Wigner – Weyl calculus
in description of non – dissipative Transport phenomena

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What is non – dissipative transport?
(~~CME~~,CSE,CVE,QHE,AQHE, …)

Appearance of current (electric, axial, energy) that flows without dissipation.

The conductivities of all known non – dissipative transport phenomena are given by topological invariants.

Why Wigner – Weyl?

The bulk topological expressions for the conductivities of non – dissipative transport are known for the uniform systems.

However, it is widely believed that the absence of spatial homogeneity does not affect robustness of the conductivities to smooth deformations of the systems.

Why Wigner – Weyl?

Example: 2D QHE without magnetic field

(ideal topological insulator, <u>uniform</u> system):

$$\sigma_H = \frac{\mathcal{N}}{2\pi}$$

$$\mathcal{N} = \frac{\epsilon_{ijk}}{3!\,4\pi^2} \int d^3p \, \mathrm{Tr} \left[G(p) \frac{\partial G^{-1}(p)}{\partial p_i} \frac{\partial G(p)}{\partial p_j} \frac{\partial G^{-1}(p)}{\partial p_k} \right]$$

QHE with magnetic field

(the presence of disorder, and varying magnetic field, <u>non-uniform</u> system):

$$\mathcal{N} = \frac{T\epsilon_{ijk}}{\mathcal{S}\,3!\,4\pi^2} \int d^3p d^3x \, \mathrm{Tr} \left[G_W(p,x) * \frac{\partial Q_W(p,x)}{\partial p_i} * \frac{\partial G_W(p,x)}{\partial p_j} * \frac{\partial Q_W(p,x)}{\partial p_k} \right]$$

Plan

1. Equilibrium theory at zero temperature.
- *Applications to Quantum Hall Effect (QHE) and*
- *chiral magnetic effect (~~CME~~)*
2. Equilibrium theory at nonzero temperature
Applications to Chiral Magnetic Effect (~~CME~~)
3. Theory out of equilibrium
- *Applications to QHE*
4. Loop corrections to QHE
- Kinetic theory
- Equilibrium theory at T=0
5. Perspectives. The other non – dissipative transport effects.
- *chiral separation effect (CSE)*

1.
Wigner – Weyl calculus in continuum theory
Equilibrium, T=0

model with fermions

$$Z = \int D\bar{\psi} D\psi \; e^{S[\psi,\bar{\psi}]}$$

typical action

$$S[\bar{\psi},\psi] = \int d^4x \bar{\psi}(x)\hat{Q}(\partial_x)\psi(x)$$

$$\hat{Q}(\partial_x) = i\gamma^\mu\partial_\mu - M$$

Green function

$$(i\gamma_\mu\partial^\mu_x - m)G(x-y) = \delta(x-y)$$

Wigner – Weyl calculus in continuum theory

average of an operator

$$\langle\Psi|\hat{A}|\Psi\rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \langle\Psi|x\rangle\langle x|\hat{A}|y\rangle\langle y|\Psi\rangle = \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq \langle\Psi|p\rangle\langle p|\hat{A}|q\rangle\langle q|\Psi\rangle$$

it may be written as

$$\langle\Psi|\hat{A}|\Psi\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp A_W(x,p)\rho_W(x,p)$$

$$\rho = |\Psi\rangle\langle\Psi|$$

Weyl symbol of operator

$$A_W(x,p) \equiv \int_{-\infty}^{\infty} dy e^{-ipy}\langle x+\frac{y}{2}|\hat{A}|x-\frac{y}{2}\rangle = \int_{-\infty}^{\infty} dq e^{iqx}\langle p+\frac{q}{2}|\hat{A}|p-\frac{q}{2}\rangle$$

Wigner – Weyl calculus in continuum theory

Moyal product

$$A_W(x,p)\star B_W(x,p)=A_W(x,p)e^{\overleftrightarrow{\Delta}}B_W(x,p)$$

$$\overleftrightarrow{\Delta}\equiv\frac{i}{2}\left(\overleftarrow{\partial}_x\overrightarrow{\partial}_p-\overleftarrow{\partial}_p\overrightarrow{\partial}_x\right)$$

Weyl symbol of the product of two operators

$$(AB)_W(x,p)\equiv A_W(x,p)\star B_W(x,p)$$

proof:

$$(\hat{A}\hat{B})_W=\int_{-\infty}^{\infty}du\int_{-\infty}^{\infty}dve^{iux}\left\langle p+\frac{u}{2}+\frac{v}{2}\right|\hat{A}\left|p-\frac{u}{2}+\frac{v}{2}\right\rangle e^{ivx}\left\langle p-\frac{u}{2}+\frac{v}{2}\right|\hat{B}\left|p-\frac{u}{2}-\frac{v}{2}\right\rangle=$$

$$\int_{-\infty}^{\infty}du\int_{-\infty}^{\infty}dv\left[e^{iux}\left\langle p+\frac{u}{2}\right|\hat{A}\left|p-\frac{u}{2}\right\rangle\right]e^{\frac{i}{2}\left(\overleftarrow{\partial}_x\overrightarrow{\partial}_p-\overleftarrow{\partial}_p\overrightarrow{\partial}_x\right)}\left[e^{ivx}\left\langle p+\frac{v}{2}\right|\hat{B}\left|p-\frac{v}{2}\right\rangle\right]$$

Wigner – Weyl calculus in continuum theory

model with fermions

$$Z = \int D\bar{\psi} D\psi \ e^{S[\psi,\bar{\psi}]}$$

typical action

$$S[\bar{\psi}, \psi] = \int d^4x \bar{\psi}(x) \hat{Q}(\partial_x) \psi(x)$$

$$\hat{Q}\,(\partial_x) = i\gamma^\mu \partial_\mu - M$$

Green function

$$(i\gamma_\mu \partial_x^\mu - m) G(x-y) = \delta(x-y)$$

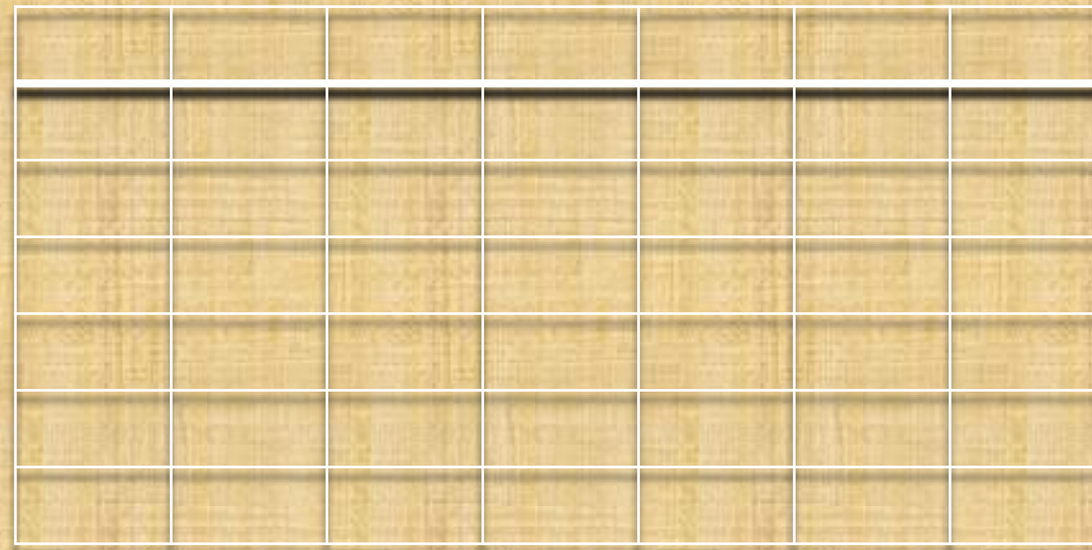
Groenewold equation

$$(\hat{Q}\hat{G})_W = Q_W \star G_W = 1$$

Lattice models

Equilibrium, T=0

fermions live on the lattice sites



typical action (Wilson fermions)

$$S_F^{(W)} = \sum_{\substack{n,m \\ \alpha,\beta}} \hat{\bar{\psi}}_\alpha(n) D_{\alpha\beta}^{(W)}(n,m)\hat{\psi}_\beta(n)$$

