General-relativistic viscous fluids

Marcelo M. Disconzi[†]
Department of Mathematics, Vanderbilt University.

Joint work with F. Bemfica, J. Graber, V. Hoang, J. Noronha, M. Radosz, C. Rodriguez, Y. Shao.

Workshop on QGP Phenomenology

Online, May 2021

[†]MMD gratefully acknowledges support from a Sloan Research Fellowship provided by the Alfred P. Sloan foundation, from NSF grant # 1812826, and from a Dean's Faculty Fellowship.

Relativistic ideal fluids

A (relativistic) ideal fluid is described by the (relativistic) Euler equations

$$\nabla_\alpha \mathcal{T}^\alpha_\beta = 0,$$
$$\nabla_\alpha J^\alpha = 0,$$

where $\mathcal{T}$ is the energy-momentum tensor of an ideal fluid given by

$$\mathcal{T}_{\alpha\beta} = (p + \varrho) u_\alpha u_\beta + p g_{\alpha\beta},$$

and J is the baryon current of an ideal fluid given by

$$J_\alpha = n u_\alpha.$$

Relativistic ideal fluids

A (relativistic) ideal fluid is described by the (relativistic) Euler equations

$$\nabla_\alpha \mathcal{T}^\alpha_\beta = 0,$$
$$\nabla_\alpha J^\alpha = 0,$$

where $\mathcal{T}$ is the energy-momentum tensor of an ideal fluid given by

$$\mathcal{T}_{\alpha\beta} = (p + \varrho)u_\alpha u_\beta + p g_{\alpha\beta},$$

and J is the baryon current of an ideal fluid given by

$$J_\alpha = n u_\alpha.$$

Above, ϱ is the fluid's (energy) density, n is the baryon density, $p = p(\varrho, n)$ is the fluid's pressure, and u is the fluid's (four-)velocity, which satisfies

$$g_{\alpha\beta} u^\alpha u^\beta = -1.$$

g is the spacetime metric and ∇ the corresponding covariant derivative.

The need for relativistic viscous fluids

The relativistic Euler equations are widely used in the study of many physical systems (astrophysics, cosmology, ...).

The need for relativistic viscous fluids

The relativistic Euler equations are widely used in the study of many physical systems (astrophysics, cosmology, ...).

There are, however, important situations where a theory or relativistic viscous fluids is needed.

The need for relativistic viscous fluids

The relativistic Euler equations are widely used in the study of many physical systems (astrophysics, cosmology, ...).

There are, however, important situations where a theory or relativistic viscous fluids is needed.

On situation where viscosity is important is in the study of QGP.

Neutron star mergers

EoS: uncertain.

Neutron star mergers

EoS: uncertain. Einstein-Euler commonly assumed (time scales for viscous transport to set in were previously estimated $>$ ten milisec $=$ scale associated with damping due to gravitational wave emission).

Neutron star mergers

EoS: uncertain. Einstein-Euler commonly assumed (time scales for viscous transport to set in were previously estimated $>$ ten milisec $=$ scale associated with damping due to gravitational wave emission).

Estimates revised by Alford-Bovard-Hanauske-Rezzolla-Schwenzer ('18):

Neutron star mergers

EoS: uncertain. Einstein-Euler commonly assumed (time scales for viscous transport to set in were previously estimated $>$ ten milisec $=$ scale associated with damping due to gravitational wave emission).

Estimates revised by Alford-Bovard-Hanauske-Rezzolla-Schwenzer ('18):

- State-of-the-art numerical simulations of general relativistic ideal fluids: estimate for characteristic macroscopic scale L associated with gradients of the fluid variables.

Neutron star mergers

EoS: uncertain. Einstein-Euler commonly assumed (time scales for viscous transport to set in were previously estimated $>$ ten milisec $=$ scale associated with damping due to gravitational wave emission).

Estimates revised by Alford-Bovard-Hanauske-Rezzolla-Schwenzer ('18):

- State-of-the-art numerical simulations of general relativistic ideal fluids: estimate for characteristic macroscopic scale L associated with gradients of the fluid variables.
- Microscopic theory arguments: estimate for the characteristic microscopic scales ℓ of the system.

Neutron star mergers

EoS: uncertain. Einstein-Euler commonly assumed (time scales for viscous transport to set in were previously estimated $>$ ten milisec $=$ scale associated with damping due to gravitational wave emission).

Estimates revised by Alford-Bovard-Hanauske-Rezzolla-Schwenzer ('18):

- State-of-the-art numerical simulations of general relativistic ideal fluids: estimate for characteristic macroscopic scale L associated with gradients of the fluid variables.
- Microscopic theory arguments: estimate for the characteristic microscopic scales ℓ of the system.

Conclusion: Knudsen number $K_n \sim \ell/L$ may not be small in some cases.

Neutron star mergers

EoS: uncertain. Einstein-Euler commonly assumed (time scales for viscous transport to set in were previously estimated $>$ ten milisec $=$ scale associated with damping due to gravitational wave emission).

Estimates revised by Alford-Bovard-Hanauske-Rezzolla-Schwenzer ('18):

- State-of-the-art numerical simulations of general relativistic ideal fluids: estimate for characteristic macroscopic scale L associated with gradients of the fluid variables.
- Microscopic theory arguments: estimate for the characteristic microscopic scales ℓ of the system.

Conclusion: Knudsen number $K_n \sim \ell/L$ may not be small in some cases.
$\Rightarrow$ viscous contributions likely to affect the gravitational wave signal.

Neutron star mergers

EoS: uncertain. Einstein-Euler commonly assumed (time scales for viscous transport to set in were previously estimated $>$ ten milisec $=$ scale associated with damping due to gravitational wave emission).

Estimates revised by Alford-Bovard-Hanauske-Rezzolla-Schwenzer ('18):

- State-of-the-art numerical simulations of general relativistic ideal fluids: estimate for characteristic macroscopic scale L associated with gradients of the fluid variables.
- Microscopic theory arguments: estimate for the characteristic microscopic scales ℓ of the system.

Conclusion: Knudsen number $K_n \sim \ell/L$ may not be small in some cases.
$\Rightarrow$ viscous contributions likely to affect the gravitational wave signal.

GW: general-relativistic framework.

From ideal to viscous fluids

Energy-momentum tensor of a relativistic viscous fluid:

$$\mathcal{T}_{\alpha\beta} := (\varrho + \mathcal{R})u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha,$$

quantities as before ($u_\alpha u^\alpha = -1$); $\Pi_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta$.

From ideal to viscous fluids

Energy-momentum tensor of a relativistic viscous fluid:

$$\mathcal{T}_{\alpha\beta} := (\varrho + \mathcal{R}) u_\alpha u_\beta + (p + \mathcal{P}) \Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha,$$

quantities as before ($u_\alpha u^\alpha = -1$); $\Pi_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta$.

Viscous fluxes: $\mathcal{R}$ = viscous correction to ϱ; $\mathcal{P}$ = viscous correction to p; $\mathcal{Q}$ =heat flow; π = viscous shear stress.

From ideal to viscous fluids

Energy-momentum tensor of a relativistic viscous fluid:

$$\mathcal{T}_{\alpha\beta} := (\varrho + \mathcal{R})u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha,$$

quantities as before ($u_\alpha u^\alpha = -1$); $\Pi_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta$.

