Quantum fluctuations of energy in subsystems of a hot relativistic gas

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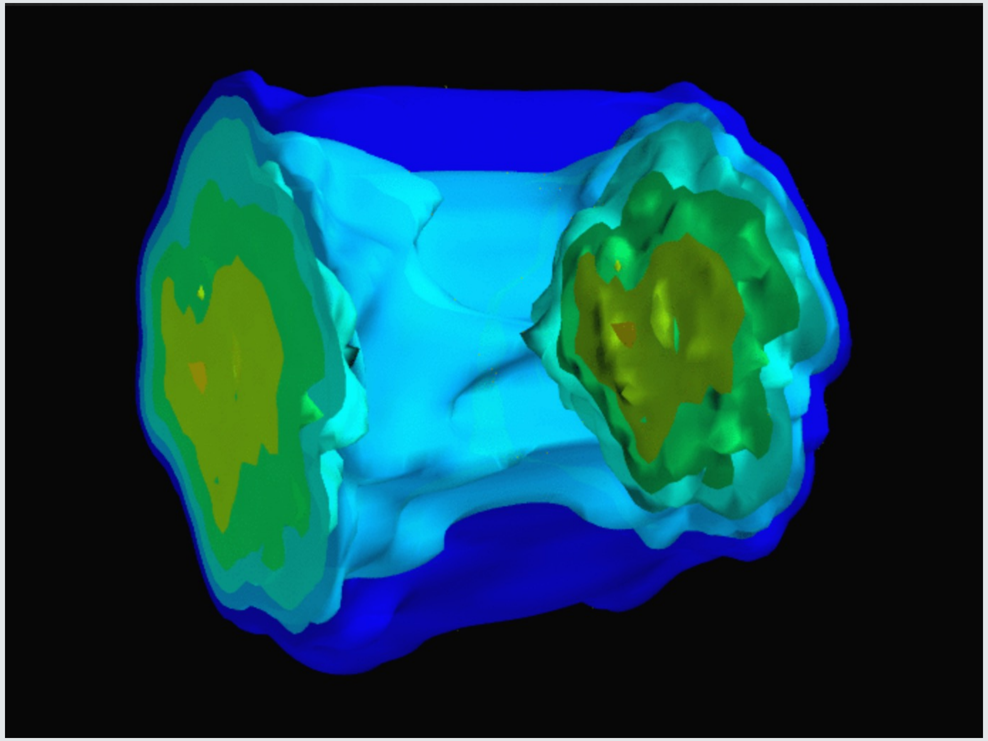
MOTIVATION:

1) Spacetime evolution of matter produced in relativistic heavy-ion collision is very well described by relativistic (dissipative) hydrodynamics.

2) One of the concepts used in hydrodynamics are those of energy density and pressure, both are defined locally – formally, the fluid element has zero size.



Isothermal lines in Au+Au collisions at the top RHIC energy, from B. Schenke

3) Successful hydro models are then used to conclude about the energy density attained in the collision processes, usually such vales are very large, exceed several GeV per fermi cubed at the initial stages (Bjorken’s model, 1983) – percolation argument for a dense system of quark and gluons (Quark Gluon Plasma).

4) I do not intend to argue with this observation but want to ask a question...

...how seriously we can accept the energy density profiles obtained with hydro codes, how real are all those detailed structures of the size of about 1 fm, especially for small systems.

5) Problem triggered by an exercise from...

Quantum Field Theory
Lectures of
SIDNEY COLEMAN

Problem 2.2b - application of dimensional analysis

(b) The canonical energy-momentum tensor is defined by (5.50), and its component T^{00} is

$$T^{00} = \pi_a \dot{\phi}^a - \mathscr{L}$$

(summation on a); this is also the Hamiltonian density H (see (4.40)). Then $[T^{00}] = [L] = [M]^d = [H]$. If we define

$$A_{\mathscr{H}}(a) = (a\sqrt{\pi})^{1-d} \int d^{d-1}\mathbf{x}\, \mathscr{H}(\mathbf{x},0) e^{-\mathbf{x}^2/a^2} \tag{S2.11}$$

we get

$$[\mathrm{var}\, A_{\mathscr{H}}(a)] = [A_{\mathscr{H}}(a)]^2 = [\mathscr{H}]^2 = [M]^{2d} \tag{S2.12}$$

$$\frac{\partial(\partial^\mu A^\nu)}{\partial(\partial^\lambda A^\sigma)} = \delta^\mu_\lambda \delta^\nu_\sigma$$

