The Measurement Problem in the Statistical Signal Processing

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Introduction

- Uncertainty relation
    - Quantum mechanics by Somerfeld and Heisenberg
    - Wave mechanics by Schrödinger
    - Communication theory by Gabor
    - Commutator relation $[Q, P] = iI$
- Statistical signal processing
    - Implementation of a stochastic process
    - Quantum informatics
    - Measurement problem

Content

- Introduction
- A Definition of the Measurement Problem
- A Paradigm of the Measurement Process
- Wavelets and Measurement Hierarchy
- Orthogonal Wavelets and Projective Measurements
- Frame Wavelets and General Measurements

A Definition of the Measurement Problem

- Foundations of QT by von Neumann
  - Reversible evolution (no entropy production)
  - Irreversible measurement (entropy production)
  - The time operator nonentity
  - Essential weakness of QT
  - Chief link between QT and RT
- Time-energy commutator relation
  - No Hamiltonian uncertainty $[T, H] \neq iI$
  - Liouvillian uncertainty $[T, L] = iI$

Quantum ensembles

- Liouville-von Neumann mechanics $P \mapsto p(P)$
  - $p(0) = 0, p(I) = 1, p(\sum_{\perp i} P_i) = \sum_i p(P_i)$
  - Density operator $\rho^\dagger = \rho, \rho \geq 0, Tr\,\rho = 1$
  - Gleason`s theorem $p(P) = \langle \rho | P \rangle$
- Koopman-von Neumann mechanics $\rho = \rho |1\rangle\langle 1|$
  - Density function $p(|1\rangle\langle 1|) = 1$
  - Hilbert space $L^2_\mu(\Omega)$, $\rho = \rho|1\rangle$, $Tr\,\rho = \int_\Omega \rho d\mu$
  - Transformation group $G^t\colon \Omega \to \Omega$, $\mu \circ G^{-1} = \mu$

The time operator formalism

- Group of evolutionary operators
  - Evolution of a variable $U^t F = F \circ G^t$
  - Evolution of a density $U^{t\dagger} F = F \circ G^{-t}$
  - Stone theorem $U^{\dagger t} = e^{iLt}$
  - Liouville equation $\frac{\partial \rho}{i \partial t} = L\rho$
- Commutator relation
  - Liouvillian operator $[T, L] = iI$
  - Evolutionary group $[T, U^t] = tU^t$
  - Cyclic group $[T, U] = U$

Complex systems physics

- Change in representation $\Lambda = \lambda(T)$
  - Lie group $U^{\dagger t}$ to the Markov semigroup $W^{\dagger t} = \Lambda U^{\dagger t} \Lambda^{-1}$
  - Irreversible evolution $W^{\dagger t}$, $\mathrm{t} < 0$ is not positivity preserving
- Terms of the change
  - Preservation of positivity $\rho \geq 0 \Rightarrow \Lambda\rho \geq 0$
  - Preservation of trace $Tr\ \rho = Tr\ \Lambda\rho$
  - Preservation of uniformity $I = \Lambda I$
  - Λ is invertible in a dense subset
    - No loss of information concerning system`s state

A Paradigm of the Measurement Process

- Euclidean algorithm $\frac{a}{b} = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\ddots}}}$
- Continued fraction sequence

$$\xi_i = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\ddots + \cfrac{1}{n_i}}}} = \frac{h_i}{k_i}$$

