# Particle motion in Hamiltonian Formalism II 

## Or how to derive and solve equations of motion

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- $2^{\text {nd }}$ order dif. equations of motion from Newton's law (in configuration space) can be solved by transforming them to pairs of $1^{\text {st }}$ order dif. equations (in phase space)
■ Natural appearance of invariant of motion ("energy")
- Non-linear oscillators have frequencies which depend on the invariant (or "amplitude")
■ Connected invariant of motion to system's Hamiltonian (derived through Lagrangian)
$\square$ Shown that through the Hamiltonian, the equations of motions can be derived
- Poisson bracket operators are helpful for discovering integrals of motion


## Canonical transformations

## Canonical Transformations

$\square$ Find a function for transforming the Hamiltonian from variable ( $\mathbf{q}, \mathbf{p}$ ) to ( $\mathbf{Q}, \mathbf{P}$ ), so system becomes simpler to study
Transformation should be canonical (or symplectic), so that Hamiltonian properties (phase-space volume) are preserved
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These "mixed variable" generating functions are derived by

$$
\begin{aligned}
& F_{1}(\mathbf{q}, \mathbf{Q}): p_{i}=\frac{\partial F_{1}}{\partial q_{i}}, \quad P_{i}=-\frac{\partial F_{1}}{\partial Q_{i}} \quad F_{3}(\mathbf{Q}, \mathbf{p}): q_{i}=-\frac{\partial F_{3}}{\partial p_{i}}, \quad P_{i}=-\frac{\partial F_{3}}{\partial Q_{i}} \\
& F_{2}(\mathbf{q}, \mathbf{P}): p_{i}=\frac{\partial F_{2}}{\partial q_{i}}, \quad Q_{i}=\frac{\partial F_{2}}{\partial P_{i}} \quad F_{4}(\mathbf{p}, \mathbf{P}): q_{i}=-\frac{\partial F_{4}}{\partial p_{i}}, Q_{i}=\frac{\partial F_{4}}{\partial P_{i}}
\end{aligned}
$$

A general non-autonomous Hamiltonian is transformed to

$$
H(\mathbf{Q}, \mathbf{P}, t)=H(\mathbf{q}, \mathbf{p}, t)+\frac{\partial F_{j}}{\partial t}, j=1,2,3,4
$$

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$$

$\square$ One generating function can be constructed by the other through Legendre transformations, e.g.
$F_{2}(\mathbf{q}, \mathbf{P})=F_{1}(\mathbf{q}, \mathbf{Q})-\mathbf{Q} \cdot \mathbf{P}, \quad F_{3}(\mathbf{Q}, \mathbf{p})=F_{1}(\mathbf{q}, \mathbf{Q})-\mathbf{q} \cdot \mathbf{p}$, with the inner product define as $\mathbf{q} \cdot \mathbf{p}=\sum q_{i} p_{i}$

## Preservation of Phase

A fundamental property of canonical transformations is the preservation of phase space volume
This volume preservation in phase space can be represented in the old and new variables as

$$
\int \prod_{i=1}^{n} d p_{i} d q_{i}=\int \prod_{i=1}^{n} d P_{i} d Q_{i}
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$$

The volume element in old and new variables are related through the Jacobian

$$
\prod_{i=1}^{n} d p_{i} d q_{i}=\frac{\partial\left(P_{1}, \ldots, P_{n}, Q_{1}, \ldots, Q_{n}\right)}{\partial\left(p_{1}, \ldots, p_{n}, q_{1}, \ldots, q_{n}\right)} \prod_{i=1}^{n} d P_{i} d Q_{i}
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$$

These two relationships imply that the Jacobian of a canonical transformation should have determinant equal to 1
$\left|\frac{\partial\left(P_{1}, \ldots, P_{n}, Q_{1}, \ldots, Q_{n}\right)}{\partial\left(p_{1}, \ldots, p_{n}, q_{1}, \ldots, q_{n}\right)}\right|=\left|\frac{\partial\left(p_{1}, \ldots, p_{n}, q_{1}, \ldots, q_{n}\right)}{\partial\left(P_{1}, \ldots, P_{n}, Q_{1}, \ldots, Q_{n}\right)}\right|=\frac{1}{10}$

The transformation $Q=-p, P=q$, which interchanges conjugate variables is area preserving, as the Jacobian is

$$
\frac{\partial(P, Q)}{\partial(p, q)}=\left|\begin{array}{ll}
\frac{\partial P}{\partial p} & \frac{\partial Q}{\partial p} \\
\frac{\partial P}{\partial q} & \frac{\partial Q}{\partial q}
\end{array}\right|=\left|\begin{array}{rr}
0 & -1 \\
1 & 0
\end{array}\right|=1
$$

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$\square$ On the other hand, the transformation from Cartesian to polar coordinates $q=P \cos Q, \quad p=P \sin Q$ is not, since

$$
\frac{\partial(q, p)}{\partial(Q, P)}=\left|\begin{array}{cc}
-P \sin Q & P \cos Q \\
\cos Q & \sin Q
\end{array}\right|=-P
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\end{array}\right|=-P
$$