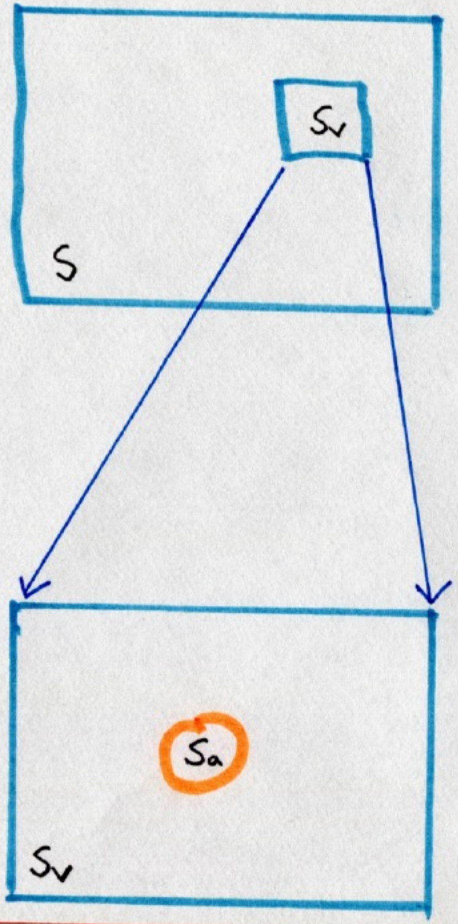
If we set

$$\mathrm{var}\, A_{\mathscr{H}}(a) = \alpha_{\mathscr{H}} a^{\beta_{\mathscr{H}}}$$

then by the previous reasoning, since a has the units of $[L] = [M]^{-1}$, we find $\beta_H = -2d$. We note that the fluctuations of the energy density grow more rapidly at small distances than those of the field itself.

Let us do it now for a hot gas!

Physics problem we would like to address



what are the fluctuations of energy in S_a?

typical situation studied in statistical physics

S – a closed/isolated system
described by the microcanonical ensemble
energy fixed

S_V – a subsystem of a closed system
in equilibrium, with volume V
described by the canonical ensemble
energy fluctuates

$$\sigma_H^2 = \frac{\langle H^2\rangle - \langle H\rangle^2}{\langle H\rangle^2} = \frac{T^2 c_V}{V \varepsilon^2} \xrightarrow[V\to\infty]{} 0$$

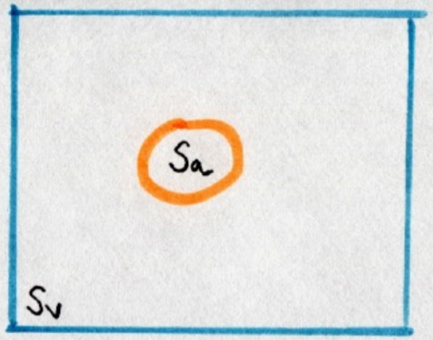
H – Hamiltonian, T – temperature,
ε – energy density, c_V – specific heat

S_a – a subsystem of S_V

for mathematical convenience described
by a "Gaussian" box

$$\frac{1}{(a\sqrt{\pi})^3}\, e^{-\vec{x}^2/a^2} \quad \leftarrow \text{Coleman}$$

Relativistic gas



S_V – relativistic gas, spinless particles with mass m

Bose-Einstein statistics included

more hadron/particle species can be included by introducing the degeneracy factor g, m – an effective/mean mass of particles

pion gas $m \sim 150$ MeV $g = 3$

perturbative QGP $m \sim 10$ MeV $g \simeq 40$

hadron gas at freeze-out $m \sim 1$ GeV $g \simeq 40$

Real scalar field (π^0)

conventions from Itzykson-Zuber : $\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a(k) e^{-ik\cdot x} + a^\dagger(k) e^{+ik\cdot x} \right]$

$$[a(k), a^\dagger(k')] = (2\pi)^3 2\omega_k \, \delta^{(3)}(\vec{k}-\vec{k}')$$

$$a(k) = (2\pi)^{3/2} \sqrt{2\omega_k}\, a_{\vec{k}} \, , \quad [a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \delta^{(3)}(\vec{k}-\vec{k}')$$

finite box $V = L^3$ periodic boundary conditions $k^i = \frac{2\pi}{L} n_{k^i}$, $\int d^3k \ldots \to \sum_{k_x, k_y, k_z} \left(\frac{2\pi}{L}\right)^3 \ldots$

$$\int \frac{d^3k}{(2\pi)^3} \ldots \to \frac{1}{V} \sum_{\vec{k}}$$

$$a(k) = \sqrt{2\omega_k V}\, A_{\vec{k}} \, , \quad a^\dagger(k') = \sqrt{2\omega_{k'} V}\, A^\dagger_{\vec{k}'} \, , \quad [A_{\vec{k}}, A^\dagger_{\vec{k}'}] = \delta_{\vec{k}, \vec{k}'}$$

$$\phi(x) = \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k V}} \left[A_{\vec{k}} e^{-ik\cdot x} + A^\dagger_{\vec{k}} e^{ik\cdot x} \right]$$

