Wigner – Weyl calculus
in description of non–dissipative Transport phenomena

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What is non–dissipative transport? (CME, CSE, CVE, QHE, AQHE, …)

Appearance of current (electric, axial, energy) that flows without dissipation.

The conductivities of all known non–dissipative transport phenomena are given by topological invariants.
Why Wigner – Weyl?

The bulk topological expressions for the conductivities of non-dissipative transport are known for the uniform systems.

However, it is widely believed that the absence of spatial homogeneity does not affect robustness of the conductivities to smooth deformations of the systems.
Why Wigner – Weyl?

Example: 2D QHE without magnetic field (ideal topological insulator, uniform system):

$$\sigma_H = \frac{N}{2\pi}$$

$$\mathcal{N} = \frac{\epsilon_{ijk}}{3! 4\pi^2} \int d^3 p \text{Tr} \left[ G(p) \frac{\partial G^{-1}(p)}{\partial p_i} \frac{\partial G(p)}{\partial p_j} \frac{\partial G^{-1}(p)}{\partial p_k} \right]$$

QHE with magnetic field (the presence of disorder, and varying magnetic field, non-uniform system):

$$\mathcal{N} = \frac{T \epsilon_{ijk}}{S 3! 4\pi^2} \int d^3 p d^3 x \text{Tr} \left[ G_W(p, x) * \frac{\partial Q_W(p, x)}{\partial p_i} * \frac{\partial G_W(p, x)}{\partial p_j} * \frac{\partial Q_W(p, x)}{\partial p_k} \right]$$
Plan

1. Equilibrium theory at zero temperature.
   - *Applications to Quantum Hall Effect (QHE) and chiral magnetic effect (CME)*

2. Equilibrium theory at nonzero temperature
   *Applications to Chiral Magnetic Effect (CME)*

3. Theory out of equilibrium
   - *Applications to QHE*

4. Loop corrections to QHE
   - Kinetic theory
   - Equilibrium theory at T=0

5. Perspectives. The other non – dissipative transport effects.
   - *Chiral separation effect (CSE)*
1. Wigner – Weyl calculus in continuum theory
Equilibrium, $T=0$

model with fermions

typical action

\[ Z = \int D\bar{\psi}D\psi \ e^{S[\psi,\bar{\psi}]} \]

\[ S[\bar{\psi}, \psi] = \int d^4 x \bar{\psi}(x) \hat{Q}(\partial_x) \psi(x) \]

\[ \hat{Q}(\partial_x) = i\gamma^\mu \partial_\mu - M \]

Green function

\[ (i\gamma_\mu \partial_x^\mu - m)G(x - y) = \delta(x - y) \]
Wigner – Weyl calculus in continuum theory

average of an operator

\[ \langle \Psi | \hat{A} | \Psi \rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \langle \Psi | x \rangle \langle x | \hat{A} | y \rangle \langle y | \Psi \rangle = \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq \langle \Psi | p \rangle \langle p | \hat{A} | q \rangle \langle q | \Psi \rangle \]

it may be written as

\[ \langle \Psi | \hat{A} | \Psi \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp A_W(x, p) \rho_W(x, p) \]

\[ \rho = |\Psi \rangle \langle \Psi| \]

Weyl symbol of operator

\[ A_W(x, p) \equiv \int_{-\infty}^{\infty} dy e^{-ipy} \langle x + \frac{y}{2} | \hat{A} | x - \frac{y}{2} \rangle = \int_{-\infty}^{\infty} dq e^{iqx} \langle p + \frac{q}{2} | \hat{A} | p - \frac{q}{2} \rangle \]
Wigner – Weyl calculus in continuum theory

**Moyal product**

\[
A_W(x, p) \star B_W(x, p) = A_W(x, p) e^{\overrightarrow{\Delta}} B_W(x, p)
\]

\[
\overrightarrow{\Delta} \equiv \frac{i}{2} \left( \overrightarrow{\partial_x \partial_p} - \overrightarrow{\partial_p \partial_x} \right)
\]

**Weyl symbol of the product of two operators**

\[
(AB)_W(x, p) \equiv A_W(x, p) \star B_W(x, p)
\]

**proof:**

\[
(\hat{A}\hat{B})_W = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv e^{iux} \langle p + \frac{u}{2} + \frac{v}{2} | \hat{A} | p - \frac{u}{2} + \frac{v}{2} \rangle e^{ivx} \langle p - \frac{u}{2} + \frac{v}{2} | \hat{B} | p - \frac{u}{2} - \frac{v}{2} \rangle =
\]

\[
\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \left[ e^{iux} \langle p + \frac{u}{2} | \hat{A} | p - \frac{u}{2} \rangle \right] e^{\frac{i}{2} \left( \overrightarrow{\partial_x \partial_p} - \overrightarrow{\partial_p \partial_x} \right)} \left[ e^{ivx} \langle p + \frac{v}{2} | \hat{B} | p - \frac{v}{2} \rangle \right]
\]
Wigner – Weyl calculus in continuum theory

model with fermions

\[ Z = \int D\bar{\psi} D\psi \ e^{S[\psi, \bar{\psi}]} \]

typical action

\[ S[\bar{\psi}, \psi] = \int d^4x \bar{\psi}(x) \hat{Q}(\partial_x) \psi(x) \]

\[ \hat{Q}(\partial_x) = i\gamma^\mu \partial_\mu - M \]