- Recurrence equation

$$h_{i+1} = n_{i+1} h_i + h_{i-1} \quad k_{i+1} = n_{i+1} k_i + k_{i-1}$$

The Ford diagram

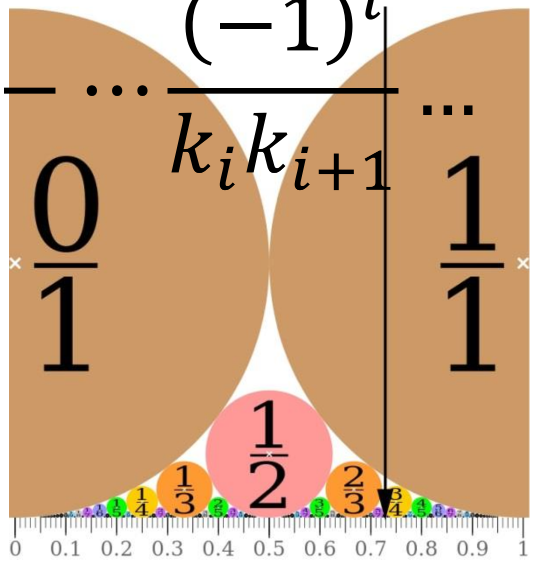
- Difference of elements

$$\Delta\xi_i = \xi_{i+1} - \xi_i = \frac{h_{i+1}}{k_{i+1}} - \frac{h_i}{k_i} = \frac{(-1)^i}{k_{i+1}k_i}$$

- Continued fraction series

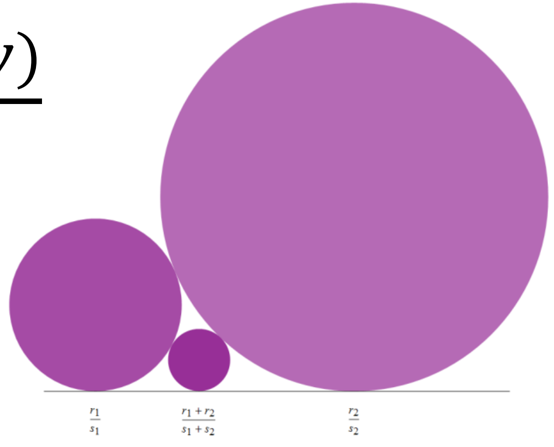
$$x = \Delta\xi_0 + \cdots + \Delta\xi_i + \cdots = \frac{1}{k_0k_1} - \cdots \frac{(-1)^i}{k_ik_{i+1}} \cdots$$

- Redundant dictionary $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$



The Minkowski function

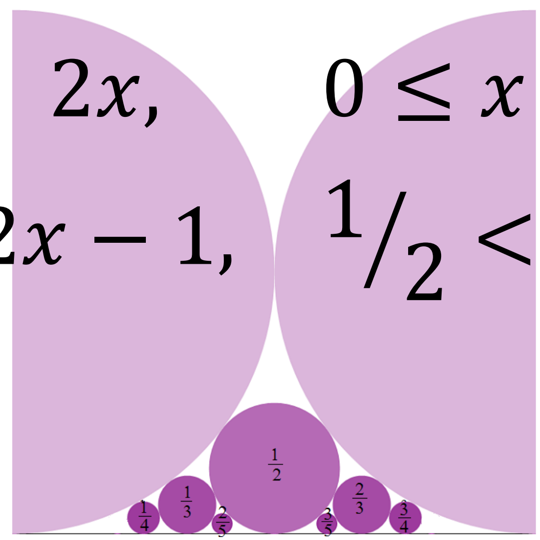
- Question mark $?: \frac{1}{n_1+\frac{1}{n_2+\frac{1}{\ddots}}} \mapsto \frac{1}{2^{n_1-1}} - \frac{1}{2^{n_1+n_2-1}} + \cdots$
- Continued to binary code $?(x) = \sum_i \frac{(-1)^i}{2^{n_1+\cdots+n_i-1}}$
- Mediant or the Farey sum $\frac{r_1}{s_1} \boxplus \frac{r_2}{s_2} = \frac{r_1+r_2}{s_1+s_2}$
- Automorphism $?(x \boxplus y) = \frac{?(x)+?(y)}{2}$