Viscous fluxes: $\mathcal{R}$ = viscous correction to ϱ; $\mathcal{P}$ = viscous correction to p; $\mathcal{Q}$ =heat flow; π = viscous shear stress. $p = p(\varrho, n)$.

From ideal to viscous fluids

Energy-momentum tensor of a relativistic viscous fluid:

$$\mathcal{T}_{\alpha\beta} := (\varrho + \mathcal{R}) u_\alpha u_\beta + (p + \mathcal{P}) \Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha,$$

quantities as before ($u_\alpha u^\alpha = -1$); $\Pi_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta$.

Viscous fluxes: $\mathcal{R}$ = viscous correction to ϱ; $\mathcal{P}$ = viscous correction to p; $\mathcal{Q}$ =heat flow; π = viscous shear stress. $p = p(\varrho, \cancel{n})$, $\cancel{\mathcal{J}}$.

From ideal to viscous fluids

Energy-momentum tensor of a relativistic viscous fluid:

$$\mathcal{T}_{\alpha\beta} := (\varrho + \mathcal{R})u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha,$$

quantities as before ($u_\alpha u^\alpha = -1$); $\Pi_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta$.

Viscous fluxes: $\mathcal{R}$ = viscous correction to ϱ; $\mathcal{P}$ = viscous correction to p; $\mathcal{Q}$ =heat flow; π = viscous shear stress. $p = p(\varrho, \cancel{n})$, $\cancel{J}$.

Theory of relativistic viscous fluids: specify the viscous fluxes.

From ideal to viscous fluids

Energy-momentum tensor of a relativistic viscous fluid:

$$\mathcal{T}_{\alpha\beta} := (\varrho + \mathcal{R})u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha,$$

quantities as before ($u_\alpha u^\alpha = -1$); $\Pi_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta$.

Viscous fluxes: $\mathcal{R}$ = viscous correction to ϱ; $\mathcal{P}$ = viscous correction to p; $\mathcal{Q}$ =heat flow; π = viscous shear stress. $p = p(\varrho, \cancel{n})$, $\cancel{\mathcal{J}}$.

Theory of relativistic viscous fluids: specify the viscous fluxes.
Parametrization of out-of-equilibrium hydro variables: choice of hydrodynamic frame.

From ideal to viscous fluids

Energy-momentum tensor of a relativistic viscous fluid:

$$\mathcal{T}_{\alpha\beta} := (\varrho + \mathcal{R}) u_\alpha u_\beta + (p + \mathcal{P}) \Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha,$$

quantities as before ($u_\alpha u^\alpha = -1$); $\Pi_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta$.

Viscous fluxes: $\mathcal{R}$ = viscous correction to ϱ; $\mathcal{P}$ = viscous correction to p; $\mathcal{Q}$ =heat flow; π = viscous shear stress. $p = p(\varrho, \cancel{n})$, $\cancel{\mathcal{J}}$.

Theory of relativistic viscous fluids: specify the viscous fluxes.
Parametrization of out-of-equilibrium hydro variables: choice of hydrodynamic frame. ∞-many choices, two classes:

From ideal to viscous fluids

Energy-momentum tensor of a relativistic viscous fluid:

$$\mathcal{T}_{\alpha\beta} := (\varrho + \mathcal{R})u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha,$$

quantities as before ($u_\alpha u^\alpha = -1$); $\Pi_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta$.

Viscous fluxes: $\mathcal{R}$ = viscous correction to ϱ; $\mathcal{P}$ = viscous correction to p; $\mathcal{Q}$ =heat flow; π = viscous shear stress. $p = p(\varrho, \cancel{n})$, $\cancel{\mathcal{J}}$.

Theory of relativistic viscous fluids: specify the viscous fluxes.
Parametrization of out-of-equilibrium hydro variables: choice of hydrodynamic frame. ∞-many choices, two classes:

- First-order: $\mathcal{R}$, $\mathcal{P}$, $\mathcal{Q}$, and π given in terms of ϱ, u, and their derivatives.

From ideal to viscous fluids

Energy-momentum tensor of a relativistic viscous fluid:

$$\mathcal{T}_{\alpha\beta} := (\varrho + \mathcal{R}) u_\alpha u_\beta + (p + \mathcal{P}) \Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha,$$

quantities as before ($u_\alpha u^\alpha = -1$); $\Pi_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta$.

Viscous fluxes: $\mathcal{R}$ = viscous correction to ϱ; $\mathcal{P}$ = viscous correction to p; $\mathcal{Q}$ =heat flow; π = viscous shear stress. $p = p(\varrho, \cancel{n})$, $\cancel{\mathcal{J}}$.

Theory of relativistic viscous fluids: specify the viscous fluxes. Parametrization of out-of-equilibrium hydro variables: choice of hydrodynamic frame. ∞-many choices, two classes:

- First-order: $\mathcal{R}$, $\mathcal{P}$, $\mathcal{Q}$, and π given in terms of ϱ, u, and their derivatives. EoM: $\nabla_\alpha \mathcal{T}^\alpha_\beta = 0$ (+Einstein).

From ideal to viscous fluids

Energy-momentum tensor of a relativistic viscous fluid:

$$\mathcal{T}_{\alpha\beta} := (\varrho + \mathcal{R}) u_\alpha u_\beta + (p + \mathcal{P}) \Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha,$$

quantities as before ($u_\alpha u^\alpha = -1$); $\Pi_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta$.

Viscous fluxes: $\mathcal{R}$ = viscous correction to ϱ; $\mathcal{P}$ = viscous correction to p; $\mathcal{Q}$ =heat flow; π = viscous shear stress. $p = p(\varrho, \cancel{n})$, $\cancel{\mathcal{J}}$.

Theory of relativistic viscous fluids: specify the viscous fluxes.
Parametrization of out-of-equilibrium hydro variables: choice of hydrodynamic frame. ∞-many choices, two classes:

- First-order: $\mathcal{R}$, $\mathcal{P}$, $\mathcal{Q}$, and π given in terms of ϱ, u, and their derivatives. EoM: $\nabla_\alpha \mathcal{T}^\alpha_\beta = 0$ (+Einstein). (Gradient expansion.)

From ideal to viscous fluids

Energy-momentum tensor of a relativistic viscous fluid:

$$\mathcal{T}_{\alpha\beta} := (\varrho + \mathcal{R}) u_\alpha u_\beta + (p + \mathcal{P}) \Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha,$$

quantities as before ($u_\alpha u^\alpha = -1$); $\Pi_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta$.

Viscous fluxes: $\mathcal{R}$ = viscous correction to ϱ; $\mathcal{P}$ = viscous correction to p; $\mathcal{Q}$ =heat flow; π = viscous shear stress. $p = p(\varrho, \cancel{n})$, $\cancel{J}$.

Theory of relativistic viscous fluids: specify the viscous fluxes. Parametrization of out-of-equilibrium hydro variables: choice of hydrodynamic frame. ∞-many choices, two classes:

- First-order: $\mathcal{R}$, $\mathcal{P}$, $\mathcal{Q}$, and π given in terms of ϱ, u, and their derivatives. EoM: $\nabla_\alpha \mathcal{T}^\alpha_\beta = 0$ (+Einstein). (Gradient expansion.)
- Second-order: $\mathcal{R}$, $\mathcal{P}$, $\mathcal{Q}$, and π are new variables treated on the same footing as ϱ, u.