Green function

\[ (i\gamma_\mu \partial^\mu_x - m)G(x - y) = \delta(x - y) \]

Groenewold equation

\[ (\hat{Q}\hat{G})_W = Q_W \ast G_W = 1 \]
Lattice models
Equilibrium, $T=0$

fermions live on the lattice sites

typical action (Wilson fermions)

\[
S_F^{(W)} = \sum_{n,m} \hat{\psi}_\alpha(n) D^{(W)}_{\alpha\beta}(n,m) \hat{\psi}_\beta(n)
\]

\[
D^{(W)}_{\alpha\beta}(n,m) = (\hat{M} + 4) \delta_{nm} \delta_{\alpha\beta} - \frac{1}{2} \sum_\mu \left[ (1 - \gamma_\mu)_{\alpha\beta} \delta_{m,n+\hat{\mu}} + (1 + \gamma_\mu)_{\alpha\beta} \delta_{m,n-\hat{\mu}} \right]
\]
fermions live on the lattice sites

Momentum space

\[ \psi(r_n) = \int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} e^{i r_n p} \psi(p) \]

For rectangular lattice Momentum space has the topology of torus
For rectangular lattice
Momentum space has the topology of torus

Action in momentum space

\[ S(\bar{\psi}, \psi) = \int \frac{d^D p}{|\mathcal{M}|} \bar{\psi}(p) Q(p) \psi(p) \]

For the case of Wilson fermions

\[ Q(p) = \sum_{k=1,2,3,4} -i \gamma^k g_k(p) + m(p) \]

\[ g_k(p) = \sin(p_k) \quad m(p) = m^{(0)} + \sum_{\nu=1}^{4} (1 - \cos(p_{\nu})) \]
**Lattice models**

**Example of Wilson fermions**

**In the presence of gauge field**

\[
S^{(W)}_{F} = \sum_{n,m,\alpha,\beta} \hat{\psi}_\alpha(n) D_{\alpha\beta}^{(W)}(n,m) \hat{\psi}_\beta(n)
\]

\[
D_{x,y} = -\frac{1}{2} \sum_i [(1 + \gamma^i)\delta_{x+e_i,y} + (1 - \gamma^i)\delta_{x-e_i,y}] U_{x,y} + (m^{(0)} + 4)\delta_{x,y}
\]

\[
U_{x,y} = Pe^{i \int_x^y d\xi A(\xi)}
\]
Lattice models

In the presence of gauge field

Action

\[
S(\bar{\psi}, \psi) = \int \frac{d^D p}{|\mathcal{M}|} \bar{\psi}(p) Q(p - A(i \partial_p)) \psi(p)
\]

Partition function

\[
Z = \int D\bar{\psi} D\psi \exp \left( \int \frac{d^D p}{|\mathcal{M}|} \bar{\psi}(p) Q(p - A(i \partial_p)) \psi(p) \right)
\]
Approximate Wigner—Weyl calculus for the lattice models

Weyl symbol of operator (momentum space)

\[
[\hat{A}]_W(x_n, p) = \int_{\mathcal{M}} dq e^{iqx_n} \langle p + \frac{q}{2} | \hat{A} | p - \frac{q}{2} \rangle
\]

Average of operator

\[
\langle \Psi | \hat{A} | \Psi \rangle = \sum x_n \int_{\mathcal{M}} \frac{dp}{\mathcal{M}} A_W(x_n, p) \rho_W(x_n, p)
\]

Density matrix

\[
[\hat{\rho}]_W(x_n, p) = W(x, p) = \int_{\mathcal{M}} dq e^{-iqx_n} \langle p - \frac{q}{2} | \hat{\rho} | p + \frac{q}{2} \rangle
\]
Approximate Wigner – Weyl calculus for the lattice models

Weyl symbol of operator (momentum space)

\[ [\hat{A}]_W(x_n, p) = \int_{\mathcal{M}} dq e^{i q x_n} (p + \frac{q}{2}) |\hat{A}| p - \frac{q}{2} \]

Weyl symbol of the product of two operators

\[ (A B)_W(x_n, p) \equiv A_W(x_n, p) \star B_W(x_n, p) \]

This identity is approximate. It is valid for the near diagonal operators
This identity is approximate. It is valid for the near diagonal operators partition function

$$Z = \int D\bar{\psi} D\psi \, e^{S[\psi, \bar{\psi}]}$$

Action

Lattice model for the description of electrons in crystals:

The typical Lattice Dirac operator $Q$ is almost diagonal if the external magnetic field strength is much smaller than 10,000 Tesla while wavelength of external electromagnetic field is much larger than 1 nanometer.
This identity is approximate. It is valid for the near diagonal operators partition function

Action

Lattice model for the regularization of continuum quantum field theory:

The typical Lattice Dirac operator $Q$ is almost diagonal when we approach continuum limit of the lattice model.
We can use the approximate Wigner – Weyl calculus dealing with any lattice regularized continuum quantum field theory and dealing with the lattice models of solid state physics if the external magnetic field strength is much smaller than 10 000 Tesla while wavelength of external electromagnetic field is much larger than 1 nanometer.
**partition function**

\[ Z = \int D\bar{\psi}D\psi \ e^{S[\psi, \bar{\psi}]} \]

**Action**

\[ S[\psi, \bar{\psi}] = \int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \bar{\psi}(p) \hat{Q}(i\partial_p, p) \psi(p) \]

**Green function**

\[ G(p_1, p_2) = \langle p_1 | G | p_2 \rangle = \frac{1}{Z} \int D\bar{\psi}D\psi \bar{\psi}(p_2) \psi(p_1) \exp \left( \int \frac{d^D p}{|\mathcal{M}|} \bar{\psi}(p) \hat{Q}(i\partial_p, p) \psi(p) \right) \]

**Groenewold equation**

\[ Q_W(p, x) \ast G_W(p, x) = 1 \]

**Moyal product**

\[ \ast_{xp} \equiv e^{\frac{i}{2} \left( \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \cdot \frac{\partial}{\partial x} \right)} \]

**Electric current**

\[ j_i(x) = \frac{\delta \log Z}{\delta A_k(x)} = -\int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \text{tr} [G_W(x, p) \partial_p Q_W(x, p)] \]
Applications to Quantum Hall Effect

Electric current orthogonal to electric field in the presence of magnetic field

Geim, Novoselov, et al., Nature 438(7065):197-200 graphene
Quantum Hall Effect

constant magnetic field, no interactions, no disorder

$k$ is Bloch vector,

$|u(k)>$ is the eigenvector of Hamiltonian

$$\sigma_H = \frac{N}{2\pi}$$

$$\sigma_{xy} = \frac{e^2}{h} \frac{1}{2\pi} \int d^2 k [\nabla \times A(k)]$$

$A(k) = -i \langle u(k)| \nabla |u(k)\rangle$.

TKNN invariant

D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs
Applications to Quantum Hall Effect

Electric current orthogonal to electric field in the presence of magnetic field

electric current $j$

electric field $E$
Intrinsic Anomalous Quantum Hall Effect

homogeneous system
no magnetic field
no interactions
no disorder


\[
N = \frac{\epsilon_{ijk}}{3! 4\pi^2} \int d^3p \text{Tr} \left[ G(p) \frac{\partial G^{-1}(p)}{\partial p_i} \frac{\partial G(p)}{\partial p_j} \frac{\partial G^{-1}(p)}{\partial p_k} \right]
\]

\[
\sigma_H = \frac{N}{2\pi}
\]

2D topological insulator
Applications to Quantum Hall Effect
Equilibrium, \( T=0 \)
non-homogeneous system
Average electric current

\[ \langle j^k \rangle = -\frac{1}{2\pi} \mathcal{N} \epsilon^{kj} E_j. \]

\[ \mathcal{N} = \frac{T \epsilon_{ijk}}{S 3! 4\pi^2} \int d^3p d^3x \text{Tr} \left[ G_W(p, x) \frac{\partial Q_W(p, x)}{\partial p_i} \frac{\partial G_W(p, x)}{\partial p_j} \frac{\partial Q_W(p, x)}{\partial p_k} \right] \]

M.A. Zubkov \(^*\), Xi Wu

Annals of Physics 418 (2020) 168179
Applications to Quantum Hall Effect

Equilibrium, $T=0$

**non-homogeneous system**

Average electric current

3 + 1 D:

$$\langle j^k \rangle = -\frac{1}{2\pi^2} \varepsilon^{kijl} \mathcal{N}_i E_j,$$

$$\mathcal{N}_i = -\frac{T \varepsilon_{ijkl}}{\sqrt{3!} 8\pi^2} \int d^4x d^4p \text{ Tr } \left[ G_W(p, x) \frac{\partial Q_W(p, x)}{\partial p_i} \frac{\partial G_W(p, x)}{\partial p_j} \frac{\partial Q_W(p, x)}{\partial p_k} \right]$$

M.A. Zubkov*, 1, Xi Wu

Annals of Physics 418 (2020) 168179
Quantum Hall Effect Equilibrium, \( T=0 \) non-homogeneous system

Average electric current

\( 2+1 \) D:

\[
\langle j^k \rangle = -\frac{1}{2\pi} \mathcal{N} \epsilon^{3kij} E_j.
\]

\[
\mathcal{N} = \frac{T \epsilon_{ijk}}{S 3! 4\pi^2} \int d^3p d^3x \text{Tr} \left[ G_W(p, x) \frac{\partial G_W(p, x)}{\partial p_i} \ast \frac{\partial G_W(p, x)}{\partial p_j} \ast \frac{\partial G_W(p, x)}{\partial p_k} \right]
\]

smooth deformation of the system

the system without disorder, elastic deformations etc, with constant magnetic field

\( N \) is not changed!