Energy density operator (Hamiltonian density)

$$\mathcal{H} = \frac{1}{2} \left(\pi^2(x) + (\vec{\nabla}\phi)^2 + m^2 \phi^2 \right)$$

$$\pi(x) = \dot{\phi}(x) = \frac{\partial \phi}{\partial t}(x) = \sum_{\vec{k}} \frac{(-i\omega_k)}{\sqrt{2\omega_k V}} \left[A_{\vec{k}} e^{-ik\cdot x} - A^\dagger_{\vec{k}} e^{ik\cdot x} \right]$$

$$\vec{\nabla}\phi(x) = \sum_{\vec{k}} \frac{i\vec{k}}{\sqrt{2\omega_k V}} \left[A_{\vec{k}} e^{-ik\cdot x} - A^\dagger_{\vec{k}} e^{ik\cdot x} \right]$$

$$\boxed{E_a \equiv \frac{1}{(a\sqrt{\pi})^3} \int d^3x \, e^{-\vec{x}^2/a^2} \mathcal{H}(\vec{x}, t)}$$

operator
energy averaged over the space region a^3

Thermal average of E_a

$$\boxed{\mathrm{Tr}(\varrho E_a) = \langle E_a \rangle_T}$$

a gas of spinless particles in thermal equilibrium

$$E_a = \frac{1}{2}\sum_{\vec{k},\vec{k}'} \frac{1}{2V}\frac{1}{\sqrt{\omega_k\omega_{k'}}}\Big[A^{+}_{\vec{k}}A_{\vec{k}'}(\omega_k\omega_{k'}+\vec{k}\cdot\vec{k}'+m^2)\,e^{i(\omega_k-\omega_{k'})t}\,e^{-\frac{a^2}{4}(\vec{k}-\vec{k}')^2}$$
$$+A_{\vec{k}}A^{+}_{\vec{k}'}(\omega_k\omega_{k'}+\vec{k}\cdot\vec{k}'+m^2)\,e^{-i(\omega_k-\omega_{k'})t}\,e^{-\frac{a^2}{4}(\vec{k}-\vec{k}')^2}$$
$$+A_{\vec{k}}A_{\vec{k}'}(-\omega_k\omega_{k'}-\vec{k}\cdot\vec{k}'+m^2)\,e^{-i(\omega_k+\omega_{k'})t}\,e^{-\frac{a^2}{4}(\vec{k}+\vec{k}')^2}$$
$$+A^{+}_{\vec{k}}A^{+}_{\vec{k}'}(-\omega_k\omega_{k'}-\vec{k}\cdot\vec{k}'+m^2)\,e^{+i(\omega_k+\omega_{k'})t}\,e^{-\frac{a^2}{4}(\vec{k}+\vec{k}')^2}\Big]$$

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thermal equilibrium
$$\boxed{\begin{aligned}\langle A^{+}_{\vec{k}}A_{\vec{k}'}\rangle_T &= \delta_{\vec{k},\vec{k}'}\,n_{\vec{k}}\\ \langle A_{\vec{k}}A^{+}_{\vec{k}'}\rangle_T &= \delta_{\vec{k},\vec{k}'}(1+n_{\vec{k}})\end{aligned}}$$
$$\langle A_{\vec{k}}A_{\vec{k}'}\rangle_T = \langle A^{+}_{\vec{k}}A^{+}_{\vec{k}'}\rangle_T = 0$$

$$\langle E_a\rangle_T = \frac{1}{2}\sum_{\vec{k}}\frac{1}{V}\big[\omega_k n_k + \omega_k(1+n_k)\big] \xrightarrow[V\to\infty]{} \int\frac{d^3k}{(2\pi)^3}\Big[\omega_{\vec{k}}\,n(\omega_{\vec{k}}) + \cancel{\tfrac{1}{2}\omega_{\vec{k}}}\Big]$$

finite medium part

infinite sum of "zero modes" removed by normal ordering

result independent of a!