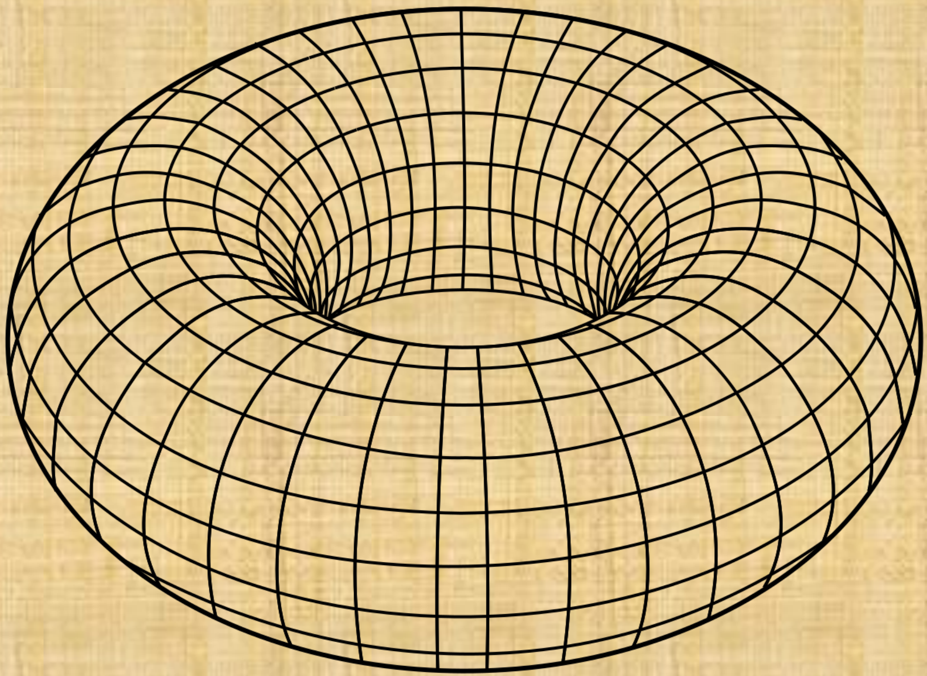
$$D_{\alpha\beta}^{(W)}(n,m) = (\hat{M}+4)\delta_{nm}\delta_{\alpha\beta} - \frac{1}{2}\sum_{\mu}\left[(1-\gamma_\mu)_{\alpha\beta}\delta_{m,n+\hat{\mu}} + (1+\gamma_\mu)_{\alpha\beta}\delta_{m,n-\hat{\mu}}\right]$$

fermions live on the lattice sites

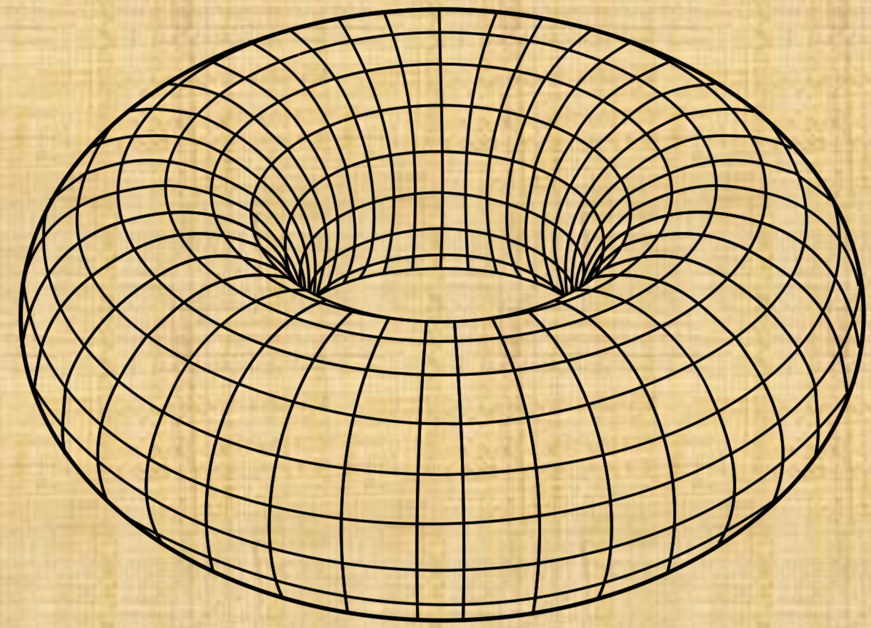
Momentum space

$$\psi(r_n) = \int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} e^{i r_n p} \psi(p)$$

For rectangular lattice Momentum space has the topology of torus



For rectangular lattice Momentum space has the topology of torus



Action in momentum space

$$S(\bar{\psi}, \psi) = \int \frac{d^D p}{|\mathcal{M}|} \bar{\psi}(p) Q(p) \psi(p)$$

For the case of Wilson fermions

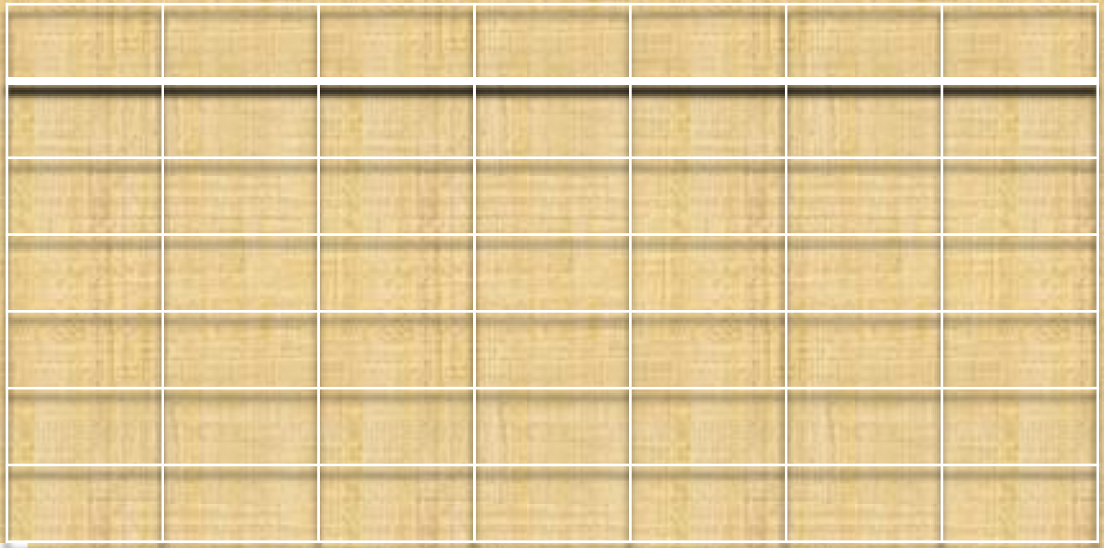
$$Q(p) = \sum_{k=1,2,3,4} -i\gamma^k g_k(p) + m(p)$$

$$g_k(p) = \sin(p_k) \qquad m(p) = m^{(0)} + \sum_{\nu=1}^{4} (1 - \cos(p_\nu))$$

Lattice models

Example of Wilson fermions

In the presence of gauge field

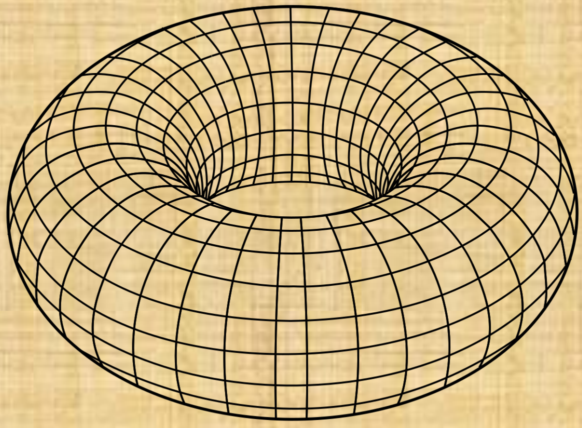


$$S_F^{(W)} = \sum_{\substack{n,m \\ \alpha,\beta}} \hat{\bar{\psi}}_\alpha(n) D^{(W)}_{\alpha\beta}(n,m) \hat{\psi}_\beta(n)$$

$$D_{\mathbf{x},\mathbf{y}} = -\frac{1}{2}\sum_i \left[(1+\gamma^i)\delta_{\mathbf{x}+\mathbf{e}_i,\mathbf{y}} + (1-\gamma^i)\delta_{\mathbf{x}-\mathbf{e}_i,\mathbf{y}}\right] U_{\mathbf{x},\mathbf{y}} + (m^{(0)}+4)\delta_{\mathbf{x},\mathbf{y}}$$

$$U_{x,y} = P e^{i\int_x^y d\boldsymbol{\xi} A(\boldsymbol{\xi})}$$

Lattice models



In the presence of gauge field

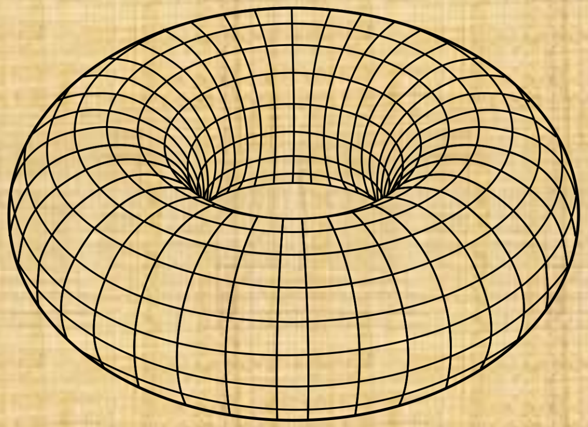
Action

$$S(\bar{\psi}, \psi) = \int \frac{d^D p}{|\mathcal{M}|} \bar{\psi}(p) Q(p - A(i\partial_p)) \psi(p)$$