If \( N \) is known for less complicated system, we know it also for the more complicated one
Applications to Chiral Magnetic Effect

non-homogeneous system, equilibrium, $T=0$

Average electric current

$3 + 1 D$: 

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D.E. Kharzeev, J. Liao, S.A. Voloshin, G. Wang,
Progress in Particle and Nuclear Physics, Volume 88, 2016, Pages 1-28,
Applications to Chiral Magnetic Effect
non-homogeneous system, equilibrium, $T=0$
Average electric current

$3 + 1 D$:

$$ \bar{J}^k = \frac{1}{4\pi^2} \epsilon^{ijkl} \mathcal{M}_l F_{ij} $$

**topological invariant:**

$$ \mathcal{M}_l = \frac{-iT \epsilon_{ijkl}}{3!V8\pi^2} \int d^D x \int d^D p \text{Tr} \left[ G_W^{(0)} \star \partial_p Q_W^{(0)} (p, x) \star G_W^{(0)} \star \partial_p Q_W^{(0)} (p, x) \star G_W^{(0)} \star \partial_p Q_W^{(0)} \right] $$

**external magnetic field:**

$$ F_{ij} = \epsilon_{ijk} B_k $$

C. Banerjee, M. Lewkowicz, M.A. Zubkov
Physics Letters B, 136457
Chiral magnetic effect **Equilibrium, \( T=0 \)**

**Non-homogeneous system**

**Average electric current**

\[
\bar{J}^k = \frac{1}{4\pi^2} \epsilon^{ijkl} M_l F_{ij}
\]

**Smooth deformation of the system**

**The system without any inhomogeneity**

\( M \) is not changed!

**We know that in homogeneous systems \( M = 0 \)**

Absence of equilibrium chiral magnetic effect, M.A. Zubkov

Physical Review D 93 (10), 105036

**No CME in non-uniform systems at \( T=0 \)**
2.

Equilibrium, $T > 0$

**QHE:** the need of kinetic theory (see below)

**CME:** the equilibrium theory may be used.

C. Banerjee, M. Lewkowicz, M.A. Zubkov

Physics Letters B, 136457
Applications to Chiral Magnetic Effect

non-homogeneous system, equilibrium, $T>0$

Average electric current

$$\bar{j}^k = \frac{1}{4\pi^2} \epsilon^{ijk} \mathcal{M}_4 F_{ij}$$

topological invariant:

$$\mathcal{M}_4 = 2\pi T \sum_\omega \mathcal{N}_4(\omega)$$

$$\omega = 2\pi T (n + 1/2), \ n \in \mathbb{Z}, \ 0 \leq n < N, \ \text{where} \ N = 1/T$$

$$\mathcal{N}_4(\omega) = \frac{-i \epsilon^{ijk4}}{3!V 8\pi^2} \int d^{D-1}x \int d^{D-1}p \text{Tr} \left[ \mathcal{G}_W^{(0)} \star \partial_p \mathcal{Q}_W^{(0)} (p, x) \star \mathcal{G}_W^{(0)} \star \partial_p \mathcal{Q}_W^{(0)} (p, x) \star \mathcal{G}_W^{(0)} \star \partial_p \mathcal{Q}_W^{(0)} \right]$$

Response of $N$ to chiral chemical potential is zero

No CME at $T>0$

The absence of CME at $T>0$ for homogeneous systems has been reported earlier in C.G. Beneventano, M. Nieto, E.M. Santangelo J. Phys. A, 53 (46) (2020), Article 465401,
3. **Keldysh technique**  
**Green functions (lower sign for fermions)**

\[
\begin{align*}
\{ \hat{G}^R \}^{(\alpha_1; \alpha_2)}(x_1; x_2) & \equiv -i \theta(t_1 - t_2) \langle \left[ \Psi_{\alpha_1}(x_1), \Psi_{\alpha_2}^\dagger(x_2) \right]_+ \rangle \\
\{ \hat{G}^A \}^{(\alpha_1; \alpha_2)}(x_1; x_2) & \equiv i \theta(t_2 - t_1) \langle \left[ \Psi_{\alpha_1}(x_1), \Psi_{\alpha_2}^\dagger(x_2) \right]_- \rangle \\
\{ \hat{G}^K \}^{(\alpha_1; \alpha_2)}(x_1; x_2) & \equiv -i \langle \left[ \Psi_{\alpha_1}(x_1), \Psi_{\alpha_2}^\dagger(x_2) \right]_\pm \rangle,
\end{align*}
\]

