensor not conserved for free fields
  $\Longrightarrow$ Global spin no Lorentz tensor
  $\Longrightarrow$ Equal to nonrelativistic spin in any frame,

$$S_C^{0\nu} = 0, \qquad n_C^\mu = (1, 0, 0, 0).$$

- HW choice:
  $\Longrightarrow$ Spin tensor conserved for free fields
  $\Longrightarrow$ Global spin is Lorentz tensor
  $\Longrightarrow$ Equal to nonrelativistic spin in rest frame,

$$p_\mu S_{HW}^{\mu\nu} = 0, \qquad n_{HW}^\mu = \frac{1}{m} p^\mu.$$

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NW, E. Speranza, X.-l. Sheng, Q. Wang, and D.H. Rischke, arXiv:2005.01506, 2103.04896

- Framework:
  Quantum field theory $\Longrightarrow$ Wigner function $\Longrightarrow$ kinetic theory $\Longrightarrow$ hydrodynamics
  See talk by Enrico Speranza
- Phase-space distribution function $f(x,p,\mathfrak{s})$, depends on spin variable $\mathfrak{s}^\mu$
- Boltzmann equation

$$p\cdot\partial f(x,p,\mathfrak{s})=\mathfrak{C}[f]$$

- Nonlocal collision term

$$\mathfrak{C}[f]=\int d\Gamma_1 d\Gamma_2 d\Gamma'\,\mathcal{W}\,[f(x+\Delta_1,p_1,\mathfrak{s}_1)\times f(x+\Delta_2,p_2,\mathfrak{s}_2)-f(x+\Delta,p,\mathfrak{s})f(x+\Delta',p',\mathfrak{s}')]$$

Particle positions displaced by Δ^μ

- Canonical energy-momentum tensor

$$T_C^{\mu\nu} = \int d\Gamma \, p^\nu \left(p^\mu + \frac{\hbar}{2} \Sigma_{\mathfrak{s}}^{\mu\lambda} \partial_\lambda \right) f(x,p,\mathfrak{s}) + \mathcal{O}(\hbar^2)$$

- Canonical spin tensor

$$S_C^{\lambda,\mu\nu} = \int d\Gamma \, \left(p^\lambda \Sigma_{\mathfrak{s}}^{\mu\nu} + p^\mu \Sigma_{\mathfrak{s}}^{\nu\lambda} + p^\nu \Sigma_{\mathfrak{s}}^{\lambda\mu} \right) f(x,p,\mathfrak{s})$$

Dipole-moment tensor

$$\Sigma_{\mathfrak{s}}^{\mu\nu} \equiv -\frac{1}{m} \epsilon^{\mu\nu\alpha\beta} p_\alpha \mathfrak{s}_\beta$$

- Global equilibrium:

$$\hbar \partial_\lambda S_{C,eq}^{\lambda,\mu\nu} = -T_{C,eq}^{[\mu\nu]} = -\frac{1}{(2\pi\hbar)^3} \frac{\hbar^2}{4} \int dP \, p^{[\nu} \varpi^{\mu]\lambda} p^\rho \varpi_{\lambda\rho} e^{-\beta\cdot p} + \mathcal{O}(\hbar^3)$$

Thermal vorticity $\varpi^{\mu\nu}$

Canonical spin tensor not conserved in global equilibrium

Inconsistent with physical picture

NW, E. Speranza, X.-l. Sheng, Q. Wang, and D.H. Rischke, arXiv:2005.01506

- Generalize HW pseudo-gauge transformation to interacting case:
- Choice of $\Phi^{\lambda,\mu\nu}$:
  - Recover HW tensors for zero interactions.
  - Obtain physically meaningful equations of motion (see next slide).
- Result:

$$T^{\mu\nu}_{HW} = \int d\Gamma p^\mu p^\nu f(x,p,\mathfrak{s}) + \mathcal{O}(\hbar^2)\,,$$

$$S^{\lambda,\mu\nu}_{HW} = \int d\Gamma p^\lambda \left(\frac{1}{2}\Sigma^{\mu\nu}_{\mathfrak{s}} - \frac{\hbar}{4m^2} p^{[\mu}\partial^{\nu]}\right) f(x,p,\mathfrak{s}) + \mathcal{O}(\hbar^2)\,.$$

- Global equilibrium:

$$S^{\lambda,\mu\nu}_{HW,eq} = \frac{\hbar}{4} u^\lambda \varpi^{\mu\nu} n^{(0)}$$

Particle density $n^{(0)}$

Form of spin tensor widely used in effective approaches

W. Florkowski, B. Friman, A. Jaiswal, and E. Speranza, PRC 97, no. 4, 041901 (2018)
K. Hattori, M. Hongo, X.-G. Huang, M. Matsuo, H. Taya, PLB 795 (2019) 100-106
S. Li, M. Stephanov, H.-U. Yee, arXiv: 2011.12318
A.D. Gallegos, U. Gürsoy, A. Yarom, arXiv: 2101.04759

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- Using Boltzmann equation

$$\partial_\mu T^{\mu\nu}_{\text{HW}} = \int d\Gamma\, p^\nu\, \mathfrak{C}[f] = 0\,,$$

$$\hbar\, \partial_\lambda S^{\lambda,\mu\nu}_{\text{HW}} = \int d\Gamma\, \frac{\hbar}{2} \Sigma^{\mu\nu}_{\mathfrak{s}}\, \mathfrak{C}[f] = T^{[\nu\mu]}_{\text{HW}}\,.$$