Partition function

$$Z = \int D\bar{\psi} D\psi exp \left(\int \frac{d^D p}{|\mathcal{M}|} \bar{\psi}(p) Q(p - A(i\partial_p)) \psi(p) \right)$$

Approximate Wigner – Weyl calculus for the lattice models



Weyl symbol of operator (momentum space)

$$[\hat{A}]_W(x_n,p) = \int_{\mathcal{M}} dq e^{iqx_n} \langle p+\frac{q}{2}|\,\hat{A}\,|p-\frac{q}{2}\rangle$$

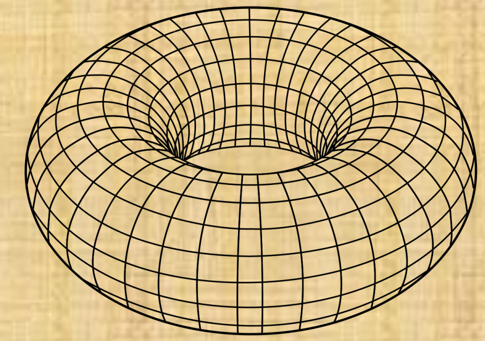
Average of operator

$$\langle\Psi|\,\hat{A}\,|\Psi\rangle = \sum_{x_n}\int_{\mathcal{M}} \frac{dp}{\mathcal{M}} A_W(x_n,p)\rho_W(x_n,p)$$

Density matrix

$$[\hat{\rho}]_W(x_n,p) = W(x,p) = \int_{\mathcal{M}} dq e^{-iqx_n} \langle p-\frac{q}{2}|\,\hat{\rho}\,|p+\frac{q}{2}\rangle$$

Approximate Wigner – Weyl calculus for the lattice models



Weyl symbol of operator (momentum space)

$$[\hat{A}]_W(x_n, p) = \int_{\mathcal{M}} dq e^{iqx_n} \langle p + \frac{q}{2} | \hat{A} | p - \frac{q}{2} \rangle$$

Weyl symbol of the product of two operators

$$(AB)_W(x_n, p) \equiv A_W(x_n, p) \star B_W(x_n, p)$$

This identity is approximate. It is valid for the near diagonal operators

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It is valid for the near diagonal operators

partition function

$$Z = \int D\bar{\psi}D\psi \; e^{S[\psi,\bar{\psi}]}$$

Action

$$S[\psi,\bar{\psi}] = \int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \bar{\psi}(p)\hat{Q}(i\partial_p, p)\psi(p)$$

Lattice model for the description of electrons in crystals:

The typical Lattice Dirac operator ***Q*** *is almost diagonal if the external magnetic field strength is much smaller than 10 000 Tesla while wavelength of external electromagnetic field is much larger than 1 nanometer*

This identity is approximate.

$$(AB)_W(x_n, p) \equiv A_W(x_n, p) \star B_W(x_n, p)$$

It is valid for the near diagonal operators

partition function

$$Z = \int D\bar{\psi} D\psi \; e^{S[\psi, \bar{\psi}]}$$

Action

$$S[\psi, \bar{\psi}] = \int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \bar{\psi}(p) \hat{Q}(i\partial_p, p) \psi(p)$$

Lattice model for the regularization of continuum quantum field theory:

The typical Lattice Dirac operator ***Q*** *is almost diagonal when we approach continuum limit of the lattice model.*

We can use the approximate Wigner – Weyl calculus dealing with ***any lattice regularized continuum quantum field theory*** *and dealing with the lattice models of solid state physics* ***if the external magnetic field strength is much smaller than 10 000 Tesla*** *while wavelength of external electromagnetic field is much larger than* ***1 nanometer*** *partition function*

$$Z = \int D\bar{\psi} D\psi \; e^{S[\psi,\bar{\psi}]}$$

Action

$$S[\psi,\bar{\psi}] = \int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \bar{\psi}(p) \hat{Q}(i\partial_p, p) \psi(p)$$

Green function

$$G(p_1,p_2) = \langle p_1 | \, G \, | p_2 \rangle = \frac{1}{Z} \int D\bar{\psi} D\psi \bar{\psi}(p_2) \psi(p_1) \exp\left(\int \frac{d^D p}{|\mathcal{M}|} \bar{\psi}(p) \hat{Q}(i\partial_p, p) \psi(p) \right)$$

Groenewold equation

$$Q_W(p,x) \star G_W(p,x) = 1$$

Moyal product

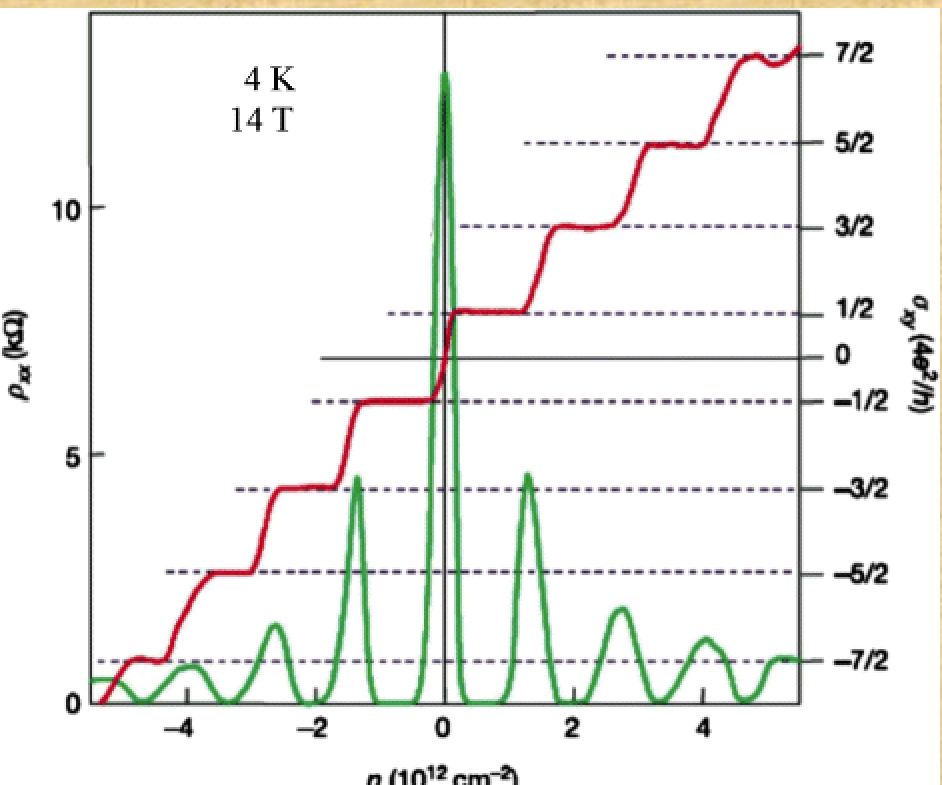
$$\star_{xp} \equiv e^{\frac{i}{2}\left(\overleftarrow{\partial_x} \overrightarrow{\partial_p} - \overleftarrow{\partial_p} \overrightarrow{\partial_x} \right)}$$

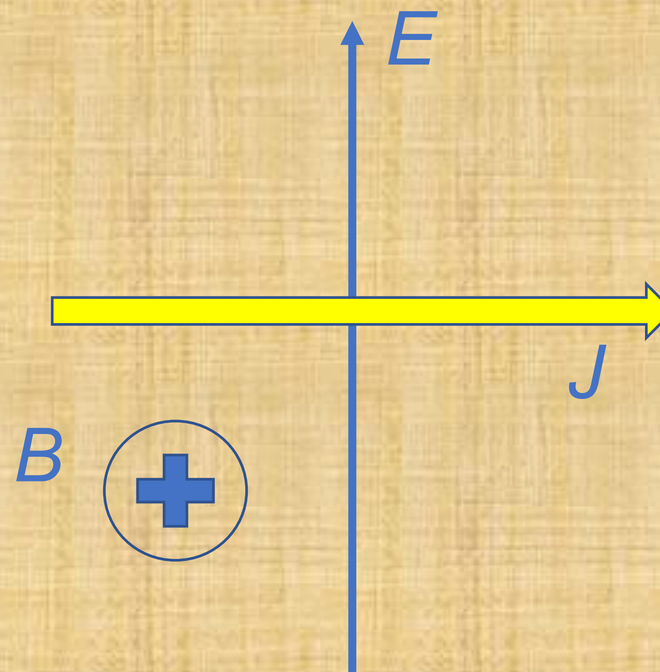
Electric current

$$j_i(x) = \frac{\delta \log Z}{\delta A_k(x)} = -\int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \operatorname{tr} \left[G_W(x,p) \partial_{p_i} Q_W(x,p) \right]$$