**Keldysh Green function**

\[
\hat{G}(t, x | t', x') = -i \begin{pmatrix} 
\langle T \Phi(t, x) \Phi^+(t', x') \rangle & -\langle \Phi^+(t', x') \Phi(t, x) \rangle \\
\langle \Phi(t, x) \Phi^+(t', x') \rangle & \langle \bar{T} \Phi(t, x) \Phi^+(t', x') \rangle 
\end{pmatrix}
\]

\[
\begin{pmatrix}
{G}^{--} & {G}^{-+} \\
{G}^{+-} & {G}^{++}
\end{pmatrix}
\]

\[
{G}^A = {G}^{--} - {G}^{+-} = {G}^{-+} - {G}^{++}
\]

\[
{G}^R = {G}^{--} - {G}^{--} = {G}^{+-} - {G}^{++}
\]

\[
{G}^{<} = {G}^{-+}
\]
3. **Keldysh technique and Wigner – Weyl calculus. Keldysh Green function**

\[
\hat{G}(t, x|t', x') = -i \left( \frac{\langle T\Phi(t, x)\Phi^+(t', x') \rangle - \langle \Phi^+(t', x')\Phi(t, x) \rangle}{\langle \Phi(t, x)\Phi^+(t', x') \rangle} \right)
\]

\[
= \begin{pmatrix}
G^{--} & G^{-+} \\
G^{+-} & G^{++}
\end{pmatrix}
\]

**Wigner transformation**

\[
\hat{G}(X_1, X_2) = \langle X_1 | \hat{G} | X_2 \rangle
\]

\[
A(X_1, X_2) = \langle X_1 | \hat{A} | X_2 \rangle
\]

\[
A_W(X|P) = \int d^{D+1}Y \ e^{iY^\mu P_\mu} A(X + Y/2, X - Y/2)
\]

**Moyal product**

\[
(A \star B)(X|P) = A(X|P) e^{-i(\vec{\partial}_x \mu \vec{\partial}_{P_\mu} - \vec{\partial}_{P_\mu} \vec{\partial}_x \mu)/2} B(X|P)
\]
Lesser representation

\[ \hat{G}(<) = U \hat{G} V. \]

\[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}, \quad V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \]

\[ \hat{G}(<) = \begin{pmatrix} G^R & 2G^< \\ 0 & G^A \end{pmatrix} \]

\[ G^A = G^{--} - G^{+-} = G^{--} - G^{++}, \]
\[ G^R = G^{--} - G^{-+} = G^{+-} - G^{++}, \]
\[ G^< = G^{-+} \]

The inverse Q of Green function

\[ \hat{Q} \hat{G} = 1 \]

After Wigner transformation

\[ \hat{Q} \ast \hat{G} = 1 \]
In non-interacting systems

\[ G^R = (i\partial_t - \hat{H}e^{+\epsilon\partial_t})^{-1} = (i\partial_t - \hat{H} + i\epsilon)^{-1} \]
\[ G^A = (i\partial_t - \hat{H}e^{-\epsilon\partial_t})^{-1} = (i\partial_t - \hat{H} - i\epsilon)^{-1} \]
\[ G^< = (G^A - G^R) \frac{\rho}{\rho + 1}. \]

distribution function

In general case without interactions electric current

\[
J^i(X) = -\frac{i}{2} \int \frac{d^{D+1}\pi}{(2\pi)^{D+1}} \text{tr} \left( (\partial_i \hat{Q}) \hat{G} \right)^< - \frac{i}{2} \int \frac{d^{D+1}\pi}{(2\pi)^{D+1}} \text{tr} \left( \hat{G} (\partial_i \hat{Q}) \right)^<
\]


product of triangle matrices is triangle matrix

lesser component for any matrix is defined as

\[
\hat{Q}^< = \begin{pmatrix} Q^R & 2Q^< \\ 0 & Q^A \end{pmatrix}
\]
Response of electric current to external field strength

\[ J^i = -\frac{1}{4} \int \frac{d^{D+1}\pi}{(2\pi)^{D+1}} \text{tr} \left( \hat{G} \star \partial_{\pi^\mu} \hat{Q} \star \hat{G} \star \partial_{\pi^\nu} \hat{Q} \star \hat{G} \partial_{\pi^i} \hat{Q} \right) < F^{\mu\nu} \]

\[ -\frac{1}{4} \int \frac{d^{D+1}\pi}{(2\pi)^{D+1}} \text{tr} \left( \partial_{\pi^i} \hat{Q} \hat{G} \star \partial_{\pi^\mu} \hat{Q} \star \hat{G} \star \partial_{\pi^\nu} \hat{Q} \star \hat{G} \right) < F^{\mu\nu} \]

Electric conductivity tensor for non–homogeneous systems

\[ J^i = \sigma^{ij} F_{0j} \]

\[ \sigma^{ij} = \frac{1}{4} \int \frac{d^{D+1}\pi}{(2\pi)^{D+1}} \text{tr} \left( \partial_{\pi^i} \hat{Q} \left[ \hat{G} \star \partial_{\pi_{[0}} \hat{Q} \star \partial_{\pi_{j]} \hat{G}} \right] \right) < + c.c. \]