There are actually "polar" coordinates that are canonical, given by $q=-\sqrt{2 P} \cos Q, \quad p=\sqrt{2 P} \sin Q \quad$ for which

$$
\frac{\partial(q, p)}{\partial(Q, P)}=\left|\begin{array}{cc}
\sqrt{2 P} \sin Q & \sqrt{2 P} \cos Q \\
-\frac{\cos Q}{\sqrt{2 P}} & \frac{\sin Q}{\sqrt{2 P}}
\end{array}\right|=1
$$

# The Relativistic Hamiltonian for electromagnetic fields 

$\square$ Neglecting self fields and radiation, motion can be described by a "single-particle" Hamiltonian

$$
\begin{aligned}
H(\mathbf{x}, \mathbf{p}, t) & =c \sqrt{\left(\mathbf{p}-\frac{e}{c} \mathbf{A}(\mathbf{x}, t)\right)^{2}+m^{2} c^{2}}+e \Phi(\mathbf{x}, t) \\
\square \mathbf{x}=(x, y, z) & \text { Cartesian positions } \\
\square \mathbf{p}=\left(p_{x}, p_{y}, p_{z}\right) & \text { conjugate momenta } \\
\square \mathbf{A}=\left(A_{x}, A_{y}, A_{z}\right) & \text { magnetic vector potential } \\
\square \Phi & \text { electric scalar potential }
\end{aligned}
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\end{aligned}
$$

The ordinary kinetic momentum vector is written

$$
\mathbf{P}=\gamma m \mathbf{v}=\mathbf{p}-\frac{e}{c} \mathbf{A}
$$

with $\mathbf{V}$ the velocity vector and $\gamma=\left(1-v^{2} / c^{2}\right)^{-1 / 2}$ the relativistic factor

## Single-particle relativistic Hamiltonian

$$
H(\mathbf{x}, \mathbf{p}, t)=c \sqrt{\left(\mathbf{p}-\frac{e}{c} \mathbf{A}(\mathbf{x}, t)\right)^{2}+m^{2} c^{2}}+e \Phi(\mathbf{x}, t)
$$

It is generally a 3 degrees of freedom one plus time (i.e., 4 degrees of freedom)

- The Hamiltonian represents the total energy

$$
H \equiv E=\gamma m c^{2}+e \Phi
$$

$$
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$\square$ It is generally a 3 degrees of freedom one plus time (i.e., 4 degrees of freedom)

- The Hamiltonian represents the total energy

$$
H \equiv E=\gamma m c^{2}+e \Phi
$$

- The total kinetic momentum is

$$
P=\left(\frac{H^{2}}{c^{2}}-m^{2} c^{2}\right)^{1 / 2}
$$

$\square$ Using Hamilton's equations

$$
(\dot{\mathbf{x}}, \dot{\mathbf{p}})=[(\mathbf{x}, \mathbf{p}), H]
$$

it can be shown that motion is governed by Lorentz equations

## From Cartesian to "curved"

$\square$ It is useful (especially for rings) to transform the Cartesian coordinate system to the Frenet-Serret system moving
 to a closed curve, with path length $S$
$\square$ The position coordinates in the two systems are connected by $\mathbf{r}=\mathbf{r}_{\mathbf{0}}(s)+X \mathbf{n}(s)+Y \mathbf{b}(s)=x \mathbf{u}_{\mathbf{x}}+y \mathbf{u}_{\mathbf{y}}+z \mathbf{u}_{\mathbf{z}}$

## From Cartesian to

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The Frenet-Serret unit vectors and their derivatives are defined as $(\mathbf{t}, \mathbf{n}, \mathbf{b})=\left(\frac{d}{d s} \mathbf{r}_{\mathbf{0}}(s),-\rho(s) \frac{d^{2}}{d s^{2}} \mathbf{r}_{\mathbf{0}}(s), \mathbf{t} \times \mathbf{n}\right)$

$$
\frac{d}{d s}\left(\begin{array}{l}
\mathbf{t} \\
\mathbf{n} \\
\mathbf{b}
\end{array}\right)=\left(\begin{array}{ccc}
0 & -\frac{1}{\rho(s)} & 0 \\
\frac{1}{\rho(s)} & 0 & -\tau(s) \\
0 & 0 & \tau(s)
\end{array}\right)\left(\begin{array}{l}
\mathbf{t} \\
\mathbf{n} \\
\mathbf{b}
\end{array}\right)
$$

with $\rho(s)$ the radius of curvature and $\tau(s)$ the torsion which vanishes in case of planar motion
$\square W e$ are seeking a canonical transformation between

$$
\begin{aligned}
(\mathbf{q}, \mathbf{p}) & \mapsto(\mathbf{Q}, \mathbf{P}) \text { or } \\
\left(x, y, z, p_{x}, p_{y}, p_{z}\right) & \mapsto\left(X, Y, s, P_{x}, P_{y}, P_{s}\right)
\end{aligned}
$$

## $\square$ The generating function is

$$
(\mathbf{q}, \mathbf{P})=-\left(\frac{\partial F_{3}(\mathbf{p}, \mathbf{Q})}{\partial \mathbf{p}}, \frac{\partial F_{3}(\mathbf{p}, \mathbf{Q})}{\partial \mathbf{Q}}\right)
$$