Applications to Quantum Hall Effect

Electric current orthogonal to electric field in the presence of magnetic field

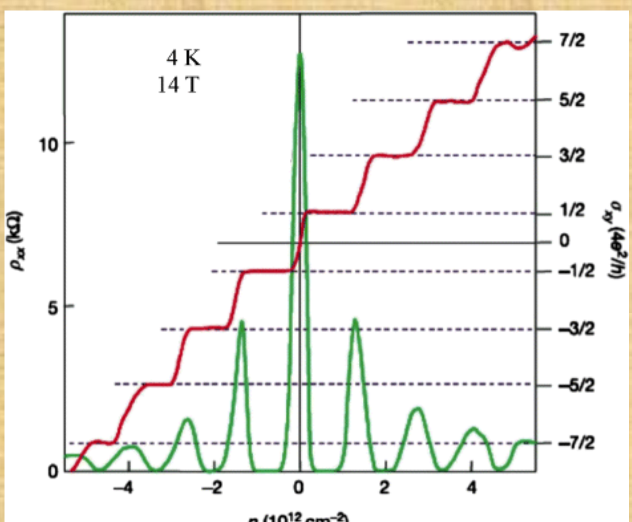




Geim, Novoselov, et all, Nature 438(7065):197-200 *graphene*

Quantum Hall Effect

constant magnetic field, no interactions, no disorder
k is Bloch vector,
|u(k)> is the eigenvector of
Hamiltonian



$$\sigma_H = \frac{\mathcal{N}}{2\pi}$$

$$\sigma_{xy} = \frac{e^2}{h}\frac{1}{2\pi}\int d^2k[\boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{k})]$$

$$\boldsymbol{A}(\boldsymbol{k}) = -\mathrm{i}\,\langle u(\boldsymbol{k})|\boldsymbol{\nabla}|u(\boldsymbol{k})\rangle\,.$$

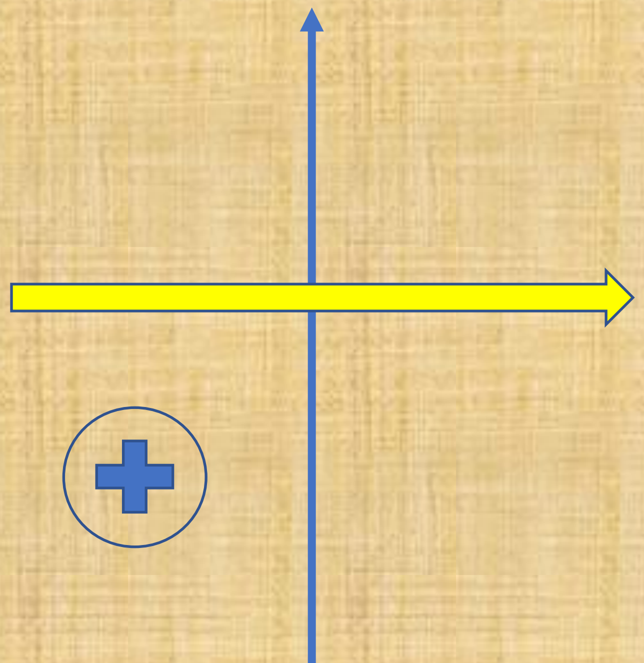
TKNN invariant

D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs
Phys. Rev. Lett. 49, 405 (1982)

Applications to Quantum Hall Effect

Electric current orthogonal to electric field in the presence of magnetic field

electric current j



electric field E

Intrinsic Anomalous Quantum Hall Effect

homogeneous system

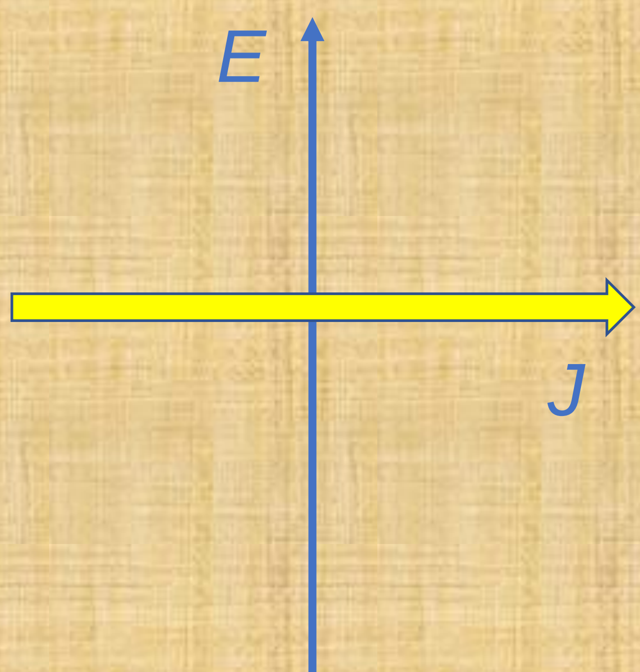
no magnetic field

no interactions

no disorder

T. Matsuyama, Quantization of Conductivity Induced by Topological Structure of Energy Momentum Space in Generalized
QED in Three-dimensions, Prog. Theor. Phys 77, 711 (1987)

$$\mathcal{N} = \frac{\epsilon_{ijk}}{3!\,4\pi^2} \int d^3p \,\mathrm{Tr}\left[G(p)\frac{\partial G^{-1}(p)}{\partial p_i}\frac{\partial G(p)}{\partial p_j}\frac{\partial G^{-1}(p)}{\partial p_k}\right]$$



$$\sigma_H = \frac{\mathcal{N}}{2\pi}$$

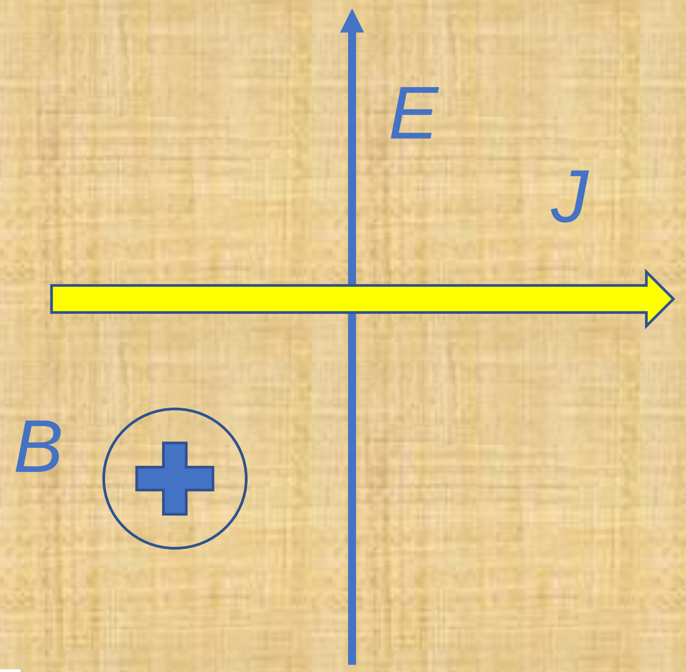
2D topological insulator

Applications to Quantum Hall Effect

Equilibrium, T=0

non-homogeneous system

Average electric current



2+1 D:

$$\langle j^k \rangle = -\frac{1}{2\pi} \mathcal{N} \epsilon^{3kj} E_j,$$

$$\mathcal{N} = \frac{T\epsilon_{ijk}}{\mathcal{S}\, 3!\, 4\pi^2} \int d^3p d^3x \, \mathrm{Tr} \left[G_W(p,x) * \frac{\partial Q_W(p,x)}{\partial p_i} * \frac{\partial G_W(p,x)}{\partial p_j} * \frac{\partial Q_W(p,x)}{\partial p_k} \right]$$

M.A. Zubkov [*,1], Xi Wu

Annals of Physics 418 (2020) 168179

Applications to Quantum Hall Effect

Equilibrium, T=0

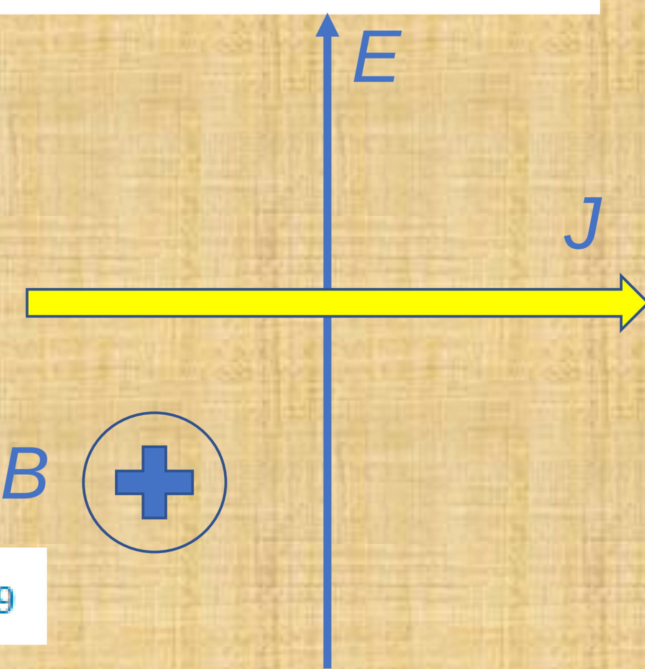
<u>non-homogeneous system</u>

Average electric current

3 + 1 D:

$$\langle j^k \rangle = -\frac{1}{2\pi^2}\epsilon^{kjl4}\mathcal{N}_l E_j,$$

$$\mathcal{N}_l = -\frac{T\epsilon_{ijkl}}{\mathcal{V}\, 3!\, 8\pi^2} \int d^4x d^4p \, \mathrm{Tr}\left[G_W(p,x) * \frac{\partial Q_W(p,x)}{\partial p_i} * \frac{\partial G_W(p,x)}{\partial p_j} * \frac{\partial Q_W(p,x)}{\partial p_k}\right]$$