From ideal to viscous fluids

Energy-momentum tensor of a relativistic viscous fluid:

$$\mathcal{T}_{\alpha\beta} := (\varrho + \mathcal{R}) u_\alpha u_\beta + (p + \mathcal{P}) \Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha,$$

quantities as before ($u_\alpha u^\alpha = -1$); $\Pi_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta$.

Viscous fluxes: $\mathcal{R}$ = viscous correction to ϱ; $\mathcal{P}$ = viscous correction to p; $\mathcal{Q}$ =heat flow; π = viscous shear stress. $p = p(\varrho, \cancel{n})$, $\cancel{J}$.

Theory of relativistic viscous fluids: specify the viscous fluxes. Parametrization of out-of-equilibrium hydro variables: choice of hydrodynamic frame. ∞-many choices, two classes:

- First-order: $\mathcal{R}$, $\mathcal{P}$, $\mathcal{Q}$, and π given in terms of ϱ, u, and their derivatives. EoM: $\nabla_\alpha \mathcal{T}^\alpha_\beta = 0$ (+Einstein). (Gradient expansion.)
- Second-order: $\mathcal{R}$, $\mathcal{P}$, $\mathcal{Q}$, and π are new variables treated on the same footing as ϱ, u. EoM: $\nabla_\alpha \mathcal{T}^\alpha_\beta = 0$ (+Einstein) supplemented by further equations satisfied by the viscous fluxes.

From ideal to viscous fluids

Energy-momentum tensor of a relativistic viscous fluid:

$$\mathcal{T}_{\alpha\beta} := (\varrho + \mathcal{R}) u_\alpha u_\beta + (p + \mathcal{P}) \Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha,$$

quantities as before ($u_\alpha u^\alpha = -1$); $\Pi_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta$.

Viscous fluxes: $\mathcal{R}$ = viscous correction to ϱ; $\mathcal{P}$ = viscous correction to p; $\mathcal{Q}$ =heat flow; π = viscous shear stress. $p = p(\varrho, \cancel{n})$, $\cancel{\mathcal{J}}$.

Theory of relativistic viscous fluids: specify the viscous fluxes. Parametrization of out-of-equilibrium hydro variables: choice of hydrodynamic frame. ∞-many choices, two classes:

- First-order: $\mathcal{R}$, $\mathcal{P}$, $\mathcal{Q}$, and π given in terms of ϱ, u, and their derivatives. EoM: $\nabla_\alpha \mathcal{T}^\alpha_\beta = 0$ (+Einstein). (Gradient expansion.)
- Second-order: $\mathcal{R}$, $\mathcal{P}$, $\mathcal{Q}$, and π are new variables treated on the same footing as ϱ, u. EoM: $\nabla_\alpha \mathcal{T}^\alpha_\beta = 0$ (+Einstein) supplemented by further equations satisfied by the viscous fluxes. (Moments method.)

From ideal to viscous fluids

Energy-momentum tensor of a relativistic viscous fluid:

$$\mathcal{T}_{\alpha\beta} := (\varrho + \mathcal{R})u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha,$$

quantities as before ($u_\alpha u^\alpha = -1$); $\Pi_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta$.

Viscous fluxes: $\mathcal{R}$ = viscous correction to ϱ; $\mathcal{P}$ = viscous correction to p; $\mathcal{Q}$ =heat flow; π = viscous shear stress. $p = p(\varrho, \cancel{n})$, $\cancel{J}$.

Theory of relativistic viscous fluids: specify the viscous fluxes. Parametrization of out-of-equilibrium hydro variables: choice of hydrodynamic frame. ∞-many choices, two classes:

- First-order: $\mathcal{R}$, $\mathcal{P}$, $\mathcal{Q}$, and π given in terms of ϱ, u, and their derivatives. EoM: $\nabla_\alpha \mathcal{T}^\alpha_\beta = 0$ (+Einstein). (Gradient expansion.)
- Second-order: $\mathcal{R}$, $\mathcal{P}$, $\mathcal{Q}$, and π are new variables treated on the same footing as ϱ, u. EoM: $\nabla_\alpha \mathcal{T}^\alpha_\beta = 0$ (+Einstein) supplemented by further equations satisfied by the viscous fluxes. (Moments method.)

First-order theory: $\pi = \pi(\varrho, u, \partial\varrho, \partial u, \dots)$ etc. (linear in $\partial(\varrho, u)$).

From ideal to viscous fluids

Energy-momentum tensor of a relativistic viscous fluid:

$$\mathcal{T}_{\alpha\beta} := (\varrho + \mathcal{R}) u_\alpha u_\beta + (p + \mathcal{P}) \Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha,$$

quantities as before ($u_\alpha u^\alpha = -1$); $\Pi_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta$.

Viscous fluxes: $\mathcal{R}$ = viscous correction to ϱ; $\mathcal{P}$ = viscous correction to p; $\mathcal{Q}$ =heat flow; π = viscous shear stress. $p = p(\varrho, \cancel{n})$, $\cancel{\mathcal{J}}$.

Theory of relativistic viscous fluids: specify the viscous fluxes. Parametrization of out-of-equilibrium hydro variables: choice of hydrodynamic frame. ∞-many choices, two classes:

- First-order: $\mathcal{R}$, $\mathcal{P}$, $\mathcal{Q}$, and π given in terms of ϱ, u, and their derivatives. EoM: $\nabla_\alpha \mathcal{T}^\alpha_\beta = 0$ (+Einstein). (Gradient expansion.)
- Second-order: $\mathcal{R}$, $\mathcal{P}$, $\mathcal{Q}$, and π are new variables treated on the same footing as ϱ, u. EoM: $\nabla_\alpha \mathcal{T}^\alpha_\beta = 0$ (+Einstein) supplemented by further equations satisfied by the viscous fluxes. (Moments method.)

First-order theory: $\pi = \pi(\varrho, u, \partial\varrho, \partial u, \dots)$ etc. (linear in $\partial(\varrho, u)$).

Second-order theory: $u^\mu \nabla_\mu \pi + \cdots = 0$ etc.

The Eckart and Landau-Lifshitz theories

Starting from:

$$\mathcal{T}_{\alpha\beta} = (\varrho + \mathcal{R})u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha.$$

Eckart ('40) and Landau-Lifshitz ('50) (first-order): $\mathcal{R} = 0$,

$$\pi_{\alpha\beta} := -2\eta \Pi^\mu_\alpha \Pi^\nu_\beta (\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3}\nabla_\lambda u^\lambda g_{\mu\nu}),\ \mathcal{P} := -\zeta \nabla_\mu u^\mu,\ (\mathcal{Q}_\alpha = ...),$$

where $\eta = \eta(\varrho), \zeta = \zeta(\varrho)$ are the coefficients of shear and bulk viscosity.

The Eckart and Landau-Lifshitz theories

Starting from:

$$\mathcal{T}_{\alpha\beta} = (\varrho + \mathcal{R})u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha.$$

Eckart ('40) and Landau-Lifshitz ('50) (first-order): $\mathcal{R} = 0$,

$$\pi_{\alpha\beta} := -2\eta\Pi_\alpha^\mu \Pi_\beta^\nu(\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3}\nabla_\lambda u^\lambda g_{\mu\nu}),\ \mathcal{P} := -\zeta\nabla_\mu u^\mu,\ (\mathcal{Q}_\alpha = ...),$$

where $\eta = \eta(\varrho), \zeta = \zeta(\varrho)$ are the coefficients of shear and bulk viscosity.