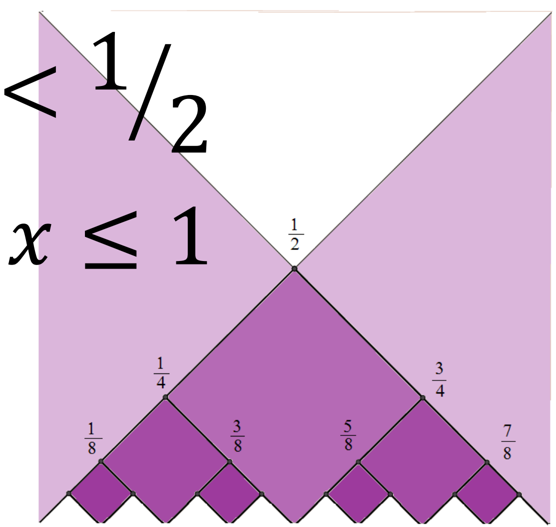


The binary tree

- Coordinates of a node $x = \frac{2k-1}{2^{j+1}}$ and $y = \frac{1}{2^{j+1}}$
- Interval $[x - y, x + y] = \left[\frac{k-1}{2^j}, \frac{k}{2^j}\right]$
- Rényi map

$$Rx = \begin{cases} 2x, & 0 \le x < {}^1/_2 \\ 2x - 1, & {}^1/_2 < x \le 1 \end{cases}$$





Wavelets and the Measurement Hierarchy

- Domain $\mathbb{I} = [0,1]$
- Autoduality $\Sigma = \Delta = L^2(\mathbb{I})$
- Wavelet base $\psi_{j,k}$ of $L^2(\mathbb{I}) \ominus \mathbb{1}$
- Resolution of identity

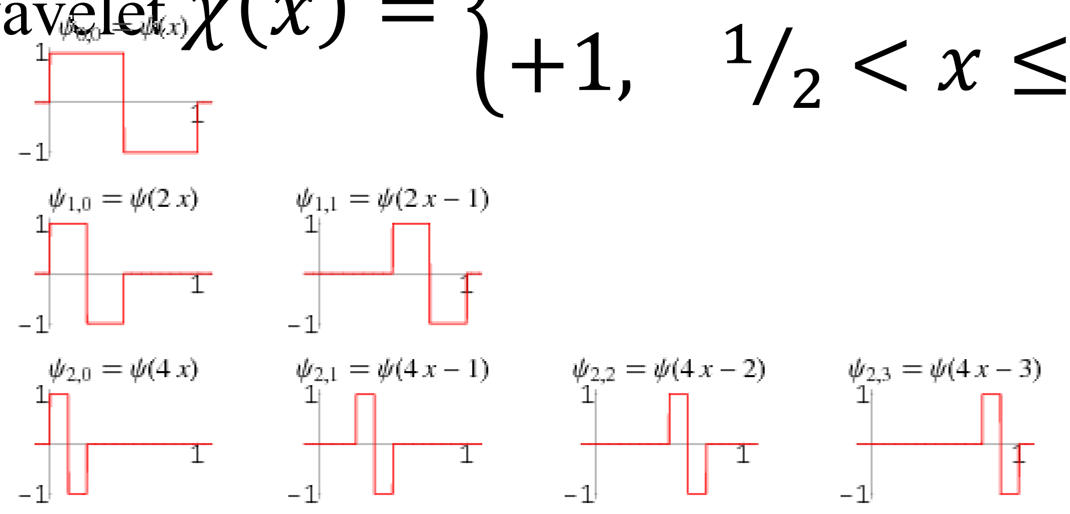
$$I = |1\rangle\langle 1| + \sum_{j\geq 0} \sum_{k=1}^{2^j} |\psi_{j,k}\rangle \left\langle \psi_{j,k} \right|$$

  - $\langle\cdot|$ state
  - $|\cdot\rangle$ device

The Haar base

- Wavelets $\chi_{j,k}(x) = \begin{cases} -2^{j/2}, & \frac{k}{2^j} \leq x < \frac{k+1/2}{2^j} \\ +2^{j/2}, & \frac{k+1/2}{2^j} < x \leq \frac{k+1}{2^j} \end{cases}$