So the right thing to look at is

$$\boxed{\langle :E_a:\rangle_T = \int\frac{d^3k}{(2\pi)^3}\,\omega_{\vec{k}}\,n_{\vec{k}} = \langle :T^{00}:\rangle = \varepsilon}$$

Fluctuation of the energy in the subsystem S_a

$$\sigma^2(a,m,T) \equiv \langle :E_a::E_a: \rangle - \langle :E_a: \rangle^2$$ variance

normalized standard deviation

$$\sigma_n(a,m,T) = \frac{\left(\langle :E_a:.E_a: \rangle - \langle :E_a: \rangle^2\right)^{1/2}}{\langle :E_a: \rangle}$$

we need $\langle A^+_{\vec{k}} A^+_{\vec{k}'} A_{\vec{p}} A_{\vec{p}'} \rangle = \left(\delta_{\vec{k}\vec{p}}\,\delta_{\vec{k}'\vec{p}'} + \delta_{\vec{k}\vec{p}'}\,\delta_{\vec{k}'\vec{p}}\right) n(\omega_k)\, n(\omega_{k'})$

this form is valid (only) for a gas

main result

$$\boxed{\sigma^2(a,m,T) = \int dK\, dK'\, n(\omega_k)(1+n(\omega_{k'}))\left[(\omega_{\vec{k}}\omega_{\vec{k}'} + \vec{k}\cdot\vec{k}' + m^2)^2 e^{-\frac{a^2}{2}(\vec{k}-\vec{k}')^2} + (\omega_{\vec{k}}\omega_{\vec{k}'} + \vec{k}\cdot\vec{k}' - m^2)^2 e^{-a^2/2(\vec{k}+\vec{k}')^2}\right] + \boxed{\sigma_0^2(a,m)}}$$

$$dK = \frac{d^3k}{(2\pi)^3 2\omega_k}$$

$$\boxed{\sigma_0^2(a,m)} = \frac{1}{8}\int dK\, dK'\, (\omega_{\vec{k}}\omega_{\vec{k}'} + \vec{k}\cdot\vec{k}' - m^2)^2 e^{-\frac{a^2}{2}(\vec{k}+\vec{k}')^2} = +\infty$$

T independent vacuum term, divergent, neglected

One may try to use an alternative definition

$$\tilde{\sigma}^2(a, m, T) \equiv \langle :E_a E_a: \rangle - \langle :E_a: \rangle^2$$

$$:E_a E_a: \; = \; ::E_a::E_a:: \quad \text{extra normal ordering}$$

$$\boxed{\begin{aligned}\tilde{\sigma}^2(a, m, T) = \int dK\, dK'\, n(\omega_K)\, n(\omega_{K'}) \Big[&(\omega_{\vec{k}}\omega_{\vec{k}'} + \vec{k}\cdot\vec{k}' + m^2)^2\, e^{-\frac{a^2}{2}(\vec{k}-\vec{k}')^2} \\ &+ (\omega_{\vec{k}}\omega_{\vec{k}'} + \vec{k}\cdot\vec{k}' - m^2)^2\, e^{-a^2/2\,(\vec{k}+\vec{k}')^2} \Big]\end{aligned}}$$

$\tilde{\sigma}^2 = \sigma^2$ with $\sigma_0^2 = 0$ and $1 + n(\omega_{K'}) \to n(\omega_{\vec{k}'})$

we remove σ_0^2 but throw away too much (as we will shortly see)

Degeneracy factor

π^0

π^0, π^+, π^- $\quad g = 3$ $\qquad \varepsilon \to g\varepsilon \;, \quad \sigma^2 \to g\sigma^2$

↑ $\qquad$ simple rescaling by g

creation/annihilation operators for different species commute

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Semiclassical gravity theory and quantum fluctuations

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I. INTRODUCTION

A natural proposal to describe the gravitational field of a quantum system is the semiclassical theory in which the expectation value of the stress tensor is the source. The semiclassical Einstein equation is

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N \langle T_{\mu\nu} \rangle. \qquad (1.1)$$

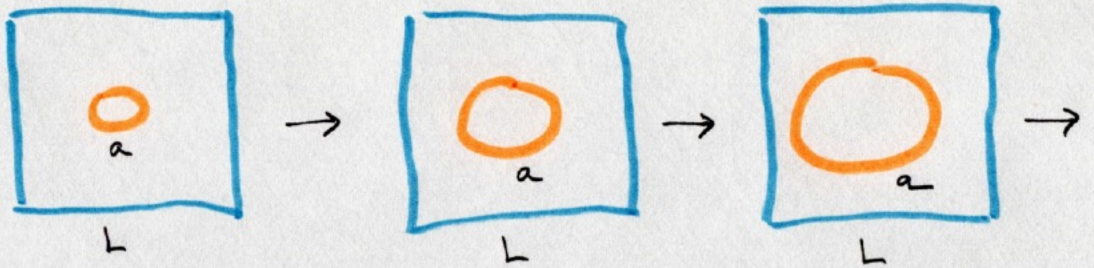
This theory is almost certain to fail at the Planck scale, where the quantum nature of gravity becomes important. However, it can also fail far away from the Planck scale if the fluctuations in the stress tensor become important. This was discussed some time ago by one of us [1] and illustrated by the problem of graviton emission by a box of particles in a general quantum state. It was found that the semiclassical theory gives reliable results when the fluctuations in the stress tensor are not too large, that is, when

$$\langle T_{\alpha\beta}(x)\, T_{\mu\nu}(y) \rangle \approx \langle T_{\alpha\beta}(x) \rangle \langle T_{\mu\nu}(y) \rangle. \qquad (1.2)$$