M.A. Zubkov [*,1], Xi Wu

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Quantum Hall Effect ***Equilibrium, T=0***

non-homogeneous system

Average electric current

2+1 D:

$$\langle j^k \rangle = -\frac{1}{2\pi} \mathcal{N} \epsilon^{3kj} E_j,$$

$$\mathcal{N} = \frac{T\epsilon_{ijk}}{\mathcal{S}\, 3!\, 4\pi^2} \int d^3p d^3x \,\mathrm{Tr}\, \Big[G_W(p,x) * \frac{\partial Q_W(p,x)}{\partial p_i} * \frac{\partial G_W(p,x)}{\partial p_j} * \frac{\partial Q_W(p,x)}{\partial p_k} \Big]$$

smooth deformation of the system

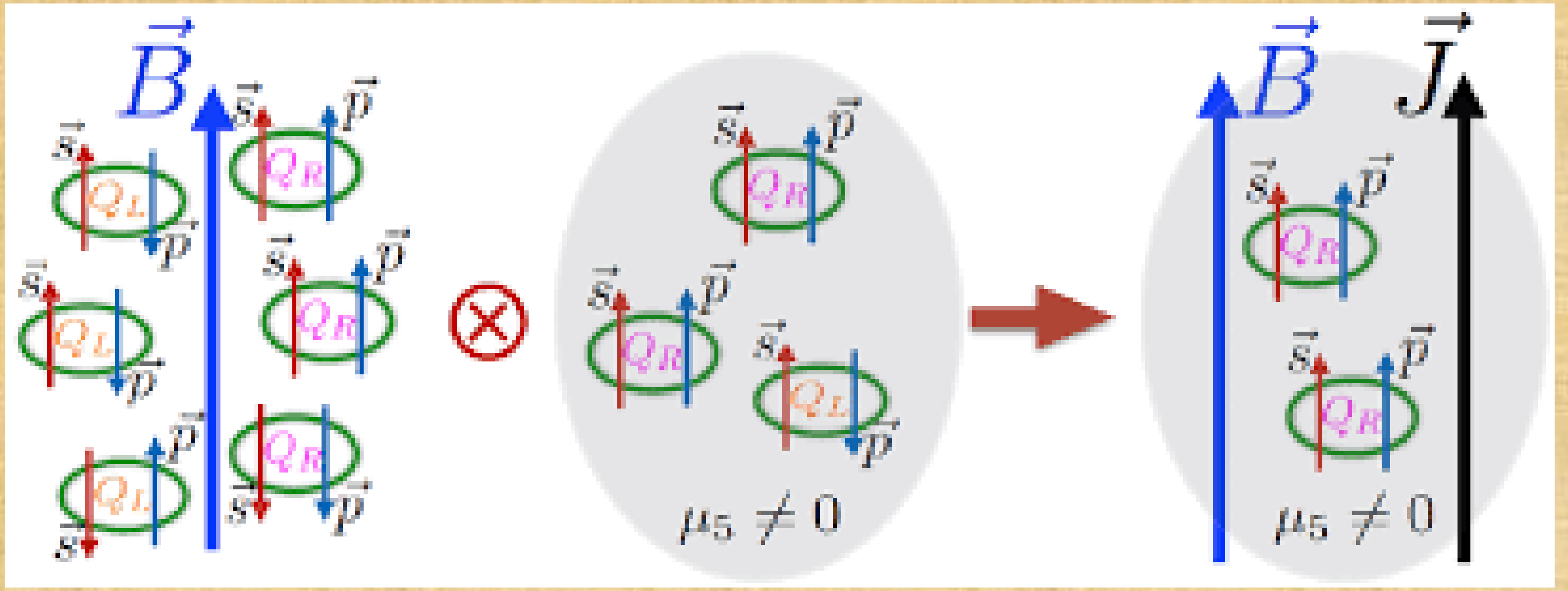
the system without disorder, elastic deformations etc, with constant magnetic field

N is not changed!

If N is known for less complicated system, we know it also for the more complicated one

Applications to Chiral Magnetic Effect
non-homogeneous system, equilibrium, T=0
Average electric current

3 + 1 D:



D.E. Kharzeev, J. Liao, S.A. Voloshin, G. Wang,
Progress in Particle and Nuclear Physics, Volume 88, 2016, Pages 1-28,

Applications to Chiral Magnetic Effect

non-homogeneous system, equilibrium, T=0

Average electric current

3 + 1 D:

$$\bar{J}^k = \frac{1}{4\pi^2}\epsilon^{ijkl}\mathcal{M}_l F_{ij}$$

topological invariant:

$$\mathcal{M}_l = \frac{-iT\epsilon_{ijkl}}{3!V\,8\pi^2}\int d^D x \int_{\mathcal{M}} d^D p \,\mathrm{Tr}\left[G_W^{(0)} \star \partial_{p_i} Q_W^{(0)}(p,x) \star G_W^{(0)} \star \partial_{p_j} Q_W^{(0)}(p,x) \star G_W^{(0)} \star \partial_{p_k} Q_W^{(0)}\right]$$

external magnetic field:

$$F_{ij} = \epsilon_{ijk} B_k$$

C. Banerjee, M. Lewkowicz, M.A. Zubkov

Physics Letters B, 136457

Chiral magnetic effect **Equilibrium, T=0**

non-homogeneous system

Average electric current

$$\bar{J}^k = \frac{1}{4\pi^2}\epsilon^{ijkl}\mathcal{M}_l F_{ij}$$

$$\mathcal{M}_l = \frac{-iT\epsilon_{ijkl}}{3!V8\pi^2}\int d^D x \int_{\mathcal{M}} d^D p \,\mathrm{Tr}\left[G_W^{(0)} \star \partial_{p_i} Q_W^{(0)}(p,x) \star G_W^{(0)} \star \partial_{p_j} Q_W^{(0)}(p,x) \star G_W^{(0)} \star \partial_{p_k} Q_W^{(0)}\right]$$

smooth deformation of the system

↓

the system without any inhomogeneity

M is not changed!

We know that in homogeneous systems M = 0

Absence of equilibrium chiral magnetic effect, M.A. Zubkov
Physical Review D 93 (10), 105036

↓

No CME in non – uniform systems at T=0

2.

Equilibrium, $T>0$

QHE: the need of kinetic theory (see below)

CME: the equilibrium theory may be used.

C. Banerjee, M. Lewkowicz, M.A. Zubkov
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Applications to Chiral Magnetic Effect

non-homogeneous system, equilibrium, T>0

Average electric current

$$\bar{j}^k = \frac{1}{4\pi^2}\epsilon^{ijk4}\mathcal{M}_4 F_{ij}$$

topological invariant:

$$\mathcal{M}_4 = 2\pi T \sum_{\omega} \mathcal{N}_4(\omega)$$

$\omega = 2\pi T(n + 1/2)$, $n \in Z$, $0 \leq n < N$, where $N = 1/T$

$$\mathcal{N}_4(\omega) = \frac{-i\epsilon_{ijk4}}{3!V\,8\pi^2}\int d^{D-1}x \int_{\mathcal{B}} d^{D-1}p\,\mathrm{Tr}\left[\mathrm{G}_{\mathrm{W}}^{(0)} \star \partial_{\mathrm{p_i}}\mathrm{Q}_{\mathrm{W}}^{(0)}(\mathrm{p},\mathrm{x}) \star \mathrm{G}_{\mathrm{W}}^{(0)} \star \partial_{\mathrm{p_j}}\mathrm{Q}_{\mathrm{W}}^{(0)}(\mathrm{p},\mathrm{x}) \star \mathrm{G}_{\mathrm{W}}^{(0)} \star \partial_{\mathrm{p_k}}\mathrm{Q}_{\mathrm{W}}^{(0)}\right]$$

Response of N to chiral chemical potential is zero

⇩

No CME at T>0

The absence of CME at T>0 **for homogeneous** systems has been reported earlier in
C.G. Beneventano, M. Nieto, E.M. Santangelo J. Phys. A, 53 (46) (2020), Article 465401,

3.