- Mother wavelet $\chi(x) = \begin{cases} -1, & 0 \leq x < 1/2 \\ +1, & 1/2 < x \leq 1 \end{cases}$



Wavelets in the interval domain

- Orthonormal base $\Psi_{j,k}(x) = 2^{\frac{j}{2}}\Psi_0\,(2^j x - k)$
- Wavelets for $L^2(\mathbb{I})$ $\psi_{j,k}(x) = \sum_n \Psi_{j,k}(x+n)$
- Periodization axiom $\psi_{j,k+2^j} = \psi_{j,k}$
- Annihilation axiom $j < 0 \Rightarrow \psi_{j,k} = 0$
- Translation axiom $\psi_{j,k+m}(x) = \psi_{j,k}(x - \frac{m}{2^j})$

Wavelets and stochastic processes

- Evolutionary operator $Uf(x) = f(Rx)$
- Adjoint operator $U^{\dagger}f(x) = \frac{f\left(\frac{x}{2}\right)+f\left(\frac{x+1}{2}\right)}{2}$
- Evolution axiom $U^{\dagger}\psi_{j,k} = \frac{1}{\sqrt{2}}\psi_{j-1,k}$
- Equivalent formulation

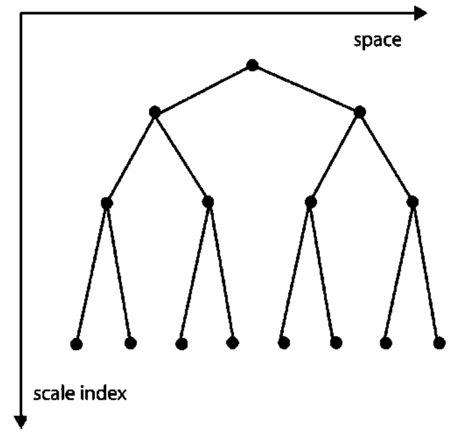
$$U\psi_{j,k} = \frac{1}{\sqrt{2}}\psi_{j+1,k} + \frac{1}{\sqrt{2}}\psi_{j+1,k+2^j}$$

Wavelet variables

- Equal distribution within scale $\psi_{j,k} \doteq \psi_{j,k+m}$
- Equal distribution across scale $\psi_{j,k} \doteq \mathrm{U}\psi_{j,k}$
- Zero mean $E\psi_{j,k} = \langle 1 | \psi_{j,k} \rangle = 0$
- Unit variance $D\psi_{j,k} = \left\| \psi_{j,k} \right\|^2 = 1$
- Mutual independence $\psi_{j,k} \neq \psi_{l,m}$

  $E\overline{\psi_{j,k}}\psi_{l,m} = \langle \psi_{j,k} | \psi_{l,m} \rangle = 0 = E\psi_{j,k}E\psi_{l,m}$

A measurement hierarchy of the wavelet base



- Distribution density $\left|\psi_{j,k}\right|^2$
- Estimation $\int_0^1 x\left|\psi_{j,k}(x)\right|^2 dx \approx \frac{2k-1}{2^{j+1}}$
- Time operator

$$T = \sum_{j\geq 0}\sum_{k=1}^{2^j} j\left|\psi_{j,k}\right\rangle\left\langle\psi_{j,k}\right|$$

- Commutator relation $\left[U^\dagger, T\right] = U^\dagger \Rightarrow [T, U] = U$

The space of signal ensembles

- Haar`s extension $U_\chi F = F \circ B$
- Baker map $B(x,y) =$

$$\begin{cases} \left(2x, \frac{y}{2}\right), & 0 \le x < 1/2 \\ \left(2x-1, \frac{y+1}{2}\right), & 1/2 < x \le 1 \end{cases}$$