$\square$ By using the relationship for the positions,

$$
\mathbf{r}=\mathbf{r}_{\mathbf{0}}(s)+X \mathbf{n}(s)+Y \mathbf{b}(s)=x \mathbf{u}_{\mathbf{x}}+y \mathbf{u}_{\mathbf{y}}+z \mathbf{u}_{\mathbf{z}}
$$

the generating function is

$$
F_{3}(\mathbf{p}, \mathbf{Q})=-\mathbf{p} \cdot \mathbf{r}
$$

## From Cartesian to "curved" planar motion, the momenta are

$$
\mathbf{P}=\left(P_{X}, P_{Y}, P_{s}\right)=\mathbf{p} \cdot\left(\frac{\partial F_{3}}{\partial X}, \frac{\partial F_{3}}{\partial Y}, \frac{\partial F_{3}}{\partial s}\right)=\mathbf{p} \cdot\left(\mathbf{n}, \mathbf{b},\left(1+\frac{X}{\rho}\right) \mathbf{t}\right)
$$

Taking into account that the vector potential is also transformed in the same way

$$
\left(A_{X}, A_{Y}, A_{s}\right)=\mathbf{A} \cdot\left(\mathbf{n}, \mathbf{b},\left(1+\frac{X}{\rho}\right) \mathbf{t}\right)
$$

the new Hamiltonian is given by
$\mathcal{H}(\mathbf{Q}, \mathbf{P}, t)=c \sqrt{\left(P_{X}-\frac{e}{c} A_{X}\right)^{2}+\left(P_{Y}-\frac{e}{c} A_{Y}\right)^{2}+\frac{\left(P_{s}-\frac{e}{c} A_{s}\right)^{2}}{\left(1+\frac{X}{\rho(s)}\right)^{2}}+m^{2} c^{2}}+e \Phi$
$\square$ It is more convenient to use the path length $s$, instead of the time as independent variable
$\square$ The Hamiltonian can be considered as having 4 degrees of freedom, where the $4^{\text {th }}$ "position" is time and its conjugate momentum is $P_{t}=-\mathcal{H}$

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$\square$ The Hamiltonian can be considered as having 4 degrees of freedom, where the $4^{\text {th }}$ "position" is time and its conjugate momentum is $P_{t}=-\mathcal{H}$
$\square$ In the same way, the new Hamiltonian with the path length as the independent variable is just $P_{s}=-\tilde{\mathcal{H}}\left(X, Y, t, P_{X}, P_{Y}, P_{t}, s\right)$ with
$\tilde{\mathcal{H}}=-\frac{e}{c} A_{s}-\left(1+\frac{X}{\rho(s)}\right) \sqrt{\left(\frac{P_{t}+e \Phi}{c}\right)^{2}-m^{2} c^{2}-\left(P_{x}-\frac{e}{c} A_{X}\right)^{2}-\left(P_{Y}-\frac{e}{c} A_{Y}\right)^{2}}$
$\square$ It can be proved that this is indeed a canonical transformation
$\square$ Note the existence of the reference orbit for zero vector potential, for which $\left(X, Y, P_{X}, P_{Y}, P_{s}\right)=\left(0,0,0,0, P_{0}\right)_{24}$

## Neglecting electric fields

$\square$ Due to the fact that longitudinal (synchrotron) motion is much slower than the transverse (betatron) one, the electric field can be set to zero and the Hamiltonian is written as

$$
\left.\tilde{\mathcal{H}}=-\frac{e}{c} A_{s}-\left(1+\frac{X}{\rho(s)}\right) \sqrt{P^{2}} \sqrt{\left(\frac{\mathcal{H}}{c}\right)^{2}-m^{2} c^{2}}-\left(P_{x}-\frac{e}{c} A_{X}\right)^{2}-\left(P_{Y}-\frac{e}{c} A_{Y}\right)^{2}\right)
$$

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$$

$\square$ The Hamiltonian is then written as
$\tilde{\mathcal{H}}=-\frac{e}{c} A_{s}-\left(1+\frac{X}{\rho(s)}\right) \sqrt{\left(P^{2}-\left(P_{x}-\frac{e}{c} A_{X}\right)^{2}-\left(P_{Y}-\frac{e}{c} A_{Y}\right)^{2}\right.}$
$\square$ If static magnetic fields are considered, the time dependence is also dropped, and the system is having 2 degrees of freedom + "time" (path length)
$\square$ Due to the fact that total momentum is much larger than the transverse ones, another transformation may be considered, where the transverse momenta are rescaled

$$
\begin{aligned}
(\mathbf{Q}, \mathbf{P}) & \mapsto(\overline{\mathbf{q}}, \overline{\mathbf{p}}) \text { or } \\
\left(X, Y, t, P_{X}, P_{Y}, P_{t}\right) & \mapsto\left(\bar{x}, \bar{y}, \bar{t}, \bar{p}_{x}, \bar{p}_{y}, \bar{p}_{t}\right)=\left(X, Y,-c t, \frac{P_{X}}{P_{0}}, \frac{P_{Y}}{P_{0}},-\frac{P_{t}}{P_{0} c}\right)
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\end{aligned}
$$