In essence:

1. Covariant generalization of Navier-Stokes.

The Eckart and Landau-Lifshitz theories

Starting from:

$$\mathcal{T}_{\alpha\beta} = (\varrho + \mathcal{R})u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha.$$

Eckart ('40) and Landau-Lifshitz ('50) (first-order): $\mathcal{R} = 0$,

$$\pi_{\alpha\beta} := -2\eta\Pi^\mu_\alpha\Pi^\nu_\beta(\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3}\nabla_\lambda u^\lambda g_{\mu\nu}),\ \mathcal{P} := -\zeta\nabla_\mu u^\mu,\ (\mathcal{Q}_\alpha = ...),$$

where $\eta = \eta(\varrho), \zeta = \zeta(\varrho)$ are the coefficients of shear and bulk viscosity.

In essence:

1. Covariant generalization of Navier-Stokes.
2. Entropy production ≥ 0.

Acausality and instability of Eckart and Landau-Lifshitz

The Eckart and Landau-Lifshitz theories violate causality: faster-than-light signals (Hiscock-Lindblom, '85; Pichon, '60).

Acausality and instability of Eckart and Landau-Lifshitz

The Eckart and Landau-Lifshitz theories violate causality: faster-than-light signals (Hiscock-Lindblom, '85; Pichon, '60). Equations are not hyperbolic.

Acausality and instability of Eckart and Landau-Lifshitz

The Eckart and Landau-Lifshitz theories violate causality: faster-than-light signals (Hiscock-Lindblom, '85; Pichon, '60). Equations are not hyperbolic.

The Eckart and Landau-Lifshitz theories are also unstable. (Stability: type of mode stability.)

Acausality and instability of Eckart and Landau-Lifshitz

The Eckart and Landau-Lifshitz theories violate causality: faster-than-light signals (Hiscock-Lindblom, '85; Pichon, '60). Equations are not hyperbolic.

The Eckart and Landau-Lifshitz theories are also unstable. (Stability: type of mode stability.)

Instability/acausality results apply to large classes of first-order theories.

Acausality and instability of Eckart and Landau-Lifshitz

The Eckart and Landau-Lifshitz theories violate causality: faster-than-light signals (Hiscock-Lindblom, '85; Pichon, '60). Equations are not hyperbolic.

The Eckart and Landau-Lifshitz theories are also unstable. (Stability: type of mode stability.)

Instability/acausality results apply to large classes of first-order theories. Difficult to construct causal and stable theories of relativistic fluids with viscosity: great deal of work trying to address the issue.

The Israel-Stewart theory

$$\mathcal{T}_{\alpha\beta} = (\varrho + \mathcal{R})u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha.$$

The Israel-Stewart theory

$$\mathcal{T}_{\alpha\beta} = (\varrho + \mathcal{R})u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha.$$

Israel-Stewart: second-order theory ('70s).

The Israel-Stewart theory

$$\mathcal{T}_{\alpha\beta} = (\varrho + \mathcal{R})u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha.$$

Israel-Stewart: second-order theory ('70s). Modern versions:
Baier-Romatschke-Son-Starinets-Stephanov ('08);
Denicol-Niemi-Molnar-Rischke ('12).

The Israel-Stewart theory

$$\mathcal{T}_{\alpha\beta} = (\varrho + \mathcal{R})u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha.$$

Israel-Stewart: second-order theory ('70s). Modern versions: Baier-Romatschke-Son-Starinets-Stephanov ('08); Denicol-Niemi-Molnar-Rischke ('12). EoM: $\mathcal{R} = 0$, $\nabla_\alpha \mathcal{T}^\alpha_\beta = 0$ (+Einstein),

$$\begin{aligned} \tau_{\mathcal{P}} u^\mu \nabla_\mu \mathcal{P} + \mathcal{P} + \zeta \nabla_\mu u^\mu &= \mathcal{J}^{\mathcal{P}}, \\ \tau_\pi u^\mu \Pi^\nu_\alpha \nabla_\mu \mathcal{Q}_\nu + \mathcal{Q}_\alpha &= \mathcal{J}^{\mathcal{Q}}_\alpha, \\ \tau_\pi u^\lambda \widehat{\Pi}^{\mu\nu}_{\alpha\beta} \nabla_\lambda \pi_{\mu\nu} + \pi_{\alpha\beta} - 2\eta\sigma_{\alpha\beta} &= \mathcal{J}^\pi_{\alpha\beta}, \end{aligned}$$

where $\widehat{\Pi}$ is the $u^\perp$ 2-tensor projection onto its symmetric and trace-free part; σ is the $u^\perp$ trace-free part of ∇u, $\tau' s = \tau(\varrho)$ are relaxation times.

The Israel-Stewart theory

$$\mathcal{T}_{\alpha\beta} = (\varrho + \mathcal{R})u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha.$$

Israel-Stewart: second-order theory ('70s). Modern versions:
Baier-Romatschke-Son-Starinets-Stephanov ('08);
Denicol-Niemi-Molnar-Rischke ('12). EoM: $\mathcal{R} = 0$, $\nabla_\alpha \mathcal{T}^\alpha_\beta = 0$ (+Einstein),

$$\begin{aligned}
\tau_{\mathcal{P}} u^\mu \nabla_\mu \mathcal{P} + \mathcal{P} + \zeta \nabla_\mu u^\mu &= \mathcal{J}^{\mathcal{P}}, \\
\tau_\pi u^\mu \Pi^\nu_\alpha \nabla_\mu \mathcal{Q}_\nu + \mathcal{Q}_\alpha &= \mathcal{J}^{\mathcal{Q}}_\alpha, \\
\tau_\pi u^\lambda \widehat{\Pi}^{\mu\nu}_{\alpha\beta} \nabla_\lambda \pi_{\mu\nu} + \pi_{\alpha\beta} - 2\eta\sigma_{\alpha\beta} &= \mathcal{J}^\pi_{\alpha\beta},
\end{aligned}$$

where $\widehat{\Pi}$ is the $u^\perp$ 2-tensor projection onto its symmetric and trace-free part; σ is the $u^\perp$ trace-free part of ∇u, $\tau' s = \tau(\varrho)$ are relaxation times.

$$\mathcal{J}' s = \partial\varrho + \partial u + \partial\mathcal{P} + \partial\mathcal{Q} + \partial\pi$$

The Israel-Stewart theory

$$\mathcal{T}_{\alpha\beta} = (\varrho + \mathcal{R})u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha.$$

Israel-Stewart: second-order theory ('70s). Modern versions: Baier-Romatschke-Son-Starinets-Stephanov ('08); Denicol-Niemi-Molnar-Rischke ('12). EoM: $\mathcal{R} = 0$, $\nabla_\alpha \mathcal{T}^\alpha_\beta = 0$ (+Einstein),

$$\begin{aligned} \tau_{\mathcal{P}} u^\mu \nabla_\mu \mathcal{P} + \mathcal{P} + \zeta \nabla_\mu u^\mu &= \mathcal{J}^{\mathcal{P}}, \\ \tau_\pi u^\mu \Pi^\nu_\alpha \nabla_\mu \mathcal{Q}_\nu + \mathcal{Q}_\alpha &= \mathcal{J}^{\mathcal{Q}}_\alpha, \\ \tau_\pi u^\lambda \widehat{\Pi}^{\mu\nu}_{\alpha\beta} \nabla_\lambda \pi_{\mu\nu} + \pi_{\alpha\beta} - 2\eta\sigma_{\alpha\beta} &= \mathcal{J}^\pi_{\alpha\beta}, \end{aligned}$$

where $\widehat{\Pi}$ is the $u^\perp$ 2-tensor projection onto its symmetric and trace-free part; σ is the $u^\perp$ trace-free part of ∇u, $\tau' s = \tau(\varrho)$ are relaxation times.