- Extended space $L^2(\mathbb{I} \times \mathbb{I}) = L^2(\mathbb{I}) \otimes L^2(\mathbb{I})$
- Signal ensembles $\mathrm{F}: \mathbb{I} \to L^2(\mathbb{I})$

Extension of the time operator

- Representation $\mathrm{F} = |1\rangle\langle A| + \sum_{j,k} |\psi_{j,k}\rangle \left\langle D_{j,k} \right|$
  - Approximation $\langle A| = \langle 1|F$, detail coefficients $\left\langle D_{j,k} \right| = \left\langle \psi_{j,k} \right|F$
- Matrix multiplication $FG(x,y) = \int_0^1 F(x,t)G(t,y)dt$
- Time operator T_χ for the Haar evolution U_χ
- Extension of the time operator $T = CT_\chi C^\dagger$
- Extension of the evolutionary operator $U = CU_\chi C^\dagger$
- Change of the base $C\colon \chi_{j,k} \mapsto \psi_{j,k}$

The density evolution

- Density operator $\rho = FF^{\dagger}, \|F\| = 1$
- Evolutionary operator $(UF)(UF)^{\dagger} = U\rho U^{\dagger} = \mathfrak{U}\rho$
- Positivity preservation $\rho \geq 0 \Rightarrow \mathfrak{U}^{\dagger}\rho \geq 0$
- Time operator $[T, \mathfrak{U}]\rho = [T, U]\rho U^{\dagger} = \mathfrak{U}\rho$
- Change in representation $\Lambda = \lambda(T)$
- Markov semigroup $\mathfrak{W}^{\dagger t} = \Lambda \mathfrak{U}^{\dagger t} \Lambda^{-1}$

Orthogonal Wavelets and Projective Measurements

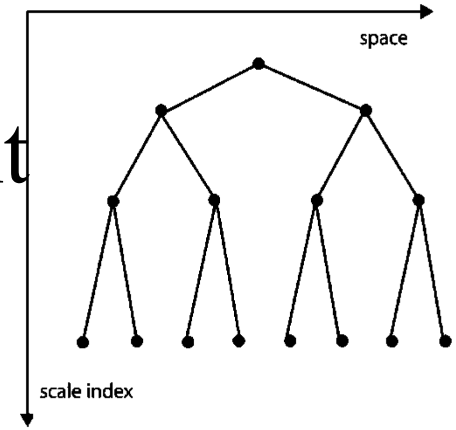
- Wavelet base of the space $L^2(\mathbb{I}) \ominus \mathbb{1}$
- Mutually independent variables $\psi_{j,k}$
- Distribution densities $\left|\psi_{j,k}\right|^2$
- The space of signal ensembles$L^2(\mathbb{I} \times \mathbb{I}) = \Delta \otimes \Sigma$
- Embedment of devices $|\cdot\rangle \hookrightarrow |\cdot\rangle\langle 1|$
- Embedment of states $\langle \cdot | \hookrightarrow |1\rangle\langle \cdot |$

The von Neumann measurement

- Orthogonal projectors $P_{j,k} = |\psi_{j,k}\rangle\langle\psi_{j,k}|$
- Probability $\langle\rho|P_{j,k}\rangle = \|D_{j,k}\|^2 = |d_{j,k}|^2$
- Expectation $|d_{j,k}|^2 = E|D_{j,k}|^2$
- Reduction of the density operator

$$\mathfrak{M}\rho = \sum_{j,k} |d_{j,k}|^2 P_{j,k} = \sum_{j,k} P_{j,k}\rho P_{j,k}$$