The new variables are indeed canonical if the Hamiltonian is also rescaled and written as

$$
\overline{\mathcal{H}}\left(\bar{x}, \bar{y}, \overline{\bar{y}} \overline{\bar{p}}_{x}, \bar{p}_{y}, \bar{p}_{t}\right)=\frac{\tilde{\mathcal{H}}}{P_{0}}=-e \bar{A}_{s}-\left(1+\frac{\bar{x}}{\rho(s)}\right) \sqrt{\bar{p}_{t}^{2}-\frac{m^{2} c^{2}}{P_{0}}-\left(\bar{p}_{x}-e \bar{A}_{x}\right)^{2}-\left(\bar{p}_{y}-e \bar{A}_{y}\right)^{2}}
$$

with $\quad\left(\bar{A}_{x}, \bar{A}_{y}, \bar{A}_{z}\right)=\frac{1}{P_{0} c}\left(\hat{A}_{x}, \hat{A}_{y}, \hat{A}_{s}\right)$
and $\quad \frac{m^{2} c^{2}}{P_{0}}=\frac{1}{\beta_{0}^{2} \gamma_{0}^{2}}$
$\square$ Along the reference trajectory $\quad \bar{p}_{t 0}=\frac{1}{\beta_{0}} \quad$ and $\left.\frac{d \bar{t}}{d s}\right|_{P=P_{0}}=\left.\frac{\partial \bar{H}}{\partial \bar{p}_{t}}\right|_{P=P_{0}}=-\bar{p}_{t 0}=-\frac{1}{\beta_{0}}$
$\square$ It is thus useful to move the reference frame to the reference trajectory for which another canonical transformation is performed

$$
(\overline{\mathbf{q}}, \overline{\mathbf{p}}) \quad \mapsto \quad(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \text { or }
$$

$\left(\bar{x}, \bar{y}, \bar{t}, \bar{p}_{x}, \bar{p}_{y}, \bar{p}_{t}\right) \quad \mapsto\left(\hat{x}, \hat{y}, \hat{t}, \hat{p}_{x}, \hat{p}_{y}, \hat{p}_{t}\right)=\left(\bar{x}, \bar{y}, \bar{t}+\frac{s-s_{0}}{\beta_{0}}, \bar{p}_{x}, \bar{p}_{y}, \bar{p}_{t}-\frac{1}{\beta_{0}}\right)$
$\square$ Along the reference trajectory $\quad \bar{p}_{t 0}=\frac{1}{\beta_{0}} \quad$ and $\left.\frac{d \bar{t}}{d s}\right|_{P=P_{0}}=\left.\frac{\partial \bar{H}}{\partial \bar{p}_{t}}\right|_{P=P_{0}}=-\bar{p}_{t 0}=-\frac{1}{\beta_{0}}$
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$\square$ The mixed variable generating function is
$(\hat{\mathbf{q}}, \overline{\mathbf{p}})=\left(\frac{\partial F_{2}(\overline{\mathbf{q}}, \hat{\mathbf{p}})}{\partial \hat{\mathbf{p}}}, \frac{\partial F_{2}(\overline{\mathbf{q}}, \hat{\mathbf{p}})}{\partial \overline{\mathbf{q}}}\right)$ providing

$$
\begin{aligned}
& F_{2}(\overline{\mathbf{q}}, \hat{\mathbf{p}})=\bar{x} \hat{p}_{x}+\bar{y} \hat{p}_{y}+\left(\bar{t}+\frac{s-s_{0}}{\beta_{0}}\right)\left(\hat{p}_{t}+\frac{1}{\beta_{0}}\right)
\end{aligned}
$$

$\square$ The Hamiltonian is then
$\hat{\mathcal{H}}\left(\hat{x}, \hat{y}, \hat{t}, \hat{p}_{x}, \hat{p}_{y}, \hat{p}_{t}\right)=\frac{1}{\beta_{0}}\left(\frac{1}{\beta_{0}}+\hat{p}_{t}\right)-e \hat{A}_{s}-\left(1+\frac{\hat{x}}{\rho(s)}\right) \sqrt{\left(\hat{p}_{t}+\frac{1}{\beta_{0}}\right)^{2}-\frac{1}{\beta_{0}^{2} \gamma_{0}^{2}}-\left(\hat{p}_{x}-e \hat{A}_{x}\right)^{2}-\left(\hat{p}_{y}-e \bar{A}_{y}\right)^{2}}$

## Relativistic and transverse field approximations

$\square$ First note that $\hat{p}_{t}=\bar{p}_{t}-\frac{1}{\beta_{0}}=\bar{p}_{t}-\bar{p}_{t 0}=\frac{P_{t}-P_{0}}{P_{0}} \equiv \delta$ and $l=\hat{t}$
In the ultra-relativistic limit $\beta_{0} \rightarrow 1, \frac{1}{\beta_{0}^{2} \gamma^{2}} \rightarrow 0$
and the Hamiltonian is written as $\mathcal{H}\left(x, y, l, p_{x}, p_{y}, \delta\right)=(1+\delta)-e \hat{A}_{s}-\left(1+\frac{x}{\rho(s)}\right) \sqrt{(1+\delta)^{2}-\left(p_{x}-e \hat{A}_{x}\right)^{2}-\left(p_{y}-e \hat{A}_{y}\right)^{2}}$ where the "hats" are dropped for simplicity