However, for quantum states in which the energy density fluctuations are large, the semiclassical theory based upon Eq. (1.1) cannot be trusted.

$$\Delta(x) \equiv \left| \frac{\langle :T_{00}{}^2(x): \rangle - \langle :T_{00}(x): \rangle^2}{\langle :T_{00}{}^2(x): \rangle} \right|.$$

Thermodynamic limit $\boxed{a \to \infty}$ $V = L^3$ $\boxed{a < L}$ L very large



$$\delta^{(3)}(\vec{k}-\vec{p}) = \lim_{a\to\infty} \frac{a^3}{(2\pi)^{3/2}} e^{-a^2/2\,(\vec{k}-\vec{p})^2}$$ representation of the Dirac delta

$$\boxed{\sigma^2 \sim \frac{g}{(2\pi)^{3/2} a^3} \int \frac{d^3k}{(2\pi)^3} \omega_{\vec{k}}^2 \, n(\omega_{\vec{k}})(1+n(\omega_{\vec{k}})) \qquad a \to \infty \text{ limit}}$$

specific heat $c_V = \frac{d\varepsilon}{dT} = \frac{g}{T^2} \int \frac{d^3k}{(2\pi)^3} \omega_{\vec{k}}^2 \, n(\omega_{\vec{k}})(1+n(\omega_{\vec{k}}))$

$$\boxed{V_a \sigma_n^2 = \frac{T^2 c_V}{\varepsilon^2} = V \frac{\langle H^2\rangle - \langle H\rangle^2}{\langle H\rangle^2} \equiv V \sigma_H^2}$$