Keldysh technique
Green functions (lower sign for fermions)

$$\left\{\hat{G}^R\right\}_{(\alpha_1;\alpha_2)}(x_1;x_2) \equiv -i\theta(t_1-t_2)\left\langle\left[\Psi_{\alpha_1}(x_1),\Psi^\dagger_{\alpha_2}(x_2)\right]_\mp\right\rangle$$

$$\left\{\hat{G}^A\right\}_{(\alpha_1;\alpha_2)}(x_1;x_2) \equiv \quad i\theta(t_2-t_1)\left\langle\left[\Psi_{\alpha_1}(x_1),\Psi^\dagger_{\alpha_2}(x_2)\right]_\mp\right\rangle$$

$$\left\{\hat{G}^K\right\}_{(\alpha_1;\alpha_2)}(x_1;x_2) \equiv -i\left\langle\left[\Psi_{\alpha_1}(x_1),\Psi^\dagger_{\alpha_2}(x_2)\right]_\pm\right\rangle,$$

$$\left\{\hat{G}^<\right\}_{(\alpha_1;\alpha_2)}(x_1;x_2) \equiv \mp i\left\langle\Psi^\dagger_{\alpha_2}(x_2)\Psi_{\alpha_1}(x_1)\right\rangle$$

Keldysh Green function

$$\hat{G}(t,x|t',x') = -i\begin{pmatrix} \langle T\Phi(t,x)\Phi^+(t',x')\rangle & -\langle\Phi^+(t',x')\Phi(t,x)\rangle \\ \langle\Phi(t,x)\Phi^+(t',x')\rangle & \langle\tilde{T}\Phi(t,x)\Phi^+(t',x')\rangle \end{pmatrix}$$

$$\begin{pmatrix} G^{--} & G^{-+} \\ G^{+-} & G^{++} \end{pmatrix}$$

$$G^{\mathrm{A}} = G^{--} - G^{+-} = G^{-+} - G^{++}$$

$$G^{\mathrm{R}} = G^{--} - G^{-+} = G^{+-} - G^{++}$$

$G^<$ $\quad$ G^{-+}

3.

Keldysh technique and Wigner – Weyl calculus.

Keldysh Green function

$$\hat{G}(t,x|t',x') = -i \begin{pmatrix} \langle T\Phi(t,x)\Phi^+(t',x')\rangle & -\langle \Phi^+(t',x')\Phi(t,x)\rangle \\ \langle \Phi(t,x)\Phi^+(t',x')\rangle & \langle \tilde{T}\Phi(t,x)\Phi^+(t',x')\rangle \end{pmatrix}$$

$$= \begin{pmatrix} G^{--} & G^{-+} \\ G^{+-} & G^{++} \end{pmatrix}$$

$$G^{\mathrm{A}} = G^{--} - G^{+-} = G^{-+} - G^{++}$$

$$G^{\mathrm{R}} = G^{--} - G^{-+} = G^{+-} - G^{++}$$

$$G^{<} = G^{-+}$$

Wigner transformation

$$\hat{G}(X_1, X_2) = \langle X_1|\hat{\mathbf{G}}|X_2\rangle \qquad A(X_1, X_2) = \langle X_1|\hat{A}|X_2\rangle$$

$$A_W(X|P) = \int d^{D+1}Y\, e^{\mathrm{i}Y^\mu P_\mu} A(X+Y/2, X-Y/2)$$

Moyal product

$$(A \star B)\,(X|P) = A(X|P)\, e^{-\mathrm{i}(\overleftarrow{\partial}_{X^\mu}\overrightarrow{\partial}_{P_\mu} - \overleftarrow{\partial}_{P_\mu}\overrightarrow{\partial}_{X^\mu})/2} B(X|P)$$

Lesser representation

$$\hat{\mathbf{G}}^{(<)} = U\hat{\mathbf{G}}V,$$

U= $\frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$ **V =** $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$

$$\hat{\mathbf{G}}^{(<)} = \begin{pmatrix} G^{\mathrm{R}} & 2G^{<} \\ 0 & G^{\mathrm{A}} \end{pmatrix}$$

$$G^{\mathrm{A}} = G^{--} - G^{+-} = G^{-+} - G^{++}$$
$$G^{\mathrm{R}} = G^{--} - G^{-+} = G^{+-} - G^{++}$$

$$G^{<} = G^{-+}$$

The inverse Q of Green function

$$\hat{\mathbf{Q}}\hat{\mathbf{G}} = 1$$

After Wigner transformation

$$\hat{Q} * \hat{G} = 1$$

In non – interacting systems

$$G^{\mathrm{R}} = (\mathrm{i}\partial_t - \hat{H}e^{+\epsilon\partial_t})^{-1} = (\mathrm{i}\partial_t - \hat{H} + \mathrm{i}\epsilon)^{-1}$$

$$G^{\mathrm{A}} = (\mathrm{i}\partial_t - \hat{H}e^{-\epsilon\partial_t})^{-1} = (\mathrm{i}\partial_t - \hat{H} - \mathrm{i}\epsilon)^{-1}$$

$$G^{<} = (G^{\mathrm{A}} - G^{\mathrm{R}})\frac{\rho}{\rho+1}.$$

← ***distribution function***

In general case without interactions electric current

$$J^i(X) = -\frac{\mathrm{i}}{2}\int\frac{d^{D+1}\pi}{(2\pi)^{D+1}}\,\mathrm{tr}\left((\partial_{\pi_i}\hat{Q})\hat{G}\right)^{<} - \frac{\mathrm{i}}{2}\int\frac{d^{D+1}\pi}{(2\pi)^{D+1}}\,\mathrm{tr}\left(\hat{G}(\partial_{\pi_i}\hat{Q})\right)^{<}$$

A.Shitade, J. Phys. Soc. Jpn. 86, 054601 (2017)

product of triangle matrices is triangle matrix

lesser component for any matrix is defined as →

$$\hat{\mathrm{Q}}^{(<)} = \begin{pmatrix} Q^{\mathrm{R}} & 2Q^{<} \\ 0 & Q^{\mathrm{A}} \end{pmatrix}$$

Response of electric current to external field strength

$$J^i = -\frac{1}{4}\int \frac{d^{D+1}\pi}{(2\pi)^{D+1}} \operatorname{tr}\Big(\hat{G}\star\partial_{\pi^\mu}\hat{Q}\star\hat{G}\star\partial_{\pi^\nu}\hat{Q}\star\hat{G}\partial_{\pi_i}\hat{Q}\Big)^< \mathcal{F}^{\mu\nu}$$
$$-\frac{1}{4}\int \frac{d^{D+1}\pi}{(2\pi)^{D+1}} \operatorname{tr}\Big(\partial_{\pi_i}\hat{Q}\hat{G}\star\partial_{\pi^\mu}\hat{Q}\star\hat{G}\star\partial_{\pi^\nu}\hat{Q}\star\hat{G}\Big)^< \mathcal{F}^{\mu\nu}.$$

Electric conductivity tensor for non – homogeneous systems

$$J^i = \sigma^{ij}\mathcal{F}_{0j}$$

$$\sigma^{ij} = \frac{1}{4}\int \frac{d^{D+1}\pi}{(2\pi)^{D+1}} \operatorname{tr}\left(\partial_{\pi_i}\hat{Q}\left[\hat{G}\star\partial_{\pi_{[0}}\hat{Q}\star\partial_{\pi_{j]}}\hat{G}\right]\right)^< + \text{c.c.}$$

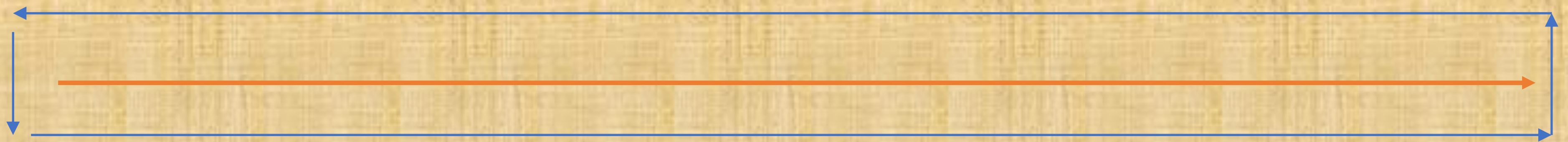
C Banerjee, IV Fialkovsky, M Lewkowicz, CX Zhang, MA Zubkov
arXiv preprint arXiv:2009.10704

2D Hall conductivity "Topological part"

$$\bar{\sigma}_H = -\frac{\mathcal{N}_f}{2\pi} + \bar{\sigma}_{H,f'}$$

$$\mathcal{N}_f = -\frac{1}{48\pi^2\,\mathcal{V}}\epsilon^{\mu\nu\rho}\oint d\pi^0 \int d^2\pi d^2x\ \mathrm{tr}\left(\partial_{\pi^\mu} Q \star \partial_{\pi^\nu} G \star \partial_{\pi^\rho} Q \star G\right) f(\pi^0) + \mathrm{c.c.}$$