First note that $\hat{p}_{t}=\bar{p}_{t}-\frac{1}{\beta_{0}}=\bar{p}_{t}-\bar{p}_{t 0}=\frac{P_{t}-P_{0}}{P_{0}} \equiv \delta$ and $l=\hat{t}$
DIn the ultra-relativistic limit $\beta_{0} \rightarrow 1, \frac{1}{\beta_{0}^{2} \gamma^{2}} \rightarrow 0$ where the "hats" are dropped for simplicity
DIf we consider only transverse field components, the vector potential has only a longitudinal component and the Hamiltonian is written as
$\mathcal{H}\left(x, y, l, p_{x}, p_{y}, \delta\right)=(1+\delta)-e \hat{A}_{s}-\left(1+\frac{x}{\rho(s)}\right) \sqrt{(1+\delta)^{2}-p_{x}^{2}-p_{y}^{2}}$

- Note that the Hamiltonian is non-linear even in the absence of any field component (i.e. for a drift)!
$\square$ Summary of canonical transformations and approximations
$\square$ From Cartesian to Frenet-Serret (rotating) coordinate system (bending in the horizontal plane)
$\square$ Changing the independent variable from time to the path length $s$
$\square$ Electric field set to zero, as longitudinal (synchrotron) motion is much slower then transverse (betatron) one
$\square$ Consider static and transverse magnetic fields
$\square$ Rescale the momentum and move the origin to the periodic orbit
$\square$ For the ultra-relativistic limit $\beta_{0} \rightarrow 1, \frac{1}{\beta_{0}^{2} \gamma^{2}} \rightarrow 0$
the Hamiltonian becomes

$$
\mathcal{H}\left(x, y, l, p_{x}, p_{y}, \delta\right)=(1+\delta)-e \hat{A}_{s}-\left(1+\frac{x}{\rho(s)}\right) \sqrt{(1+\delta)^{2}-p_{x}^{2}-p_{y}^{2}}
$$

$$
\text { with } \frac{P_{t}-P_{0}}{P_{0}} \equiv \delta
$$

## High-energy, large

$\square$ It is useful for study purposes (especially for finding an "integrable" version of the Hamiltonian) to make an extra approximation
$\square$ For this, transverse momenta (rescaled to the reference momentum) are considered to be much smaller than 1, i.e. the square root can be expanded.
$\square$ Considering also the large machine approximation $x \ll \rho$, (dropping cubic terms), the Hamiltonian is simplified to

$$
\mathcal{H}=\frac{p_{x}^{2}+p_{y}^{2}}{2(1+\delta)}-\frac{x(1+\delta)}{\rho(s)}-e \hat{A}_{s}
$$

$\square$ This expansion may not be a good idea, especially for low energy, small size rings

## Linear magnetic fields

Assume a simple case of linear transverse magnetic fields,

$$
B_{x}=b_{1}(s) y
$$

$$
B_{y}=-b_{0}(s)+b_{1}(s) x
$$

$\square$ main bending field
$\square$ normalized quadrupole gradient

$$
K(s)=b_{1}(s) \frac{e}{c P_{0}}=\frac{b_{1}(s)}{B \rho}\left[1 / \mathrm{m}^{2}\right]
$$

$\square$ magnetic rigidity

$$
-B_{0} \equiv b_{0}(s)=\frac{P_{0} c}{e \rho(s)}[\mathrm{T}]
$$

$$
B \rho=\frac{P_{0} c}{e}[\mathrm{~T} \cdot \mathrm{~m}]
$$

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$$
B \rho=\frac{P_{0} c}{e}[\mathrm{~T} \cdot \mathrm{~m}]
$$

- The vector potential has only a longitudinal component which in curvilinear coordinates is

$$
B_{x}=-\frac{1}{1+\frac{x}{\rho(s)}} \frac{\partial A_{s}}{\partial y}, \quad B_{y}=\frac{1}{1+\frac{x}{\rho(s)}} \frac{\partial A_{s}}{\partial x}
$$

- The previous expressions can be integrated to give

$$
A_{s}(x, y, s)=\frac{P_{0} c}{e}\left[-\frac{x}{\rho(s)}-\left(\frac{1}{\rho(s)^{2}}+K(s)\right) \frac{x^{2}}{2}+K(s) \frac{y^{2}}{2}\right]=P_{0} c \hat{A}_{s}(x, y, s)
$$