- Collisional invariant: p^μ

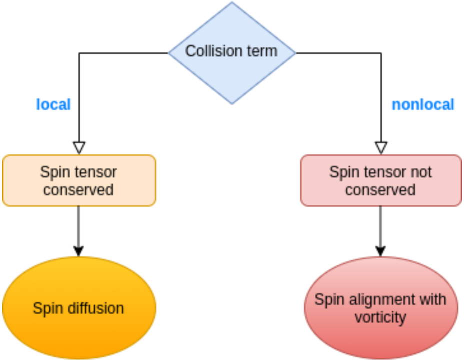
$\Longrightarrow$ Energy-momentum conserved in a collision

- Collisional invariant: total angular momentum $\Delta^{[\mu}p^{\nu]} + (\hbar/2)\Sigma^{\mu\nu}_{\mathfrak{s}}$

$\Longrightarrow$ Spin not conserved in nonlocal collisions $\Leftrightarrow$ $T^{[\nu\mu]}_{\text{HW}} \neq 0$ at $\mathcal{O}(\hbar^2)$
$\Longrightarrow$ Conversion between spin and orbital angular momentum

- $T_{\mathrm{HW}}^{[\nu\mu]} = 0$
  (i) for local collisions ($\Delta^\mu = 0$), as spin is collisional invariant
  (ii) in global equilibrium, as collision term vanishes

- With nonlocal collisions out of global equilibrium: dynamics dissipative



- Simple model with many applications: spintronics, chiral active fluids,...
  R. Takahashi, M. Matsuo, M. Ono, K. Harii, H. Chudo, S. Okayasu, J. Ieda, S. Takahashi, S. Maekawa, and E. Saitoh, Nature Physics 12, 52 (2016)
  D. Banerjee, A. Souslov, A. G. Abanov, and V. Vitelli, Nature communications 8, 1 (2017)
- Fluid of rigid, randomly oriented particles with internal angular momentum $\boldsymbol{\ell}$.
- Mass density ρ, fluid velocity $\boldsymbol{u}$,
  non-symmetric stress tensor T^{ij}.
- Conservation of total angular momentum

$$\frac{d}{dt}\int_{\Omega(t)} d^3x\,\rho(\ell^i + \epsilon^{ijk}x^j u^k) = \int_{\partial\Omega(t)} d\Sigma^l (C^{li} + \epsilon^{ijk}x^j T^{lk})$$

  Change in volume element given by surface flow described by stress for momentum, "couple stress" for internal angular momentum.
- After short calculation:

$$\rho\left(\partial^0 + u^j\partial^j\right)\ell^i = \partial^j C^{ji} + \epsilon^{ijk}T^{jk}$$

- Gain or loss of internal angular momentum: couple stress tensor and antisymmetric part of stress tensor!

- $p^\mu \to m(1, \boldsymbol{v})$, $\Sigma_{\mathfrak{s}}^{\mu\nu} \to \epsilon^{ijk}\mathfrak{s}^k$

$$T_{\mathrm{HW}}^{[ji]} = m\epsilon^{ijk}\partial^0\left\langle \frac{\hbar}{2}\mathfrak{s}^k \right\rangle + m\epsilon^{ijk}\partial^l\left\langle v^l \frac{\hbar}{2}\mathfrak{s}^k \right\rangle$$

with $\langle ... \rangle \equiv (m^2/2\pi\sqrt{3}) \int d^3v\, d^3\mathfrak{s}\, \delta(\mathfrak{s}^2 - 3)\, (...) f$

- Agreement with phenomenological result of nonrelativistic kinetic theory.

S. Hess and L. Waldmann, Zeitschrift für Naturforschung A 26, 1057 (1971)

- Comparison with micropolar fluids

G. Lukaszewicz, Micropolar Fluids, Theory and Applications (Birkhäuser Boston, 1999)

$$\rho\left(\partial^0 + u^j\partial^j\right)\ell^i = \partial^j C^{ji} + \epsilon^{ijk}T^{jk}$$

$\Longrightarrow$ Internal angular momentum

$$\rho\,\ell^i = m\left\langle \frac{\hbar}{2}\mathfrak{s}^i \right\rangle ,$$

$\Longrightarrow$ Couple stress tensor

$$C^{ji} = -\left\langle \frac{\hbar}{2}\mathfrak{s}^i p^j \right\rangle + m\left\langle \frac{\hbar}{2}\mathfrak{s}^i \right\rangle u^j .$$

Conclusions

- Definition of energy-momentum and spin tensor depends on choice of pseudo-gauge
  $\Longrightarrow$ Splitting of total angular momentum into orbital and spin part not unique
- Pseudo-gauge choice can affect results of calculations
- Issue about "physical" choice of pseudo-gauge long-standing, not finally answered
  $\Longrightarrow$ Need calculations in different pseudo-gauges and comparison to experiment
  $\Longrightarrow$ "Physical" choice may depend on context of application
- Canonical currents: derived directly from Noether's theorem
  $\Longrightarrow$ Spin tensor not conserved for free fields
- Belinfante currents: couple to gravity in conventional general relativity
  $\Longrightarrow$ Spin tensor vanishes
- HW currents: derived from Klein-Gordon Lagrangian for spinors
  $\Longrightarrow$ Spin tensor conserved for free fields
  $\Longrightarrow$ Covariant generalization of canonical global spin
  $\Longrightarrow$ Transparent interpretation of equations of motion in kinetic theory
  $\Longrightarrow$ Nonrelativistic limit