C Banerjee, IV Fialkovsky, M Lewkowicz, CX Zhang, MA Zubkov, arXiv:2009.10704



A similar expression has been obtained independently in F.R. Lux, F. Freimuth, S. Bl¨ugel, Y. Mokrousov, Physical Review Letters 124 (9), 096602 (2020)

contour in complex plane of π^0

in the case of thermal equilibrium at T->0

$$\mathcal{N}_f = -\frac{1}{24\pi^2\,\beta\,\mathcal{V}}\epsilon^{\mu\nu\rho}\int d^3X \int d^3\Pi\, \mathrm{tr}\left(\partial_{\Pi^\mu}\hat{Q}^{\mathrm{M}} \star \hat{G}^{\mathrm{M}} \star \partial_{\Pi^\nu}\hat{Q}^{\mathrm{M}} \star \hat{G}^{\mathrm{M}} \star \partial_{\Pi^\rho}\hat{Q}^{\mathrm{M}} \star \hat{G}^{\mathrm{M}}\right)$$

Matsubara Green function

G^M (we replace inside G^R : $\pi^0 \to i\,\omega$)

2D Hall conductivity

$$\bar{\sigma}_H = -\frac{\mathcal{N}_f}{2\pi} + \bar{\sigma}_{H,f'}$$

"non - topological part"

$$\bar{\sigma}_{H,f'} = +\frac{1}{8\,\mathcal{V}}\epsilon^{ij}\int \frac{d^3\pi d^2x}{(2\pi)^3}\, \mathrm{tr}\left((\partial_{\pi^i}Q^{\mathrm{R}} \star G^{\mathrm{R}} + \partial_{\pi^i}Q^{\mathrm{A}} \star G^{\mathrm{A}}) \star \partial_{\pi^j}Q^{\mathrm{R}} \star (G^{\mathrm{A}} - G^{\mathrm{R}})\right) \partial_{\pi^0} f(\pi^0) + c.c.$$

ordinary symmetric conductivity

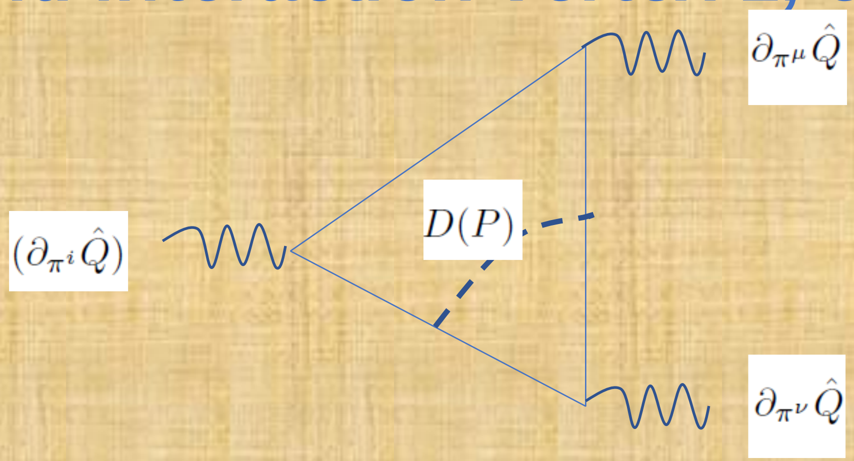
$$\bar{\sigma}^{ij}_{\parallel} = \frac{1}{8\,\mathcal{V}}\int \frac{d^3\pi d^2x}{(2\pi)^3}\, \mathrm{tr}\left((-\partial_{\pi^i}Q^{\mathrm{R}} \star G^{\mathrm{R}} + \partial_{\pi^i}Q^{\mathrm{A}} \star G^{\mathrm{A}}) \star \partial_{\pi^j}Q^{\mathrm{R}} \star (G^{\mathrm{A}} - G^{\mathrm{R}})\right) \partial_{\pi^0} f(\pi^0) + (i \leftrightarrow j) + \mathrm{c.c}$$

4.

Interaction corrections to electric conductivity

$$J^{i(1)}_{\mathcal{F}} = -\frac{\mathcal{F}^{\mu\nu}}{4\mathcal{V}}\int \frac{d^Dx d^{D+1}\pi}{(2\pi)^{D+1}}\frac{d^{D+1}P}{(2\pi)^{D+1}} D(P)\,\mathrm{tr}\Big[(\partial_{\pi^i}\hat{Q})\star\hat{G}\star\Big(\Big(G\star\partial_{\pi^\mu}\hat{Q}\star\hat{G}\star\partial_{\pi^\nu}\hat{Q}\star\hat{G}\Big)_{\pi-P}$$
$$+G\Big|_{\pi-P}\star\hat{G}\star\partial_{\pi^\mu}\hat{Q}\star G\star\partial_{\pi^\nu}\hat{Q}-\partial_{\pi^\mu}G\Big|_{\pi-P}\star G\star\partial_{\pi^\nu}\hat{Q}+\partial_{\pi^\mu}\hat{Q}\star\hat{G}\star G\Big|_{\pi-P}\star\hat{G}\star\partial_{\pi^\nu}\hat{Q}$$
$$-\partial_{\pi^\mu}\hat{Q}\star G\star\partial_{\pi^\nu}G\Big|_{\pi-P}+\partial_{\pi^\mu}\hat{Q}\star G\star\partial_{\pi^\nu}\hat{Q}\star\hat{G}\star G\Big|_{\pi-P}\Big)\star\hat{G}\Big]^{<} + \mathrm{c.c.}$$

the exchange by bosonic excitation with propagator D(P), and interaction vertex 1, one loop



C Banerjee, IV Fialkovsky, M Lewkowicz, CX Zhang, MA Zubkov, arXiv:2009.10704

Interaction corrections to electric conductivity

$$J^{i(1)}_{\mathcal{F}} = -\frac{\mathcal{F}^{\mu\nu}}{4\mathcal{V}} \int \frac{d^D x d^{D+1}\pi}{(2\pi)^{D+1}} \frac{d^{D+1}P}{(2\pi)^{D+1}} D(P)\,\mathrm{tr}\Big[(\partial_{\pi^i}\hat{Q}) \star \hat{G} \star \Big(\Big(G \star \partial_{\pi^\mu}\hat{Q} \star \hat{G} \star \partial_{\pi^\nu}\hat{Q} \star \hat{G}\Big)_{\pi-P}$$
$$+G\Big|_{\pi-P} \star \hat{G} \star \partial_{\pi^\mu}\hat{Q} \star G \star \partial_{\pi^\nu}\hat{Q} - \partial_{\pi^\mu} G\Big|_{\pi-P} \star G \star \partial_{\pi^\nu}\hat{Q} + \partial_{\pi^\mu}\hat{Q} \star \hat{G} \star G\Big|_{\pi-P} \star \hat{G} \star \partial_{\pi^\nu}\hat{Q}$$
$$-\partial_{\pi^\mu}\hat{Q} \star G \star \partial_{\pi^\nu} G\Big|_{\pi-P} + \partial_{\pi^\mu}\hat{Q} \star G \star \partial_{\pi^\nu}\hat{Q} \star \hat{G} \star G\Big|_{\pi-P}\Big) \star \hat{G}\Big]^{<} + \mathrm{c.c.}$$

If there is symmetry under cyclic permutation of operators under the trace, then one – loop contribution to antisymmetric (Hall) component of conductivity vanishes. This occurs, for example, if the initial distribution of fermions $f(\pi^0)$ is Fermi distribution with T → 0, and chemical potential is inside the gap.

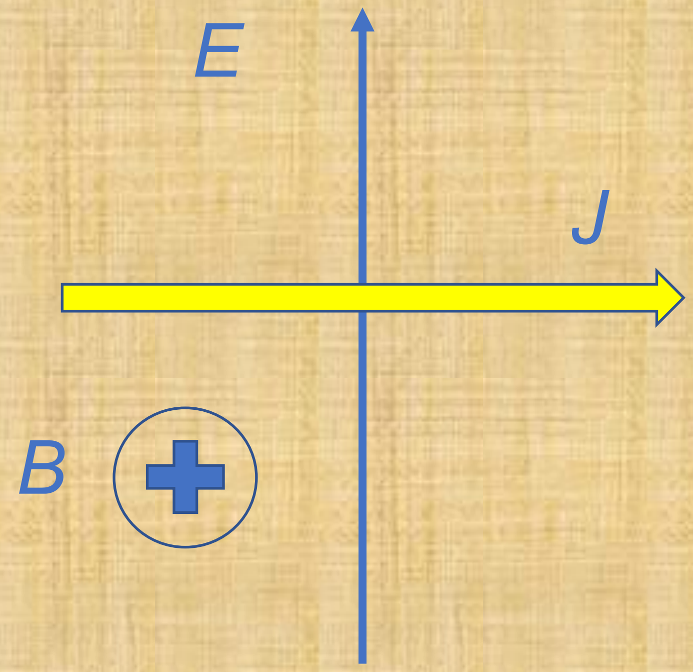
Thus in equilibrium at zero temperature there are no one – loop interaction corrections to Hall conductivity.