## The integrable

The Hamiltonian for linear fields can be finally written as
$\mathcal{H}=\frac{p_{x}^{2}+p_{y}^{2}}{2(1+\delta)}-\frac{x \delta}{\rho(s)}+\frac{x^{2}}{2 \rho(s)^{2}}+\frac{K(s)}{2}\left(x^{2}-y^{2}\right)$

$$
\frac{d x}{d s}=\frac{p_{x}}{1+\delta}, \frac{d p_{x}}{d s}=\frac{\delta}{\rho(s)}-\left(\frac{1}{\rho^{2}(s)}+K(s)\right) x
$$

$$
\frac{d y}{d s}=\frac{p_{y}}{1+\delta}, \frac{d p_{y}}{d s}=K(s) y
$$

and they can be written as two second order uncoupled differential equations, i.e. Hill's equations (see Transverse Dynamics lecture)
$x^{\prime \prime}+\frac{1}{1+\delta}(\overbrace{\frac{1}{\rho(s)^{2}}+K(s)}^{K_{x}}) x=\frac{\delta}{\rho(s)}$
with the usual solution for
$y^{\prime \prime}-\frac{1}{1+\delta} \underbrace{K(s) y=0}_{K_{y}}$ $\delta=0$ and $u=x, y$
$u(s)=\sqrt{\epsilon \beta(s)} \cos \left(\psi(s)+\psi_{0}\right)$
$u^{\prime}(s)=\sqrt{\frac{\epsilon}{\beta(s)}}\left(\sin \left(\psi(s)+\psi_{0}\right)+\alpha(s) \cos \left(\psi(s)+\psi_{0}\right)\right){ }_{37}$

## Action-angle variables

- There is a canonical transformation to some optimal set of variables which can simplify the phase-space motion
- This set of variables are the action-angle variables
- The action vector is defined as the integral $\mathbf{J}=\oint \mathbf{p} d \mathbf{q}$
over closed paths in phase space.


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The action vector is defined as the integral $\mathbf{J}=\oint \mathbf{p} d \mathbf{q}$ over closed paths in phase space.

- An integrable Hamiltonian is written as a function of only the actions, i.e. $H_{0}=H_{0}(\mathbf{J})$. Hamilton's equations give $\dot{\phi}_{i}=\frac{\partial H_{0}(\mathbf{J})}{\partial J_{i}}=\omega_{i}(\mathbf{J}) \Rightarrow \phi_{i}=\omega_{i}(\mathbf{J}) t+\phi_{i 0}$ $\dot{J}_{i}=-\frac{\partial H_{0}(\mathbf{J})}{\partial \phi_{i}}=0 \Rightarrow J_{i}=$ const.

i.e. the actions are integrals of motion and the angles are evolving linearly with time, with constant frequencies which depend on the actions
■ The actions define the surface of an invariant torus, topologically equivalent to the product of $n$ circles

■ Considering on-momentum motion, the Hamiltonian can be written as

$$
\mathcal{H}=\frac{p_{x}^{2}+p_{y}^{2}}{2}+\frac{K_{x}(s) x^{2}-K_{y}(s) y^{2}}{2}
$$

- The generating function from the original to action angle variables is
$F_{1}\left(x, y, \phi_{x}, \phi_{y} ; s\right)=-\frac{x^{2}}{2 \beta_{x}(s)}\left[\tan \phi_{x}(s)+a_{x}(s)\right]-\frac{y^{2}}{2 \beta_{y}(s)}\left[\tan \phi_{y}(s)+a_{y}(s)\right]$
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■ The old variables with respect to actions and angles are
$u(s)=\sqrt{2 \beta_{u}(s) J_{u}} \cos \phi_{u}(s), \quad p_{u}(s)=-\sqrt{\frac{2 J_{u}}{\beta_{u}(s)}}\left(\sin \phi_{u}(s)+\alpha_{u}(s) \cos \phi_{u}(s)\right)$ and the Hamiltonian takes the form

$$
\mathcal{H}_{0}\left(J_{x}, J_{y}, s\right)=\frac{J_{x}}{\beta_{x}(s)}+\frac{J_{y}}{\beta_{y}(s)}
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and the Hamiltonian takes the form

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$$

■ The "time" (longitudinal position) dependence can be eliminated by the transformation to normalized coordinate

## Linear normal forms

- Make a coordinate transformation so that we get a simpler form of the matrix, i.e. ellipses are transformed to circles (simple rotation)


$$
M=\mathcal{A} \circ \mathcal{R} \circ \mathcal{A}^{-1} \quad \text { or }: \quad \mathcal{R}=\mathcal{A}^{-1} \circ M \circ \mathcal{A}
$$

- Using linear algebra, the solution is

$$
\mathcal{A}=\left(\begin{array}{cc}
\sqrt{\beta\left(s_{0}\right)} & 0 \\
-\frac{\alpha\left(s_{0}\right)}{\sqrt{\beta\left(s_{0}\right)}} & \frac{1}{\sqrt{\beta\left(s_{0}\right)}}
\end{array}\right) \quad \text { and } \quad \mathcal{R}=\left(\begin{array}{cc}
\cos \left(\mu_{x}\right) & \sin \left(\mu_{x}\right) \\
-\sin \left(\mu_{x}\right) & \cos \left(\mu_{x}\right)
\end{array}\right)
$$

■ This transformation can be extended to a non-linear system (see Advanced course)
co̊ Appendix