The optimal measurement



- The measurement operator $\sum_{j,k} d_{j,k} P_{j,k}$
- Optimal base $F = \sum_{j,k} d^{o}_{j,k} P^{o}_{j,k} = \sum_{j,k} |\psi^{o}_{j,k}\rangle \left\langle D^{o}_{j,k} \right|$
- Decorrelation of detail coefficients $\left\langle D^{o}_{j,k} \right| = d^{o}_{j,k} \langle \psi^{o}_{j,k}|$
- Approximate decorrelation in a suboptimal base

$$\left\langle D_{l,m} \right| = \sum_{j,k} d^{o}_{j,k} \langle \psi_{l,m} | \psi^{o}_{j,k} \rangle \langle \psi^{o}_{j,k}|$$

$$\left[\frac{k-1}{2^j} \frac{k}{2^j}\right] \cap \left[\frac{l-1}{2^l} \frac{l}{2^l}\right] = \emptyset \Longrightarrow \langle \psi_{l,m} | \psi^{o}_{j,k} \rangle \approx 0$$

The Euclidean paradigm

- Optimal measurement $F|\psi_{j,k}\rangle = d_{j,k}|\psi_{j,k}\rangle$

$$\sum_j c_j U^j \psi_0 = \sum_{j,k} c_j 2^{-j/2} \psi_{j,k} = \sum_{j,k} d_{j,k} \psi_{j,k}$$

- Eigenvalues $d_j = 2^{-j/2} c_j \Longrightarrow F = d(T)$
  - Optimal time $T = \sum_{j\geq 0} \sum_{k=1}^{2^j} j|\psi_{j,k}\rangle\langle\psi_{j,k}|$
- Density operator $\rho = FF^{\dagger}/\|F\|^2$
- Normalization by $\|F\|^2 = \sum_j 2^{-j} c_j$

The Euclidean ensembles

- Operator function of optimal time $F = d(T)$
- Contribution of a digit $|d|_j^2 = {2^{-j}c_j}\Big/{\sum_j 2^{-j}c_j}$
- Detail coefficients $\left\langle D_{j,k}^{o}\right| = d_j \left\langle \psi_{j,k}\right|$
- Statistical stationarity $\left\langle D_{j,k}\right| \doteq \left\langle D_{j,m}\right|$

Evolution of the measurement process

- Projective measurement $\mathfrak{M}\rho = \sum_{j,k} P_{j,k}\rho P_{j,k}$
- Temporal decomposition $\mathfrak{M}_j = \sum_k \mathfrak{P}_{j,k}$
- Measurement evolution $\mathfrak{M}_{j+1} = 2\mathfrak{U}\mathfrak{M}_j\mathfrak{U}^\dagger$
- Measurement process $\mathfrak{M} = \sum_j 2^j \boldsymbol{\mathcal{U}}^j \mathfrak{M}_0$
- Elementary measurement $\mathfrak{M}_0 = \mathfrak{P}_0$
    - $\mathfrak{M}_0\rho = P_0\rho P_0$
- Elementary device $|\psi_0\rangle\langle 1|$

Crossing between states and devices

- Measurement evolution $\mathfrak{M}_j\rho = 2^j\mathfrak{U}^j\mathfrak{M}_0\mathfrak{U}^{\dagger j}\rho$
- Density evolution $\mathfrak{U}^\dagger\rho = (U^\dagger F)(U^\dagger F)^\dagger$
- State into device $U^\dagger|1\rangle\langle\psi_0| = |\psi_0\rangle\langle 1|$
- Measurement display $\mathfrak{M}_0\rho = (P_0F)(P_0F\,)^\dagger$
- Device into state $U|\psi_0\rangle\langle 1| = |1\rangle\langle\psi_0|$
- Root of the ensemble $M_j\colon F \mapsto 2^{j/2}U^jP_0U^{\dagger j}F$

Psychophysical parallelism

- Boundary between states and devices which is arbitrary
- Crossing of devices into states and vice versa
- Bohr, Höffding and Fechner
- Identity view
    - Outer psychophysics (sensation and stimulation)
    - Inner psychophysics (sensation and neuroactivity)
- Irreversibility and observer`s mind
- Change in representation $\Lambda = \lambda(T)$