$$\mathcal{J}' s = \partial\varrho + \partial u + \partial\mathcal{P} + \partial\mathcal{Q} + \partial\pi = \text{top order}$$

The Israel-Stewart theory

$$\mathcal{T}_{\alpha\beta} = (\varrho + \mathcal{R})u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha.$$

Israel-Stewart: second-order theory ('70s). Modern versions: Baier-Romatschke-Son-Starinets-Stephanov ('08); Denicol-Niemi-Molnar-Rischke ('12). EoM: $\mathcal{R} = 0$, $\nabla_\alpha \mathcal{T}^\alpha_\beta = 0$ (+Einstein),

$$\tau_{\mathcal{P}} u^\mu \nabla_\mu \mathcal{P} + \mathcal{P} + \zeta \nabla_\mu u^\mu = \mathcal{J}^{\mathcal{P}},$$
$$\tau_\pi u^\mu \Pi^\nu_\alpha \nabla_\mu \mathcal{Q}_\nu + \mathcal{Q}_\alpha = \mathcal{J}^{\mathcal{Q}}_\alpha,$$
$$\tau_\pi u^\lambda \widehat{\Pi}^{\mu\nu}_{\alpha\beta} \nabla_\lambda \pi_{\mu\nu} + \pi_{\alpha\beta} - 2\eta\sigma_{\alpha\beta} = \mathcal{J}^\pi_{\alpha\beta},$$

where $\widehat{\Pi}$ is the $u^\perp$ 2-tensor projection onto its symmetric and trace-free part; σ is the $u^\perp$ trace-free part of ∇u, $\tau' s = \tau(\varrho)$ are relaxation times.

$$\mathcal{J}' s = \partial\varrho + \partial u + \partial\mathcal{P} + \partial\mathcal{Q} + \partial\pi = \text{top order}$$

System is highly complex; large system with non-diagonal principal part.

Features of the Israel-Stewart theory

For the Israel-Stewart theory:

- Stability holds (Hiscock-Lindblom, '83; Olson, '89).

Features of the Israel-Stewart theory

For the Israel-Stewart theory:

- Stability holds (Hiscock-Lindblom, '83; Olson, '89).
- Linearization about the global equilibrium $(\varrho = constant, u = constant, viscous = 0)$ is casual (Hiscock-Lindblom, '83; Olson, '89).

Features of the Israel-Stewart theory

For the Israel-Stewart theory:

- Stability holds (Hiscock-Lindblom, '83; Olson, '89).
- Linearization about the global equilibrium ($\varrho = constant, u = constant, viscous = 0$) is casual (Hiscock-Lindblom, '83; Olson, '89).
- Causality in $1+1$ (Denicol-Kodama-Koide-Mota, '08) and in rotational symmetry (Pu-Koide-Rischke, '10; Floerchinger-Grossi, '18).

Features of the Israel-Stewart theory

For the Israel-Stewart theory:

- Stability holds (Hiscock-Lindblom, '83; Olson, '89).
- Linearization about the global equilibrium $(\varrho = constant, u = constant, viscous = 0)$ is casual (Hiscock-Lindblom, '83; Olson, '89).
- Causality in $1 + 1$ (Denicol-Kodama-Koide-Mota, '08) and in rotational symmetry (Pu-Koide-Rischke, '10; Floerchinger-Grossi, '18).
- Very successful in applications to the study of the QGP (numerical simulations, phenomenology) (Romatschke-Romatschke, '19).

Features of the Israel-Stewart theory

For the Israel-Stewart theory:

- Stability holds (Hiscock-Lindblom, '83; Olson, '89).
- Linearization about the global equilibrium $(\varrho = constant, u = constant, viscous = 0)$ is casual (Hiscock-Lindblom, '83; Olson, '89).
- Causality in $1+1$ (Denicol-Kodama-Koide-Mota, '08) and in rotational symmetry (Pu-Koide-Rischke, '10; Floerchinger-Grossi, '18).
- Very successful in applications to the study of the QGP (numerical simulations, phenomenology) (Romatschke-Romatschke, '19).

Good theory for many applications!

Features of the Israel-Stewart theory

For the Israel-Stewart theory:

- Stability holds (Hiscock-Lindblom, '83; Olson, '89).
- Linearization about the global equilibrium $(\varrho = constant, u = constant, viscous = 0)$ is casual (Hiscock-Lindblom, '83; Olson, '89).
- Causality in $1+1$ (Denicol-Kodama-Koide-Mota, '08) and in rotational symmetry (Pu-Koide-Rischke, '10; Floerchinger-Grossi, '18).
- Very successful in applications to the study of the QGP (numerical simulations, phenomenology) (Romatschke-Romatschke, '19).

Good theory for many applications!

But: Local well-posedness? (existence and uniqueness of solutions to the initial value problem).

Features of the Israel-Stewart theory

For the Israel-Stewart theory:

- Stability holds (Hiscock-Lindblom, '83; Olson, '89).
- Linearization about the global equilibrium $(\varrho = constant, u = constant, viscous = 0)$ is casual (Hiscock-Lindblom, '83; Olson, '89).
- Causality in $1+1$ (Denicol-Kodama-Koide-Mota, '08) and in rotational symmetry (Pu-Koide-Rischke, '10; Floerchinger-Grossi, '18).
- Very successful in applications to the study of the QGP (numerical simulations, phenomenology) (Romatschke-Romatschke, '19).

Good theory for many applications!

But: Local well-posedness? (existence and uniqueness of solutions to the initial value problem). Causality in $3+1$ without symmetry?

Features of the Israel-Stewart theory

For the Israel-Stewart theory:

- Stability holds (Hiscock-Lindblom, '83; Olson, '89).
- Linearization about the global equilibrium $(\varrho = constant, u = constant, viscous = 0)$ is casual (Hiscock-Lindblom, '83; Olson, '89).
- Causality in $1+1$ (Denicol-Kodama-Koide-Mota, '08) and in rotational symmetry (Pu-Koide-Rischke, '10; Floerchinger-Grossi, '18).
- Very successful in applications to the study of the QGP (numerical simulations, phenomenology) (Romatschke-Romatschke, '19).

Good theory for many applications!

But: Local well-posedness? (existence and uniqueness of solutions to the initial value problem). Causality in $3+1$ without symmetry?

Theorem: Causality and LWP of the Israel-Stewart equations (D-Bemfica-Noronha, '19; D-Bemfica-Hoang-Noronha-Radosz, '20)

The Israel-Stewart equations are causal. The Cauchy problem is locally well-posed in Gevrey spaces. If $\mathcal{Q} = 0$, $\pi = 0$, local well-posedness holds in Sobolev spaces. These results hold with or without coupling to Einstein's equations.

Theorem: Causality and LWP of the Israel-Stewart equations (D-Bemfica-Noronha, '19; D-Bemfica-Hoang-Noronha-Radosz, '20)

The Israel-Stewart equations are causal. The Cauchy problem is locally well-posed in Gevrey spaces. If $\mathcal{Q} = 0$, $\pi = 0$, local well-posedness holds in Sobolev spaces. These results hold with or without coupling to Einstein's equations.