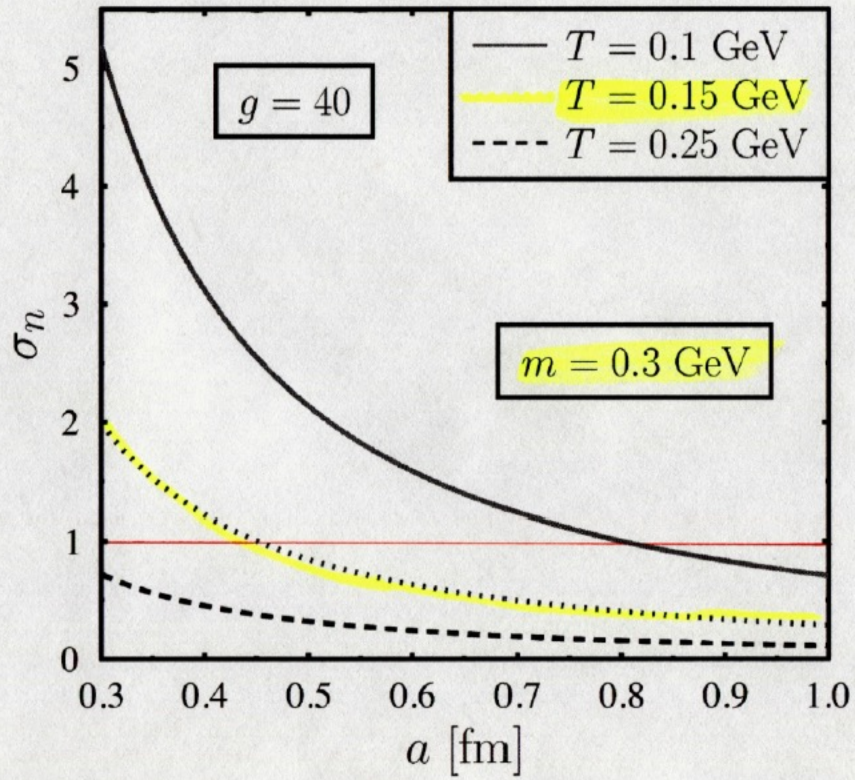
↳ $= a^3 (2\pi)^{3/2}$

↳ an artifact of using the "Gaussian" box

thanks to K. Golec-Biernat for discussion of this point

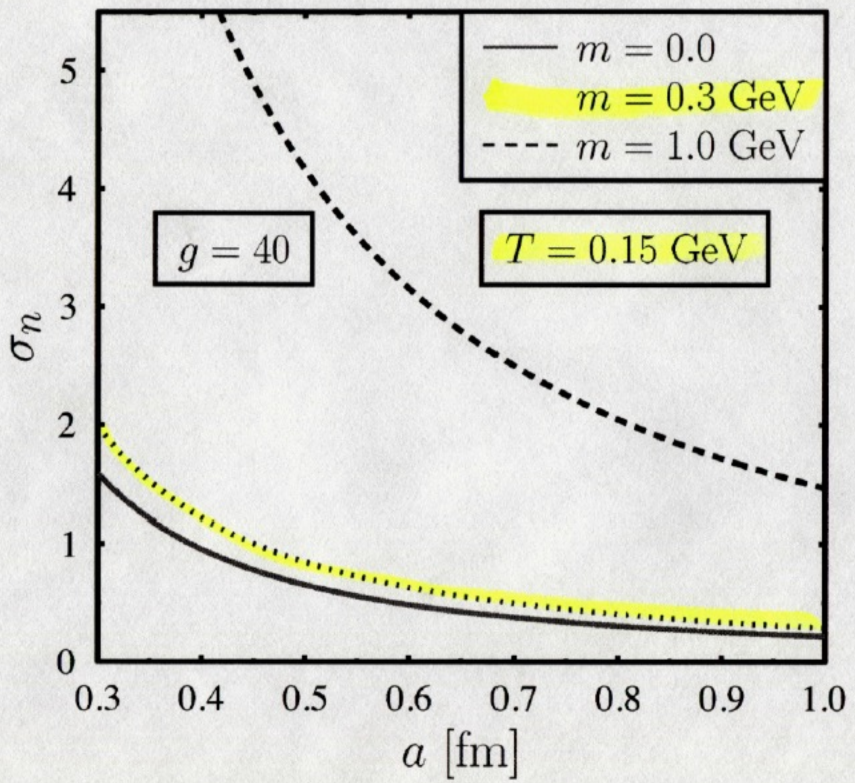
Results 1 — Temperature dependence of fluctuations