## Magnetic multipole

■ From Gauss law of magnetostatics, a vector potential exist

$$
\nabla \cdot \mathbf{B}=0 \quad \rightarrow \quad \exists \mathbf{A}: \quad \mathbf{B}=\nabla \times \mathbf{A}
$$

■ Assuming transverse 2D field, vector potential has only one component $A_{s}$. The Ampere's law in vacuum (inside the beam pipe) $\nabla \times \mathbf{B}=0 \quad \rightarrow \quad \exists V: \quad \mathbf{B}=-\nabla V$

- Using the previous equations, the relations between field components and potentials are

$$
B_{x}=-\frac{\partial V}{\partial x}=\frac{\partial A_{s}}{\partial y}, \quad B_{y}=-\frac{\partial V}{\partial y}=-\frac{\partial A_{s}}{\partial x}
$$

i.e. Riemann conditions of an analytic function


Exists complex potential of $z=x+i y$ with power series expansion convergent in a circle with radius $|z|=r_{c}$ (distance from iron yoke)

$$
\mathcal{A}(x+i y)=A_{s}(x, y)+i V(x, y)=\sum_{n=1}^{\infty} \kappa_{n} z^{n}=\sum_{n=1}^{\infty}\left(\lambda_{n}+i \mu_{n}\right)(x+i y)^{n}
$$

■ From the complex potential we can derive the fields
$B_{y}+i B_{x}=-\frac{\partial}{\partial x}\left(A_{s}(x, y)+i V(x, y)\right)=-\sum_{n=1}^{\infty} n\left(\lambda_{n}+i \mu_{n}\right)(x+i y)^{n-1}$
■ Setting $\quad b_{n}=-n \lambda_{n}, \quad a_{n}=n \mu_{n}$

$$
B_{y}+i B_{x}=\sum_{n=1}\left(b_{n}-i a_{n}\right)(x+i y)^{n-1}
$$

■ Define normalized coefficients

$$
b_{n}^{\prime}=\frac{b_{n}}{10^{-4} B_{0}} r_{0}^{n-1}, a_{n}^{\prime}=\frac{a_{n}}{10^{-4} B_{0}} r_{0}^{n-1}
$$

on a reference radius $r_{0}, 10^{-4}$ of the main field to get

$$
B_{y}+i B_{x}=10^{-4} B_{0} \sum_{n=1}^{\infty}\left(b_{n}^{\prime}-i a_{n}^{\prime}\right)\left(\frac{x+i y}{r_{0}}\right)^{n-1}
$$

■ Note: $n^{\prime}=n-1$ is the US convention

## Symplectic maps

- A generalization of the matrix (which can only describe linear systems), is a map, which transforms a system from some initial to some final coordinates

- Analyzing the map, will give useful information about the behavior of the system
- There are different ways to build the map:
$\square$ Taylor (Power) maps
$\square$ Lie transformations
$\square$ Truncated Power Series Algebra (TPSA), can generate maps from straight-forward tracking
- Preservation of symplecticity is important

■ Consider two sets of canonical variables $\mathbf{Z}, \overline{\mathbf{Z}}$ which may be even considered as the evolution of the system between two points in phase space

- A transformation from the one to the other set can be constructed through a map $\mathcal{M}: \mathbf{Z} \mapsto \overline{\mathbf{Z}}$
■ The Jacobian matrix of the map $M=M(\mathbf{z}, t)$ is composed by the elements $M_{i j} \equiv \frac{\partial \bar{z}_{i}}{\partial z_{j}}$
■ The map is symplectic if $M^{T} J M=J \quad$ where $J=\left(\begin{array}{rr}\mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0}\end{array}\right)$ ■ It can be shown that $\operatorname{det}(M)=1$
- It can be shown that the variables defined through a symplectic map $\quad\left[\bar{z}_{i}, \bar{z}_{j}\right]=\left[z_{i}, z_{j}\right]=J_{i j} \quad$ which is a known relation satisfied by canonical variables
■ In other words, symplectic maps preserve Poisson brackets
- Symplecticity guarantees that the transformations in phase space are area preserving
- To understand what deviation from symplecticity produces consider the simple case of the quadrupole with the general matrix written as

$$
\mathcal{M}_{\mathrm{Q}}=\left(\begin{array}{cc}
\cos (\sqrt{k} L) & \frac{1}{\sqrt{k}} \sin (\sqrt{k} L) \\
-\sqrt{k} \sin (\sqrt{k} L) & \cos (\sqrt{k} L)
\end{array}\right)
$$

■ Take the Taylor expansion for small lengths, up to first order

$$
\mathcal{M}_{\mathrm{Q}}=\left(\begin{array}{cc}
1 & L \\
-k L & 1
\end{array}\right)+O\left(L^{2}\right)
$$

■ This is indeed not symplectic as the determinant of the matrix is equal to $1+k L^{2}$, i.e. there is a deviation from symplecticity at $2^{\text {nd }}$ order in the quadrupole length provide the well-know elliptic trajectory in phase space

- Although the trajectory is very close to the original one, it spirals outwards towards infinity



## Lie formalism

■ The Poisson bracket properties satisfy what is mathematically called a Lie algebra

- They can be represented by (Lie) operators of the form $: f: g=[f, g]$ and $: f:{ }^{2} g=[f,[f, g]]$ etc.