C Banerjee, IV Fialkovsky, M Lewkowicz, CX Zhang, MA Zubkov
2D Hall conductivity
“Topological part”

\[ \bar{\sigma}_H = -\frac{N_f}{2\pi} + \bar{\sigma}_H, f' \]

\[ N_f = -\frac{1}{48\pi^2} \epsilon^{\mu\nu\rho} \int d\pi^0 \int d^2x \, \text{tr} \left( \partial_\mu Q \ast \partial_\nu G \ast \partial_\rho Q \ast G \right) f(\pi^0) + \text{c.c.} \]


A similar expression has been obtained independently in F.R. Lux, F. Freimuth, S. Blügel, Y. Mokrousov, Physical Review Letters 124 (9), 096602 (2020)

contour in complex plane of \( \pi^0 \)
in the case of thermal equilibrium at \( T\rightarrow 0 \)

\[ N_f = -\frac{1}{24\pi^2} \frac{1}{\beta} \epsilon^{\mu\nu\rho} \int d^3X \int d^3\Pi \, \text{tr} \left( \partial_{\Pi\mu} \hat{Q}^M \ast \hat{G}^M \ast \partial_{\Pi\nu} \hat{Q}^M \ast \hat{G}^M \ast \partial_{\Pi\rho} \hat{Q}^M \ast \hat{G}^M \right) \]

Matsubara Green function
\[ G^M \ (\text{we replace inside } G^R : \ \pi^0 \rightarrow i \omega) \]
2D Hall conductivity

“non-topological part”

\[
\bar{\sigma}_{H,f'} = \frac{1}{8 V} \varepsilon^{i j} \int \frac{d^3 \pi d^2 x}{(2\pi)^3} \text{tr} \left( (\partial_{\pi^i} Q^R \star G^R + \partial_{\pi^i} Q^A \star G^A) \star \partial_{\pi^j} Q^R \star (G^A - G^R) \right) \partial_{\pi^0} f(\pi^0) + \text{c.c.}
\]

ordinary symmetric conductivity

\[
\bar{\sigma}^{i j} = \frac{1}{8 V} \int \frac{d^3 \pi d^2 x}{(2\pi)^3} \text{tr} \left( (-\partial_{\pi^i} Q^R \star G^R + \partial_{\pi^i} Q^A \star G^A) \star \partial_{\pi^j} Q^R \star (G^A - G^R) \right) \partial_{\pi^0} f(\pi^0) + (i \leftrightarrow j) + \text{c.c.}
\]
4. Interaction corrections to electric conductivity

\[ J_i^{(1)} = -\frac{\mathcal{F}_i^{\mu\nu}}{4V} \int \frac{d^D x d^{D+1} \pi}{(2\pi)^{D+1}} \frac{d^{D+1} P}{(2\pi)^{D+1}} D(P) \text{tr} \left[ \left( \partial_{\pi^i \hat{Q}} \hat{G} \right) \star \hat{G} \right] \left( (G \star \partial_{\pi^\mu \hat{Q}} \hat{G} \star \hat{G} \star \partial_{\pi^\nu \hat{Q}} \hat{G} \star \hat{G}) \right)_{\pi-P} \\
+ G \left( \partial_{\pi^\mu \hat{Q}} \hat{G} \star \hat{G} \star \partial_{\pi^\nu \hat{Q}} \hat{G} \star \hat{G} \right)_{\pi-P} + G \left( \partial_{\pi^\nu \hat{Q}} \hat{G} \star \hat{G} \star \partial_{\pi^\mu \hat{Q}} \hat{G} \star \hat{G} \right)_{\pi-P} \\
- \partial_{\pi^\mu \hat{Q}} \hat{G} \star \hat{G} \star \partial_{\pi^\nu \hat{G}} \star \hat{G} \left[ \partial_{\pi^\mu \hat{Q}} \hat{G} \star \hat{G} \star \partial_{\pi^\nu \hat{Q}} \hat{G} \star \hat{G} \right]_{\pi-P} \hat{G} \right)^< + \text{c.c.} \\
\]

The exchange by bosonic excitation with propagator \( D(P) \), and interaction vertex 1, one loop.