m - fixed



Results 2

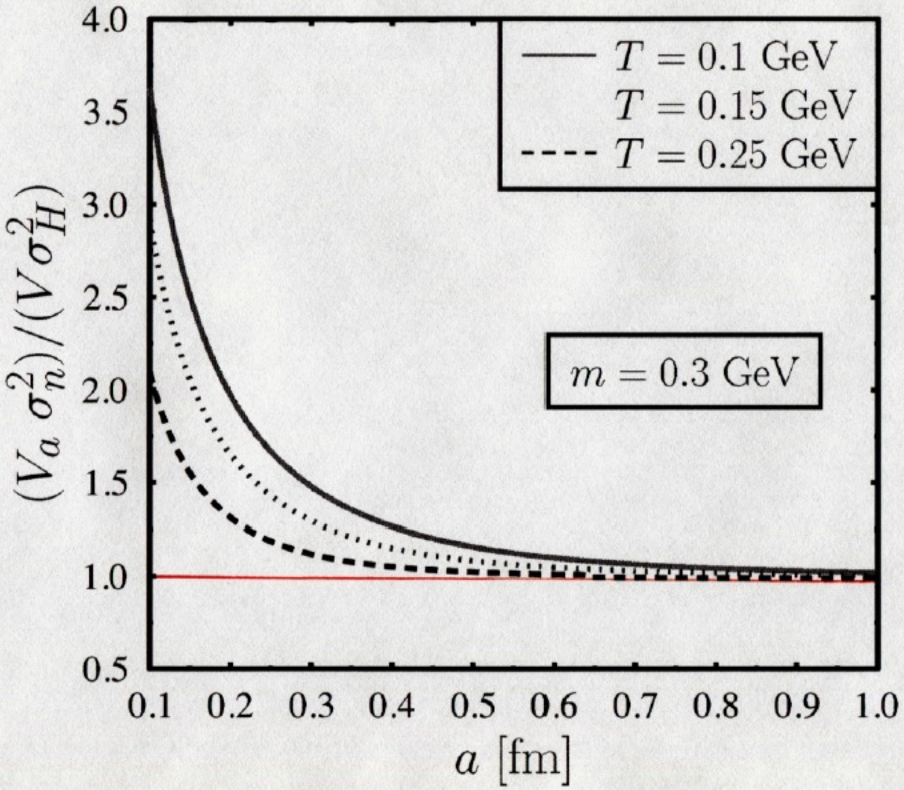
Mass dependence , T- fixed



Results 3

Approach to the thermodynamic limit

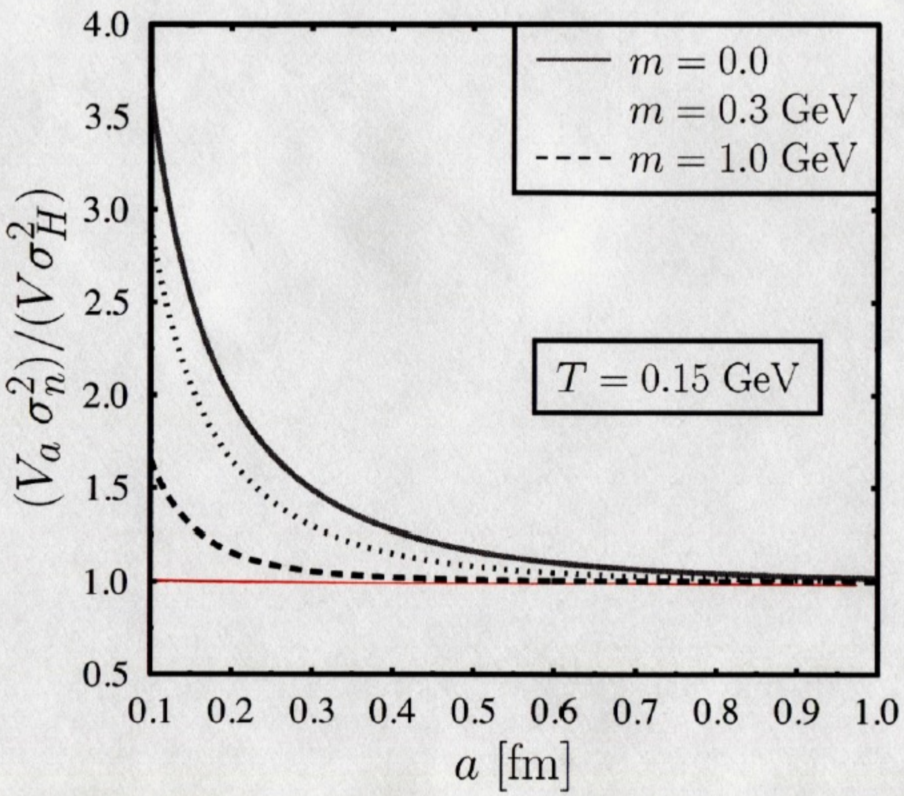
fixed mass (canonical ensemble result)



Results 4

Approach to the thermodynamic limit

fixed T



CONCLUSIONS:

Quantum fluctuations of the energy density at T=150 MeV are on the order of 20% for a~1fm.

As expected from the uncertainity relation, for a given size, the fluctuations decrease with increasing temperature. The effects are still of about 20% for a~0.2 fm at T=500 MeV.

If energy indeed fluctuates, there is a place for more energetic processes.

This finding contributes to the well established puzzle of unreasonable effectiveness of hydrodynamics in modeling of heavy-ion collisions.

Other measures of the energy density fluctuation?

Back-up slides 1: massless particles

Analytic formulas are available for m=0 and for m>>T. They serve as a check of numerics.

For m=0 the base integral is:

```
f[a_, T1_, T2_] := NIntegrate[Exp[-k/T1] Exp[-p/T2] Exp[-a^2/2 (k - p)^2], {k, 0, Infinity}, {p, 0, Infinity}];
```

$$\text{fnum[a\_, T1\_, T2\_]} := \frac{\sqrt{\frac{\pi}{2}}\, T1\, T2 \left(e^{\frac{1}{2 a^2 T1^2}} \mathrm{Erfc}\left[\frac{1}{\sqrt{2}\, a\, T1}\right] + e^{\frac{1}{2 a^2 T2^2}} \mathrm{Erfc}\left[\frac{1}{\sqrt{2}\, a\, T2}\right]\right)}{a\,(T1 + T2)};$$

```
f[12., 0.1, 0.2]
fnum[12., 0.1, 0.2]

0.00912796

0.00912796
```

For large a, the fluctuation scales like $1/a^{3/2} \sim 1/V^{1/2}$, in agreement with thermodynamic Expectations (although with a different coefficients, subsystem described by a Gaussian instead of a box)