## Lie formalism

- The Poisson bracket properties satisfy what is mathematically called a Lie algebra
- They can be represented by (Lie) operators of the form $: f: g=[f, g]$ and $: f:{ }^{2} g=[f,[f, g]]$ etc.
■ For a Hamiltonian system $H(\mathbf{z}, t)$ there is a formal solution of the equations of motion $\frac{d \mathbf{z}}{d t}=[H, \mathbf{z}]=: H: \mathbf{z}$ written as $\mathbf{z}(t)=\sum_{k=0}^{\infty} \frac{t^{k}: H:^{k}}{k!} \mathbf{z}_{0}=e^{t: H:} \mathbf{z}_{0}$ with a symplectic $\operatorname{map} \mathcal{M}=e^{: H:{ }^{k=0}}$

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- The 1-turn accelerator map can be represented by the composition of the maps of each element $\mathcal{M}=e^{: f_{2}:} e^{: f_{3}:} e^{: f_{4}:} \ldots$ where $f_{i}$ (called the generator) is the Hamiltonian for each element, a polynomial of degree $m$ in the variables $z_{1}, \ldots, z_{n}$


## Hamiltonian

■ Considering the general expression of the the longitudinal component of the vector potential is (see appendix)

- In curvilinear coordinates (curved elements)

$$
A_{s}=\left(1+\frac{x}{\rho(s)}\right) B_{0} \Re e \sum_{n=0}^{\infty} \frac{b_{n}+i a_{n}}{n+1}(x+i y)^{n+1}
$$

$\square$ In Cartesian coordinates $A_{s}=B_{0} \Re<\sum_{n=0}^{\infty} \frac{b_{n}+i a_{n}}{n+1}(x+i y)^{n+1}$ with the multipole coefficients being written as

$$
a_{n}=\left.\frac{1}{B_{0} n!} \frac{\partial^{n} B_{x}}{\partial x^{n}}\right|_{x=y=0} \text { and } b_{n}=\left.\frac{1}{B_{0} n!} \frac{\partial^{n} B_{y}}{\partial x^{n}}\right|_{x=y=0}
$$

- The general non-linear Hamiltonian can be written as

$$
\mathcal{H}\left(x, y, p_{x}, p_{y}, s\right)=\mathcal{H}_{0}\left(x, y, p_{x}, p_{y}, s\right)+\sum_{k_{x}, k_{y}} h_{k_{x}, k_{y}}(s) x^{k_{x}} y^{k_{y}}
$$

with the periodic functions $h_{k_{x}, k_{y}}(s)=h_{k_{x}, k_{y}}(s+C)$

- Dipole:

$$
H=\frac{x \delta}{\rho}+\frac{x^{2}}{2 \rho^{2}}+\frac{p_{x}^{2}+p_{y}^{2}}{2(1+\delta)}
$$

■ Quadrupole:

$$
H=\frac{1}{2} k_{1}\left(x^{2}-y^{2}\right)+\frac{p_{x}^{2}+p_{y}^{2}}{2(1+\delta)}
$$

$$
H=\frac{1}{3} k_{2}\left(x^{3}-3 x y^{2}\right)+\frac{p_{x}^{2}+p_{y}^{2}}{2(1+\delta)}
$$

- Octupole:

$$
H=\frac{1}{4} k_{3}\left(x^{4}-6 x^{2} y^{2}+y^{4}\right)+\frac{p_{x}^{2}+p_{y}^{2}}{2(1+\delta)}
$$

■ Consider the 1D quadrupole Hamiltonian

$$
H=\frac{1}{2}\left(k_{1} x^{2}+p^{2}\right)
$$

$■$ For a quadrupole of length $L$, the map is written as

$$
e^{\frac{L}{2}:\left(k_{1} x^{2}+p^{2}\right):}
$$

## Map for quadrupole

■ Consider the 1D quadrupole Hamiltonian

$$
H=\frac{1}{2}\left(k_{1} x^{2}+p^{2}\right)
$$

■ For a quadrupole of length $L$, the map is written as

$$
e^{\frac{L}{2}:\left(k_{1} x^{2}+p^{2}\right)}
$$

$\square$ Its application to the transverse variables is

$$
\begin{aligned}
e^{-\frac{L}{2}:\left(k_{1} x^{2}+p^{2}\right):} x & =\sum_{n=0}^{\infty}\left(\frac{\left(-k_{1} L^{2}\right)^{n}}{(2 n)!} x+L \frac{\left(-k_{1} L^{2}\right)^{n}}{(2 n+1)!} p\right) \\
e^{-\frac{L}{2}:\left(k_{1} x^{2}+p^{2}\right):} p & =\sum_{n=0}^{\infty}\left(\frac{\left(-k_{1} L^{2}\right)^{n}}{(2 n)!} p-\sqrt{k_{1}} \frac{\left(-k_{1} L^{2}\right)^{n}}{(2 n+1)!} p\right)
\end{aligned}
$$