(General-relativistic simulations: Sobolev good, Gevrey bad).

Theorem: Causality and LWP of the Israel-Stewart equations (D-Bemfica-Noronha, '19; D-Bemfica-Hoang-Noronha-Radosz, '20)

The Israel-Stewart equations are causal. The Cauchy problem is locally well-posed in Gevrey spaces. If $\mathcal{Q} = 0$, $\pi = 0$, local well-posedness holds in Sobolev spaces. These results hold with or without coupling to Einstein's equations.

(General-relativistic simulations: Sobolev good, Gevrey bad).

Proof:

- Causality: computation of the system's characteristics. Intractable by brute force. Think geometrically: develop calculation techniques guided by would-be acoustical metrics.
- Local well-posedness: derive estimates using techniques of weakly hyperbolic systems (Leray-Ohya, '60s). Gevrey: avoid loss of derivatives. If $\mathcal{Q} = 0$, $\pi = 0$, estimates close in Sobolev spaces.

Causality conditions in Israel-Stewart

Our theorem holds under suitable assumptions, including inequalities for certain scalar quantities (transport coefficients, eigenvalues of π, etc.):

$$(\varrho + p + \mathcal{P} - |\Lambda_1|) - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\mathcal{P}}\mathcal{P}) - \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3 \geq 0, \ldots$$

Causality conditions in Israel-Stewart

Our theorem holds under suitable assumptions, including inequalities for certain scalar quantities (transport coefficients, eigenvalues of π, etc.):

$$(\varrho + p + \mathcal{P} - |\Lambda_1|) - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\mathcal{P}}\mathcal{P}) - \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3 \geq 0, \ldots$$

We also derive some necessary conditions for causality,

$$(3\eta + \lambda_{\pi\mathcal{P}}\mathcal{P}) - \frac{1}{2}|\Lambda_1| \geq 0, \ldots$$

Causality conditions in Israel-Stewart

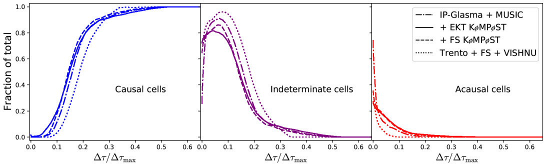
Our theorem holds under suitable assumptions, including inequalities for certain scalar quantities (transport coefficients, eigenvalues of π, etc.):

$$(\varrho + p + \mathcal{P} - |\Lambda_1|) - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\mathcal{P}}\mathcal{P}) - \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3 \geq 0, \ldots$$

We also derive some necessary conditions for causality,

$$(3\eta + \lambda_{\pi\mathcal{P}}\mathcal{P}) - \frac{1}{2}|\Lambda_1| \geq 0, \ldots$$

Necessary conditions: test at each time-step of simulations (Plumberg-Almaalol-Dore-Noronha-Noronha-Hostler, '21; Cheng-Shen, '21):



Causality violation over time, Plumberg et al.

Up to 30% of initial fluid cells violate causality; many follow-up questions.

Limitations of the Israel-Stewart theory

Despite its successes, we should keep in mind some (potential) limitations of the Israel-Stewart theory:

Limitations of the Israel-Stewart theory

Despite its successes, we should keep in mind some (potential) limitations of the Israel-Stewart theory:

- Not known whether it is applicable to the study of (viscous effects on) neutron star mergers.

Limitations of the Israel-Stewart theory

Despite its successes, we should keep in mind some (potential) limitations of the Israel-Stewart theory:

- Not known whether it is applicable to the study of (viscous effects on) neutron star mergers. LWP in Sobolev?

Limitations of the Israel-Stewart theory

Despite its successes, we should keep in mind some (potential) limitations of the Israel-Stewart theory:

- Not known whether it is applicable to the study of (viscous effects on) neutron star mergers. LWP in Sobolev?
- Not known whether it is applicable to low energy heavy-ion collisions when vorticity effects and the dynamics of the baryon current are relevant.

Limitations of the Israel-Stewart theory

Despite its successes, we should keep in mind some (potential) limitations of the Israel-Stewart theory:

- Not known whether it is applicable to the study of (viscous effects on) neutron star mergers. LWP in Sobolev?
- Not known whether it is applicable to low energy heavy-ion collisions when vorticity effects and the dynamics of the baryon current are relevant.
- It lacks the degree of universality expected in hydrodynamics (equations of motion change depending on the derivation).

Limitations of the Israel-Stewart theory

Despite its successes, we should keep in mind some (potential) limitations of the Israel-Stewart theory:

- Not known whether it is applicable to the study of (viscous effects on) neutron star mergers. LWP in Sobolev?
- Not known whether it is applicable to low energy heavy-ion collisions when vorticity effects and the dynamics of the baryon current are relevant.
- It lacks the degree of universality expected in hydrodynamics (equations of motion change depending on the derivation).
- Reassessment of numerical simulations in light of causality conditions?

Limitations of the Israel-Stewart theory

Despite its successes, we should keep in mind some (potential) limitations of the Israel-Stewart theory:

- Not known whether it is applicable to the study of (viscous effects on) neutron star mergers. LWP in Sobolev?
- Not known whether it is applicable to low energy heavy-ion collisions when vorticity effects and the dynamics of the baryon current are relevant.
- It lacks the degree of universality expected in hydrodynamics (equations of motion change depending on the derivation).
- Reassessment of numerical simulations in light of causality conditions?
- Difficulties in describing shocks (Geroch-Lindblom, '91; Olson-Hiscock, '91).

Limitations of the Israel-Stewart theory

Despite its successes, we should keep in mind some (potential) limitations of the Israel-Stewart theory:

- Not known whether it is applicable to the study of (viscous effects on) neutron star mergers. LWP in Sobolev?
- Not known whether it is applicable to low energy heavy-ion collisions when vorticity effects and the dynamics of the baryon current are relevant.
- It lacks the degree of universality expected in hydrodynamics (equations of motion change depending on the derivation).
- Reassessment of numerical simulations in light of causality conditions?
- Difficulties in describing shocks (Geroch-Lindblom, '91; Olson-Hiscock, '91).

Motivation for alternative theories.

The BDNK theory

The BDNK theory is a first-order theory defined by (D-Bemfica-Noronha, '18, '19, '20; Kovtun, '19; Hoult-Kovtun, '20):

$$\mathcal{T}_{\alpha\beta} = (\varrho + \mathcal{R})u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha,$$

with

$$\begin{aligned} \mathcal{R} &:= \tau_{\mathcal{R}}(u^\mu \nabla_\mu \varrho + (\varrho + p)\nabla_\mu u^\mu), \\ \mathcal{P} &:= -\zeta \nabla_\mu u^\mu + \tau_{\mathcal{P}}(u^\mu \nabla_\mu \varrho + (\varrho + p)\nabla_\mu u^\mu), \\ \mathcal{Q}_\alpha &:= \tau_{\mathcal{Q}}(\varrho + p)u^\mu \nabla_\mu u_\alpha + \beta_{\mathcal{Q}} \Pi_\alpha^\mu \nabla_\mu \varrho, \\ \pi_{\alpha\beta} &:= -2\eta \Pi_\alpha^\mu \Pi_\beta^\nu (\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3}\nabla_\lambda u^\lambda g_{\mu\nu}), \end{aligned}$$

where $\tau' s, \beta_{\mathcal{Q}} = \tau(\varrho), \beta_{\mathcal{Q}}(\varrho)$.