Interaction corrections to electric conductivity

\[ J^{(1)}_{\mathcal{F}} = -\frac{\mathcal{F}^{\mu\nu}}{4V} \int \frac{d^D x d^{D+1}P}{(2\pi)^{D+1}} \frac{d^{D+1}P}{(2\pi)^{D+1}} D(P) \text{tr} \left[ (\partial_{\pi^i} \hat{Q}) \ast \hat{G} \ast \left( (G \ast \partial_{\pi^\mu} \hat{Q} \ast \hat{G} \ast \partial_{\pi^\nu} \hat{Q} \ast \hat{G} \right) \right]_{\pi^- P} \]

\[ + G \left|_{\pi^- P} \right. \ast \hat{G} \ast \partial_{\pi^\mu} \hat{Q} \ast G \ast \partial_{\pi^\nu} \hat{Q} \ast \partial_{\pi^\mu} G \left|_{\pi^- P} \right. \ast \hat{G} \ast \partial_{\pi^\nu} \hat{Q} \ast \hat{G} \ast \hat{G} \left|_{\pi^- P} \right. \ast \hat{G} \ast \partial_{\pi^\nu} \hat{Q} \ast \hat{G} \ast \hat{G} \left|_{\pi^- P} \right. \ast \hat{G} \right]^{<} + c.c. \]

If there is symmetry under cyclic permutation of operators under the trace, then one-loop contribution to antisymmetric (Hall) component of conductivity vanishes. This occurs, for example, if the initial distribution of fermions \( f(\pi^0) \) is Fermi distribution with \( T \to 0 \), and chemical potential is inside the gap.

Thus in equilibrium at zero temperature there are no one-loop interaction corrections to Hall conductivity.
The absence of interaction corrections to Quantum Hall Effect equilibrium, $T=0$

Electric current orthogonal to electric field in the presence of magnetic field

C. X. Zhang$^a$ and M. A. Zubkov$^a$, *

The absence of interaction corrections to Quantum Hall Effect

Electric current orthogonal to electric field in the presence of magnetic field

as an example:

\[ S = \int d\tau \sum_{x,x'} [\bar{\psi}_x (i(i\partial_\tau - A_3(i\tau, x))\delta_{x,x'} - i\mathcal{D}_{x,x'})\psi_x \\
+ \alpha \bar{\psi}(\tau, x)\psi(\tau, x)\theta(y)V(x - x')\theta(y')\bar{\psi}(\tau, x')\psi(\tau, x')] \]

without interactions:

\[ \mathcal{D}_{x,x'} = -i \sum_{i=1,2} \left[ (1 + \sigma^i)\delta_{x+e_i,x'}e^{iA_{x+e_i,x}} \\
+ (1 - \sigma^i)\delta_{x-e_i,x'}e^{iA_{x-e_i,x}} \right] \sigma_3 + i(m + 2)\delta_{x,x}\sigma_3 \]

\[ \sigma_{xy} = \frac{N}{2\pi} \]

\[ N = \frac{T}{S3!4\pi^2} \epsilon_{ijk} \int d^3x \int d^3p \text{Tr} G_W(p, x)^* \frac{\partial Q_W(p, x)}{\partial p_i} \\
* \frac{\partial G_W(p, x)}{\partial p_j} * \frac{\partial Q_W(p, x)}{\partial p_k} \]
Gedankenexperiment: we consider the system on the torus and divide it into the two pieces
we consider the system on the torus and divide it into the two pieces

\[ \sigma_{xy} = \frac{N}{2\pi} \]

is zero in the part II, \( E(I) = -E(II) \)

\[ I_{tot} = (I_1 + I_2) / 2 = (\bar{\sigma}_1 E + \bar{\sigma}_2 (-E)) / 2 + I_{tot} \big|_{E=0} \]

We prove that the total current remains zero with the interaction corrections

\[ \bar{\sigma}_1(0) = \bar{\sigma}_2(0) \]

\[ \bar{\sigma}_1(g) = \bar{\sigma}_2(0) \]

\[ \bar{\sigma}_1(0) = \bar{\sigma}_1(g) \]

no interaction corrections
is zero in the part II, $E(I) = -E(II)$

$I(g) = \sum_{n=0}^{\infty} I^{(n)}$

$I^{(n)} = \int \frac{d^3R}{\beta S} \int \frac{d^3p}{(2\pi)^3} \text{Tr} \left[ G_{g,w}(R, p) \star \frac{\partial}{\partial p_x} Q_W(R, p) \right]^n G_W \star \frac{\partial Q_W}{\partial p_x}$

$G_{g,w} = G_W + G_W \star \Sigma_W \star G_W + ...$

an example: 1-loop

$I_1 = -\int \frac{d^3R}{\beta S} \int \frac{d^3p}{(2\pi)^3} \text{Tr} \left[ \int \frac{d^3q}{(2\pi)^3} G_W(R, p - q) \mathcal{D}(q) \right] \star \frac{\partial}{\partial p_x} G_W(R, p)$