From the outer to the inner psychophysics

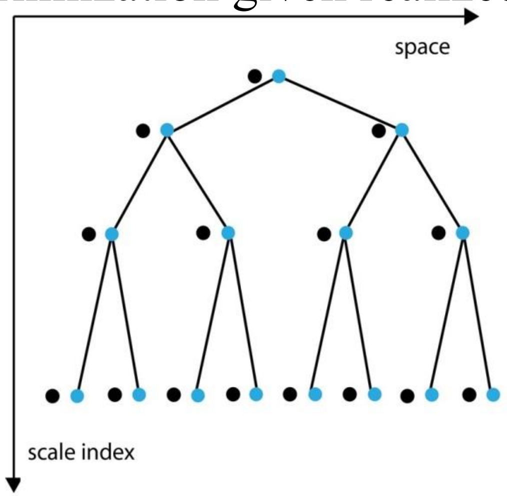
- Measurement process $M_j: F \mapsto \sum_k 2^{j/2} P_{j,k} F$
- Evolution of projectors $\mathfrak{U} \sum_k P_{j,k} = \sum_k P_{j+1,k}$
- Semigroup $\mathfrak{W}^\dagger = \Lambda \mathfrak{U}^\dagger \Lambda^{-1}$

$$M_{j+1} = \sqrt{2} \sum_k \mathfrak{U} 2^{j/2} P_{j,k} F = \sqrt{2} \sum_k \Lambda^\dagger \mathfrak{W} \Lambda^{\dagger -1} 2^{j/2} P_{j,k} F$$

- Detail coefficients $P_{j,k} F = |\psi_{j,k}\rangle \left\langle D_{j,k} \right|$
- Hidden variables $S_{j,k} = \Lambda^{\dagger -1} 2^{j/2} P_{j,k} F$
- Markov process $\sqrt{2} \mathfrak{W} \sum_k S_{j,k} = \sum_k S_{j+1,k}$

Wavelet domain hidden Markov model

- Approximate decorrelation of coefficients $\mathbf{D} = \left(D_{j,k}\right)$
- Markovian tree of hidden variables $\mathbf{S} = \left(S_{j,k}\right)$
- Statistical stationarity within each scale
- Baum-Welch algorithm for parameters estimation
  - Expectation maximization given realized values of coefficients



Canonical relation

- Global entropy of the Markovian tree $H(\mathbf{S})$
- Increase of local entropy $H(S_{j,k}) \nearrow$
- Entropy of coefficients $H(\mathbf{D}) = H(\mathbf{S}) + H(\mathbf{D}|\mathbf{S})$
- Outer psychophysical information

$$H(C\mathbf{D}) = H(\mathbf{D}) + \log|\det C| = H(\mathbf{D})$$

- Inner psychophysical information $H(\mathbf{S})$
- Irreducible randomness $H(\mathbf{D}|\mathbf{S})$

The Fechner law

- Logarithmic dependence between outer and inner scales
- Eigenvalues $d_j \ = 2^{-j/2} c_j$
- Uniform distribution of c_j for normal numbers
- Exponential decay of detail coefficients across scale
    - Almost all ensembles of the Euclidean paradigm

Frame Wavelets and General Measurements

- Frame $\psi_{j,k}$

$$AI \leq \sum_{j,k} |\psi_{j,k}\rangle\langle\psi_{j,k}| \leq BI$$

- Parseval frame $A = B = 1$

$$I = \sum_{j,k} |\psi_{j,k}\rangle\langle\psi_{j,k}|$$

- Dual frame $\widetilde{\psi}_{j,k}$

$$I = \sum_{j,k} |\psi_{j,k}\rangle\langle\widetilde{\psi}_{j,k}|$$

Canonical dual

- $\vDash$ such that $\vDash \psi_{j,k}\rangle$ is Parseval frame

$$I = \sum_{j,k} \vDash \psi_{j,k}\rangle\langle\psi_{j,k} \Dashv$$

$$\vDash^{-1}\Dashv^{-1} = \#^{-1} = \sum_{j,k} |\psi_{j,k}\rangle\langle\psi_{j,k}|$$