The BDNK theory

The BDNK theory is a first-order theory defined by (D-Bemfica-Noronha, '18, '19, '20; Kovtun, '19; Hoult-Kovtun, '20):

$$\mathcal{T}_{\alpha\beta} = (\varrho + \mathcal{R}) u_\alpha u_\beta + (p + \mathcal{P}) \Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha,$$

with

$$\begin{aligned}
\mathcal{R} &:= \tau_{\mathcal{R}} (u^\mu \nabla_\mu \varrho + (\varrho + p) \nabla_\mu u^\mu), \\
\mathcal{P} &:= -\zeta \nabla_\mu u^\mu + \tau_{\mathcal{P}} (u^\mu \nabla_\mu \varrho + (\varrho + p) \nabla_\mu u^\mu), \\
\mathcal{Q}_\alpha &:= \tau_{\mathcal{Q}} (\varrho + p) u^\mu \nabla_\mu u_\alpha + \beta_{\mathcal{Q}} \Pi^\mu_\alpha \nabla_\mu \varrho, \\
\pi_{\alpha\beta} &:= -2\eta \Pi^\mu_\alpha \Pi^\nu_\beta (\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3} \nabla_\lambda u^\lambda g_{\mu\nu}),
\end{aligned}$$

where $\tau's, \beta_{\mathcal{Q}} = \tau(\varrho), \beta_{\mathcal{Q}}(\varrho)$.

Lots of terms: need them to fix the causality and instability problems of Eckart and Landau-Lifshitz.

Theorem: Causality, stability, and LWP of the BDNK theory (D-Bemfica-Rodriguez-Shao, '19; D-Bemfica-Graber, '20; D-Bemfica-Noronha, '20)

The BDNK equations are causal and stable. The Cauchy problem is locally well-posed in Sobolev spaces. These results hold with or without coupling to Einstein's equations.

Theorem: Causality, stability, and LWP of the BDNK theory (D-Bemfica-Rodriguez-Shao, '19; D-Bemfica-Graber, '20; D-Bemfica-Noronha, '20)

The BDNK equations are causal and stable. The Cauchy problem is locally well-posed in Sobolev spaces. These results hold with or without coupling to Einstein's equations.

(Theorem in fact valid with baryon current J and $p = p(\varrho, n)$.)

Theorem: Causality, stability, and LWP of the BDNK theory (D-Bemfica-Rodriguez-Shao, '19; D-Bemfica-Graber, '20; D-Bemfica-Noronha, '20)

The BDNK equations are causal and stable. The Cauchy problem is locally well-posed in Sobolev spaces. These results hold with or without coupling to Einstein's equations.

(Theorem in fact valid with baryon current J and $p = p(\varrho, n)$.)

Proof:

- Causality: system's characteristics; think geometrically.
- Stability: analysis of the roots guided by causality.
- LWP: Diagonalize the principal part of the system; can do it because we understand the characteristics. Diagonalization at the level of symbols. Rational functions, pass to the PDE: pseudo-differential operators. Quasilinear problem: pseudo-differential calculus for symbols with limited smoothness.

Where does the BDNK tensor come from?

Idea from effective theories: start with the most general energy-momentum tensor compatible with symmetries:

$$\mathcal{T}_{\alpha\beta} = \mathcal{E} u_\alpha u_\beta + \mathcal{S}\Pi_{\alpha\beta} + u_\alpha \mathcal{Q}_\beta + u_\beta \mathcal{Q}_\alpha + \mathscr{T}_{\alpha\beta}.$$

Where does the BDNK tensor come from?

Idea from effective theories: start with the most general energy-momentum tensor compatible with symmetries:

$$\mathcal{T}_{\alpha\beta} = \mathcal{E} u_\alpha u_\beta + \mathcal{S} \Pi_{\alpha\beta} + u_\alpha \mathcal{Q}_\beta + u_\beta \mathcal{Q}_\alpha + \mathscr{T}_{\alpha\beta}.$$

Irreducible components, u-splitting: $\mathcal{E} = u^\alpha u^\beta \mathcal{T}_{\alpha\beta}$, $\mathcal{Q}_\alpha = -\Pi^{\alpha\beta} u^\gamma \mathcal{T}_{\beta\gamma}$, etc.

Where does the BDNK tensor come from?

Idea from effective theories: start with the most general energy-momentum tensor compatible with symmetries:

$$\mathcal{T}_{\alpha\beta} = \mathcal{E} u_\alpha u_\beta + \mathcal{S}\Pi_{\alpha\beta} + u_\alpha \mathcal{Q}_\beta + u_\beta \mathcal{Q}_\alpha + \mathscr{T}_{\alpha\beta}.$$

Irreducible components, u-splitting: $\mathcal{E} = u^\alpha u^\beta \mathcal{T}_{\alpha\beta}$, $\mathcal{Q}_\alpha = -\Pi^{\alpha\beta} u^\gamma \mathcal{T}_{\beta\gamma}$, etc.

Reduction to ideal case: $\mathcal{E} = \varrho + \mathcal{R}$, $\mathcal{S} = p + \mathcal{P}$, etc.

Where does the BDNK tensor come from?

Idea from effective theories: start with the most general energy-momentum tensor compatible with symmetries:

$$\mathcal{T}_{\alpha\beta} = \mathcal{E} u_\alpha u_\beta + \mathcal{S}\Pi_{\alpha\beta} + u_\alpha \mathcal{Q}_\beta + u_\beta \mathcal{Q}_\alpha + \mathscr{T}_{\alpha\beta}.$$

Irreducible components, u-splitting: $\mathcal{E} = u^\alpha u^\beta \mathcal{T}_{\alpha\beta}$, $\mathcal{Q}_\alpha = -\Pi^{\alpha\beta} u^\gamma \mathcal{T}_{\beta\gamma}$, etc.

Reduction to ideal case: $\mathcal{E} = \varrho + \mathcal{R}$, $\mathcal{S} = p + \mathcal{P}$, etc.

Viscous corrections linear in $\partial(\varrho, u)$: $\mathcal{R} = \chi_1 u^\mu \nabla_\mu \varrho + \chi_2 \nabla_\mu u^\mu$, etc.

Where does the BDNK tensor come from?

Idea from effective theories: start with the most general energy-momentum tensor compatible with symmetries:

$$\mathcal{T}_{\alpha\beta} = \mathcal{E} u_\alpha u_\beta + \mathcal{S}\Pi_{\alpha\beta} + u_\alpha \mathcal{Q}_\beta + u_\beta \mathcal{Q}_\alpha + \mathscr{T}_{\alpha\beta}.$$

Irreducible components, u-splitting: $\mathcal{E} = u^\alpha u^\beta \mathcal{T}_{\alpha\beta}$, $\mathcal{Q}_\alpha = -\Pi^{\alpha\beta} u^\gamma \mathcal{T}_{\beta\gamma}$, etc.

Reduction to ideal case: $\mathcal{E} = \varrho + \mathcal{R}$, $\mathcal{S} = p + \mathcal{P}$, etc.

Viscous corrections linear in $\partial(\varrho, u)$: $\mathcal{R} = \chi_1 u^\mu \nabla_\mu \varrho + \chi_2 \nabla_\mu u^\mu$, etc.