$-\int \frac{d^3R}{\beta S} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \text{Tr} \frac{\partial}{\partial p_x} \left[ G_W(R, p - q) \star G_W(R, p) \right] \mathcal{D}(q) = 0$
an example: 1-loop diagram

\[ I(g) = \sum_{n=0}^{\infty} I^{(n)} \]

\[ I^{(n)} = \int \frac{d^3 R}{\beta S} \int \frac{d^3 p}{(2\pi)^3} \text{Tr} \left( G_W \ast \Sigma_W \ast \right)^n G_W \ast \frac{\partial Q_W}{\partial p_x} \]

\[ \frac{\partial}{\partial p_x} \]

\[ = 2 \]

\[ = 0 \]

\[ I_1 = -\int \frac{d^3 R}{\beta S} \int \frac{d^3 p}{(2\pi)^3} \text{Tr} \left[ \int \frac{d^3 q}{(2\pi)^3} G_W(R, p - q) \mathcal{D}(q) \right] \ast \frac{\partial}{\partial p_x} G_W(R, p) \]

\[ -\int \frac{d^3 R}{\beta S} \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} \text{Tr} \frac{\partial}{\partial p_x} \left[ G_W(R, p - q) \ast G_W(R, p) \right] \mathcal{D}(q) = 0 \]
In the presence of interactions the sum of the currents in the two pieces is zero ➔ the electric conductivity receives no corrections in the part I
Another example of diagram technique

\[ \int \int \left[ G_1(R, p) \circ_1 \star G_2(R, p - k_1) \circ_2 \star G_3(R, p - k_1 - k_2) \star_1 \circ G_4(R, p - k_2) \star_2 \circ G_5(R, p) \right] \]

\[ D^{(1)}_W (R, k_1) D^{(2)}_W (R, k_2) dk_1 dk_2. \]

\[ \circ j = e^{-i \partial_p \partial_R^{(j)} / 2} \quad \text{and} \quad j \circ = e^{i \partial_R^{(j)} \partial_p / 2}. \partial_R^{(j)} \text{ acts on } D^{(j)} \text{ only.} \]
5. Wigner – Weyl calculus may also be used for the investigation of the other non-dissipative transport phenomena.

An example is Chiral Separation Effect
Axial current along magnetic field in the presence of chemical potential

D.E. Kharzeev, J. Liao, S.A. Voloshin, G. Wang, Progress in Particle and Nuclear Physics, Volume 88, 2016, Pages 1-28,

\[ J_5^k = -\frac{1}{4\pi^2}\epsilon^{ijk0} \mu F_{ij} \]

A. Metlitski and Ariel R. Zhitnitsky, Phys. Rev. D 72 (2005), 045011
Lattice Dirac operator $Q$ is a $4 \times 4$ matrix expressed through the Gamma matrices.

$$j_{k}^{5}(x) = -\int_{\mathcal{M}} \frac{d^{D}p}{|\mathcal{M}|} \text{tr} [\gamma^{5}G_{W}(x,p)\partial_{p_k}Q_{W}(x,p)]$$

The system with Fermi surface of arbitrary complicated form.

Fermi surface of gold from http://exciting.wikidot.com/nitrogen-fermisurface
Lattice Dirac operator $Q$ is $4 \times 4$ matrix expressed through the Gamma matrices.

$$j^5_k(x) = -\int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \text{tr} \left[ \gamma^5 G_W(x, p) \partial_{p_k} Q_W(x, p) \right]$$

The system with Fermi surface of arbitrary complicated form

$$\mathcal{J}^k = -\frac{N}{4\pi^2} \epsilon^{ijk0} \mu F_{ij}$$

$$\mathcal{N} = -\frac{1}{48\pi^2 V} \int_{\Sigma_3} \int d^3 x \text{tr} \left[ \gamma^5 G_W * dQ_W * G_W \wedge * dQ_W * G_W * \wedge dQ_W \right]$$

Surface $\Sigma_3$ surrounds the singularities of

$$\left[ \gamma^5 G_W^{(0)} * dQ_W^{(0)} * G_W^{(0)} \wedge * dQ_W^{(0)} * G_W^{(0)} * \wedge dQ_W^{(0)} \right]$$

$\gamma^5$ commutes/anticommutes with $Q$ in small vicinity of $\Sigma_3$

M. Suleymanov, M. Zubkov, Physical Review D 102 (7), 076019
Lattice Dirac operator $Q$ is $4 \times 4$ matrix expressed through the Gamma matrices.

The system with Fermi surface of arbitrary complicated form.

Irrespective of the form of the Fermi surface the value of $\mathcal{N}$ is equal to the number of chiral $4$–component Dirac fermions.

M. Suleymanov, M. Zubkov, Physical Review D 102 (7), 076019
Conclusions

• Wigner – Weyl calculus allows to represent in compact form the conductivities of non–dissipative transport phenomena in non–uniform systems.

• Equilibrium systems at zero temperature: QHE conductivity is given by topological invariant composed of the Wigner transformed two-point Green functions.

• Equilibrium systems at finite temperatures: CME response of electric current to magnetic field is the topological invariant in phase space.
Conclusions

• Non – equilibrium systems, Keldysh technique and Wigner – Weyl calculus allow to express in compact form electric conductivity.

• Out of equilibrium already in one loop the interaction corrections to Hall conductivity do not vanish. In equilibrium theory at zero temperature these corrections disappear to all orders.

• This technique may be applied also to the other non – dissipative transport effects, say, to CSE.