Massless case with Boltzman statistics

$m = 0 \qquad n_k(\omega_k) = \exp(-\omega_k/T)$

$$\langle :E_a: \rangle^2 = \frac{9g^2}{\pi^4} T^4$$

$$\sigma_n^2 = \frac{1}{4320g} \Big[2970\,\xi^4 - 540\,\xi^6 - 96\,\xi^8 - 28\,\xi^{10} - 2\,\xi^{12}$$

$$+ \sqrt{2\pi}\; e^{\xi^2/2} \left(1458\,\xi^3 - 765\,\xi^5 + 300\,\xi^7 + 60\,\xi^9 + 15\,\xi^{11} + \xi^{13}\right) \operatorname{erfc}\left(\frac{\xi}{\sqrt{2}}\right) \Big]$$

exact result, $\xi = \dfrac{1}{aT}$

$$\lim_{a\to\infty} V_a \sigma_n^2 = \frac{11\pi^2}{8g} \frac{1}{T^3} = V\sigma_H^2$$

Back-up slides 2: thermal averages

Thermal averages (recapitulation from statistical mechanics)

$$\langle a_i^\dagger a_j \rangle_T = \frac{\sum_{n_1 n_2 \ldots} \langle n_1 n_2 \ldots n_\infty | a_i^\dagger a_j | n_1 n_2 \ldots n_\infty \rangle \exp\left[-\beta \sum_{k=1}^{\infty} \varepsilon_k n_k\right]}{\sum_{n_1 n_2 \ldots} \langle n_1 n_2 \ldots n_\infty | n_1 n_2 \ldots n_\infty \rangle \exp\left[-\beta \sum_{k=1}^{\infty} \varepsilon_k n_k\right]}$$

zero chem. potential

$\beta = \frac{1}{T}$

if $i \neq j$ the result is zero, so we do the calculation for $i = j$

$a_i^\dagger a_i$ – the occupation number operator $\hat{n}_i$

$$\langle a_i^\dagger a_j \rangle_T = \delta_{ij} \frac{\sum_{n_i=0}^{\infty} n_i e^{-\beta \varepsilon_i n_i}}{\sum_{n_i=0}^{\infty} e^{-\beta \varepsilon_i n_i}} = \delta_{ij} \frac{1}{e^{\beta \varepsilon_i} - 1} \longrightarrow \delta_{ij} \underbrace{e^{-\beta \varepsilon_i}}_{n_i}$$

↑

Bose statistics

I am sorry for sloppy notation $a_i \sim A_i$ $\hat{n}_i \sim n_i$

Thermal averages of four creation/annihilation operators

$\langle a_i^\dagger a_k^\dagger a_j a_l \rangle_T$ if we know this, we know other combinations from the commutation rules

not zero if $i = k = j = l$ then $\langle a_i^\dagger a_k^\dagger a_j a_l \rangle_T = \underline{\delta_{ik}\, \overset{j=l}{\delta_{ij}}\, \delta_{il} \langle n_i (n_i - 1) \rangle_T}$

the other possibility is that $j \neq l$ where either $i = j$ and $k = l$
or $i = l$ and $k = j$

$$\langle a_i^\dagger a_k^\dagger a_j a_l \rangle_T = (1 - \underline{\delta_{jl}})(\delta_{ij}\, \delta_{kl} + \delta_{il}\, \delta_{kj}) \langle n_i \rangle_T \langle n_k \rangle_T$$

the orange terms cancel if $\langle n(n-1) \rangle = 2 \langle n \rangle^2$
which holds for a gas $P(n) \sim \exp(-\beta \varepsilon_0 n)$ (ε_0 - energy of the considered state)

so we may use $\boxed{\langle a_i^\dagger a_k^\dagger a_j a_l \rangle = (\delta_{ij}\, \delta_{kl} + \delta_{il}\, \delta_{kj}) \langle n_i \rangle_T \langle n_k \rangle_T}$

thanks M. Sadzikowski for discussion

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we get other terms needed

$$\langle A_{\vec{k}}^\dagger A_{\vec{k}'} A_{\vec{p}}^\dagger A_{\vec{p}'} \rangle_T = (\delta_{\vec{k}\vec{k}'}\, \delta_{\vec{p}\vec{p}'} + \delta_{\vec{k}\vec{p}'}\, \delta_{\vec{k}'\vec{p}}) \langle n_k \rangle \langle n_p \rangle + \langle n_k \rangle \delta_{\vec{k}\vec{p}}\, \delta_{\vec{k}'\vec{p}}$$

$$\langle A_{\vec{k}} A_{\vec{k}'} A_{\vec{p}}^\dagger A_{\vec{p}'}^\dagger \rangle_T = (\delta_{\vec{p}\vec{k}}\, \delta_{\vec{p}'\vec{k}'} + \delta_{\vec{p}\vec{k}'}\, \delta_{\vec{p}'\vec{k}})(1 + \langle n_{\vec{k}} \rangle \langle n_{\vec{k}'} \rangle + \langle n_{\vec{k}} \rangle \langle n_{\vec{k}'} \rangle)$$