$$I = \sum_{j,k} |\psi_{j,k}\rangle\langle\psi_{j,k}|\#$$

Wavelet frame

- Periodization axiom $\psi_{j,k+2^j} = \psi_{j,k}$
- Annihilation axiom $j < 0 \Rightarrow \psi_{j,k} = 0$
- Translation axiom $\psi_{j,k+m}(x) = \psi_{j,m}(x - \frac{m}{2^j})$
- Evolution axiom $U^\dagger \psi_{j,k} = \frac{1}{\sqrt{2}} \psi_{j-1,k}$
- General measurement $\mathfrak{M}\rho = \sum_{j,k} M_{j,k} \rho\, {M_{j,k}}^\dagger$

General measurement

- Measurement operators $I = \sum_{j,k} M_{j,k} \, M_{j,k}{}^{\dagger}$
- Resolution of identity $I = \sum_{j,k} |\psi_{j,k}\rangle \left\langle \widetilde{\psi}_{j,k} \right|$
  - $P_{j,k} = |\psi_{j,k}\rangle \left\langle \widetilde{\psi}_{j,k} \right|$
  - $\left\| \widetilde{\psi}_{j,k} \right\| = 1$
  - $\sum_{j,k} P_{j,k} \, P_{j,k}{}^{\dagger} = \sum_{j,k} |\psi_{j,k}\rangle \left\langle \psi_{j,k} \right| = \#^{-1} = \vDash^{-1} \dashv^{-1}$
  - $M_{j,k} = \vDash \psi_{j,k}\rangle \left\langle \widetilde{\psi}_{j,k} \right|$
- Euclidean frames

The time operator

- Time of wavelets $T = \sum_{j\geq 0} \sum_{k=1}^{2^j} j|\psi_{j,k}\rangle \langle\psi_{j,k}|$

$$\vDash T \Dashv = \sum_{j\geq 0} \sum_{k=1}^{2^j} j \vDash \psi_{j,k}\rangle \langle\psi_{j,k} \Dashv$$

- Commutator relation $[\vDash T \Dashv, U] = U$

An ancillary extension

- Frame $|\psi_{j,k}\rangle$
  - Dual frame $\langle\tilde{\psi}_{j,k}|$
- Parseval frame $\vDash \psi_{j,k}\rangle$
- Riesz base $|\psi_{j,k} \gg$
  - Biorthogonal base $\ll\tilde{\psi}_{j,k}|$
- Orthonormal base $\vDash \psi_{j,k}\gg$
  - Orthonoraml base $\ll\psi_{j,k}\Dashv$

The optimal representation

- Optimal frame $F = \sum_{j,k} d_j \ |\psi^o_{j,k}\rangle \left\langle \psi^o_{j,k} \right| = d(T)$
    - Optimal time $\mathrm{T} = \sum_{j,k} j|\psi^o_{j,k}\rangle \left\langle \psi^o_{j,k} \right|$
- Ancillary extension

$$\gg F \ll = \sum_{j,k} d^o_j \left| \psi^o_{j,k} \gg \ll \psi^o_{j,k} \right|$$

$$\ll D^o_{j,k} \mid = \ll \widetilde{\psi}^o_{j,k} \mid \gg F \ll = d^o_j \ll \psi^o_{j,k} \mid$$

- Independent variables $\ll D^o_{j,k} \dashv = d^o_j \ll \psi^o_{j,k} \dashv$

Conclusion

- Measurement problem in terms of mathematical physics
- Statistical signal processing and quantum informatics
- Time operator formalism of complex systems
- Euclidean paradigm of the measurement process
- Psychophysical parallelism
- Wavelet domain hidden Markov model
- General measurements and Euclidean frames