Physics: $\chi_1 u^\mu \nabla_\mu \varrho + \chi_2 \nabla_\mu u^\mu = \tau_\varrho (u^\mu \nabla_\mu \varrho + (\varrho + p) \nabla_\mu u^\mu)$, etc.

Where does the BDNK tensor come from?

Idea from effective theories: start with the most general energy-momentum tensor compatible with symmetries:

$$\mathcal{T}_{\alpha\beta} = \mathcal{E} u_\alpha u_\beta + \mathcal{S}\Pi_{\alpha\beta} + u_\alpha \mathcal{Q}_\beta + u_\beta \mathcal{Q}_\alpha + \mathscr{T}_{\alpha\beta}.$$

Irreducible components, u-splitting: $\mathcal{E} = u^\alpha u^\beta \mathcal{T}_{\alpha\beta}$, $\mathcal{Q}_\alpha = -\Pi^{\alpha\beta} u^\gamma \mathcal{T}_{\beta\gamma}$, etc.

Reduction to ideal case: $\mathcal{E} = \varrho + \mathcal{R}$, $\mathcal{S} = p + \mathcal{P}$, etc.

Viscous corrections linear in $\partial(\varrho, u)$: $\mathcal{R} = \chi_1 u^\mu \nabla_\mu \varrho + \chi_2 \nabla_\mu u^\mu$, etc.

Physics: $\chi_1 u^\mu \nabla_\mu \varrho + \chi_2 \nabla_\mu u^\mu = \tau_\varrho (u^\mu \nabla_\mu \varrho + (\varrho + p)\nabla_\mu u^\mu)$, etc.

Find relations among the transport coefficients that ensure causality, stability, LWP.

Where does the BDNK tensor come from?

Idea from effective theories: start with the most general energy-momentum tensor compatible with symmetries:

$$\mathcal{T}_{\alpha\beta} = \mathcal{E} u_\alpha u_\beta + \mathcal{S} \Pi_{\alpha\beta} + u_\alpha \mathcal{Q}_\beta + u_\beta \mathcal{Q}_\alpha + \mathscr{T}_{\alpha\beta}.$$

Irreducible components, u-splitting: $\mathcal{E} = u^\alpha u^\beta \mathcal{T}_{\alpha\beta}$, $\mathcal{Q}_\alpha = -\Pi^{\alpha\beta} u^\gamma \mathcal{T}_{\beta\gamma}$, etc.

Reduction to ideal case: $\mathcal{E} = \varrho + \mathcal{R}$, $\mathcal{S} = p + \mathcal{P}$, etc.

Viscous corrections linear in $\partial(\varrho, u)$: $\mathcal{R} = \chi_1 u^\mu \nabla_\mu \varrho + \chi_2 \nabla_\mu u^\mu$, etc.

Physics: $\chi_1 u^\mu \nabla_\mu \varrho + \chi_2 \nabla_\mu u^\mu = \tau_\varrho (u^\mu \nabla_\mu \varrho + (\varrho + p) \nabla_\mu u^\mu)$, etc.

Find relations among the transport coefficients that ensure causality, stability, LWP.

One should let the fundamental principle of causality constrain which terms are allowed in the theory rather than decide the possible terms and then try to establish causality.

Physical significance of the BDNK theory

Need to connect the BDNK theory with known physics.

- Entropy production is ≥ 0 within the limit of validity of the theory.

Physical significance of the BDNK theory

Need to connect the BDNK theory with known physics.

- Entropy production is ≥ 0 within the limit of validity of the theory.
- The BDNK tensor is derivable (formally) from kinetic theory in some specific limits (e.g., barotropic theory).

Physical significance of the BDNK theory

Need to connect the BDNK theory with known physics.

- Entropy production is ≥ 0 within the limit of validity of the theory.
- The BDNK tensor is derivable (formally) from kinetic theory in some specific limits (e.g., barotropic theory).
- Test-cases in conformal fluids reproduce expected behavior (Bjorken and Gubser flows).

Physical significance of the BDNK theory

Need to connect the BDNK theory with known physics.

- Entropy production is ≥ 0 within the limit of validity of the theory.
- The BDNK tensor is derivable (formally) from kinetic theory in some specific limits (e.g., barotropic theory).
- Test-cases in conformal fluids reproduce expected behavior (Bjorken and Gubser flows).
- Simulations, comparison to Israel-Stewart (Pandya-Pretorius, '21): BDNK agrees with Israel-Stewart for small viscosity and it presents smoothing effects.

Physical significance of the BDNK theory

Need to connect the BDNK theory with known physics.

- Entropy production is ≥ 0 within the limit of validity of the theory.
- The BDNK tensor is derivable (formally) from kinetic theory in some specific limits (e.g., barotropic theory).
- Test-cases in conformal fluids reproduce expected behavior (Bjorken and Gubser flows).
- Simulations, comparison to Israel-Stewart (Pandya-Pretorius, '21): BDNK agrees with Israel-Stewart for small viscosity and it presents smoothing effects.

The BDNK theory has many of the good features of the Israel-Stewart theory plus a good local well-posedness theory in Sobolev spaces, which is lacking for Israel-Stewart (applications to neutron star mergers).

Shocks and singularities in BDNK and Israel-Stewart

Fluid dynamics: shocks, singularities.

Fluid dynamics: shocks, singularities.

- Israel-Stewart: No physically acceptable strong viscous shock profile solutions (Geroch-Lindblom, '91; Olson-Hiscock, '91). Weak solutions? (Not in divergence form.)

Shocks and singularities in BDNK and Israel-Stewart

Fluid dynamics: shocks, singularities.

- Israel-Stewart: No physically acceptable strong viscous shock profile solutions (Geroch-Lindblom, '91; Olson-Hiscock, '91). Weak solutions? (Not in divergence form.)
- BDNK: first investigations (Freistühler, '21; Pandya-Pretorius, '21), potential advantage over Israel-Stewart.

Shocks and singularities in BDNK and Israel-Stewart

Fluid dynamics: shocks, singularities.

- Israel-Stewart: No physically acceptable strong viscous shock profile solutions (Geroch-Lindblom, '91; Olson-Hiscock, '91). Weak solutions? (Not in divergence form.)
- BDNK: first investigations (Freistühler, '21; Pandya-Pretorius, '21), potential advantage over Israel-Stewart.

Can singularities form in finite time from smooth initial data in Israel-Stewart and BDNK?

Theorem: Breakdown of smooth solutions to the Israel-Stewart equations (D-Hoang-Radosz, '20)

There exists an open set of smooth initial data for the Israel-Stewart equations for which the corresponding unique smooth solutions to the Cauchy problem break down in finite time. Such data consists of localized (large) perturbations of constant states.

Theorem: Breakdown of smooth solutions to the Israel-Stewart equations (D-Hoang-Radosz, '20)

There exists an open set of smooth initial data for the Israel-Stewart equations for which the corresponding unique smooth solutions to the Cauchy problem break down in finite time. Such data consists of localized (large) perturbations of constant states.

Proof:

- Known strategy (Guo-Tahvildar-Zadeh, '99): assume the solution exists for all time.
- Derive some quantitatively precise estimates for the evolution of $\mathcal{P}$.
- Derive a contradiction. (Proof by contradiction: it does not reveal the nature of the singularity; first breakdown result for Israel-Stewart.)

Looking ahead

Exciting time for relativistic viscous fluids.

Looking ahead

Exciting time for relativistic viscous fluids.

Excellent field for those interested in collaborative research among mathematicians and physicists.

– Thank you for your attention –