The absence of interaction corrections to Quantum Hall Effect equilibrium, T=0

Electric current orthogonal to electric field in the presence of magnetic field

C. X. Zhang[a] and M. A. Zubkov[a, *]

JETP Letters, 2019, Vol. 110, No. 7, pp. 487–494.



The absence of interaction corrections to Quantum Hall Effect

Electric current orthogonal to electric field in the presence of magnetic field

$$S = \int d\tau \sum_{\mathbf{x},\mathbf{x}'} [\bar{\psi}_{\mathbf{x}'}(i(i\partial_\tau - A_3(i\tau,\mathbf{x}))\delta_{\mathbf{x},\mathbf{x}'} - i\mathcal{D}_{\mathbf{x},\mathbf{x}'})\psi_{\mathbf{x}}$$
$$+ \alpha\bar{\psi}(\tau,\mathbf{x})\psi(\tau,\mathbf{x})\theta(y)V(\mathbf{x}-\mathbf{x}')\theta(y')\bar{\psi}(\tau,\mathbf{x}')\psi(\tau,\mathbf{x}')]$$

as an example:

$$\mathcal{D}_{\mathbf{x},\mathbf{x}'} = -\frac{i}{2}\sum_{i=1,2}[(1+\sigma^i)\delta_{x+e_i,x'}e^{iA_{x+e_i,x}}$$
$$+(1-\sigma^i)\delta_{x-e_i,x'}e^{iA_{x-e_i,x}}]\sigma_3 + i(m+2)\delta_{\mathbf{x},\mathbf{x}'}\sigma_3$$

without interactions:

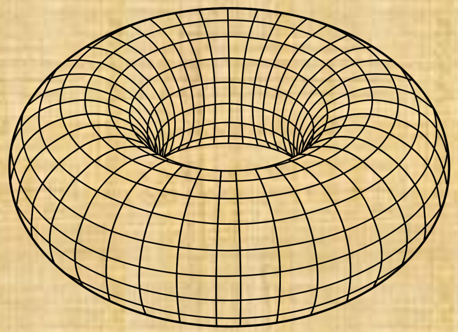
$$\sigma_{xy} = \frac{\mathcal{N}}{2\pi}$$

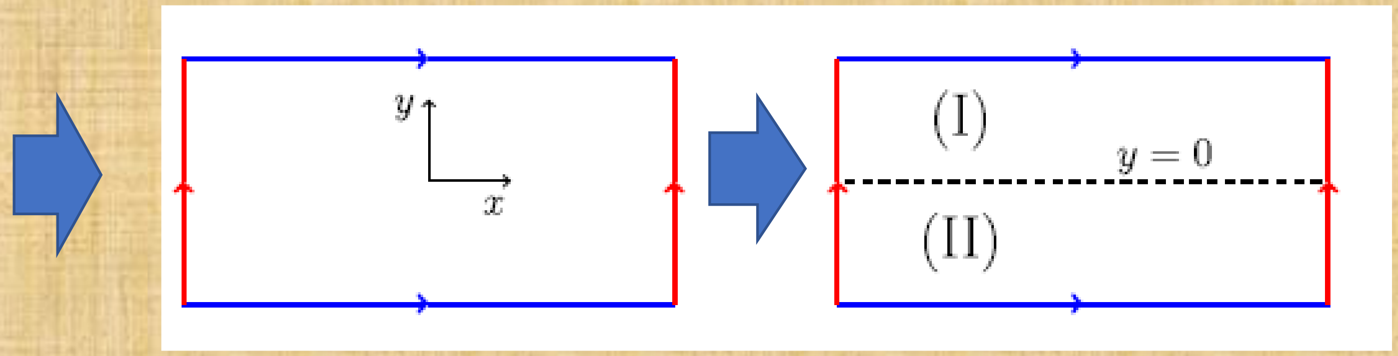
$$\mathcal{N} = \frac{T}{S\,3!\,4\pi^2}\epsilon_{ijk}\int d^3x \int d^3p \,\mathrm{Tr}\, G_W(p,x) * \frac{\partial Q_W(p,x)}{\partial p_i} * \frac{\partial G_W(p,x)}{\partial p_j} * \frac{\partial Q_W(p,x)}{\partial p_k}.$$

$$\sigma_{xy} = \frac{\mathcal{N}}{2\pi}$$

$$\mathcal{N} = \frac{T}{S3!4\pi^2}\epsilon_{ijk}\int d^3x \int d^3p \,\mathrm{Tr}\, G_W(p,x) * \frac{\partial Q_W(p,x)}{\partial p_i} * \frac{\partial G_W(p,x)}{\partial p_j} * \frac{\partial Q_W(p,x)}{\partial p_k}.$$

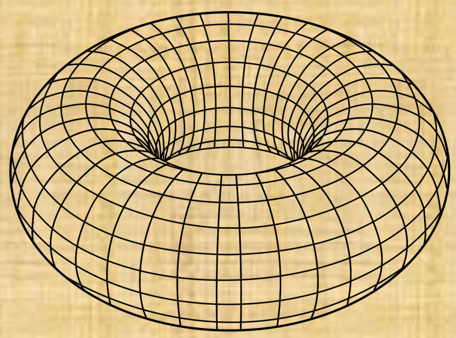
Gedankenexperiment:
we consider the system on the torus and divide it into the two pieces

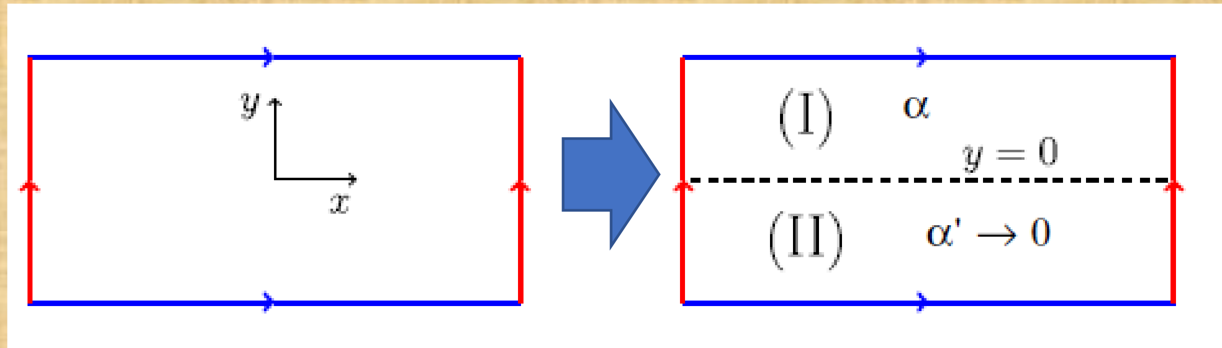




we consider the system on the torus and divide it into the two pieces

$$\sigma_{xy} = \frac{\mathcal{N}}{2\pi}$$





α *is zero in the part II, E(I) = - E(II)*

$$I_{tot} = (I_1 + I_2)/2 = (\bar{\sigma}_1 E + \bar{\sigma}_2(-E))/2 + I_{tot}\Big|_{E=0}$$

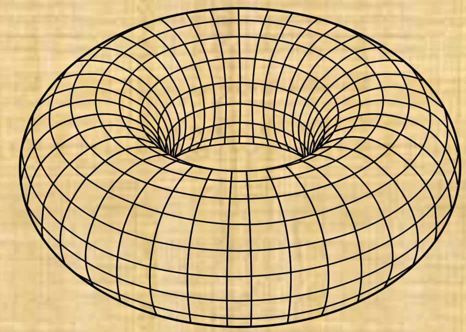
$$\bar{\sigma}_1(0) = \bar{\sigma}_2(0),$$

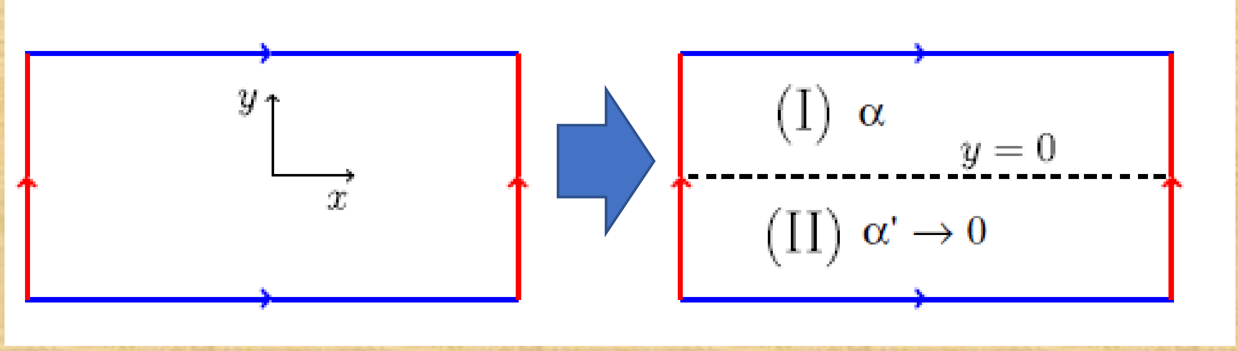
We prove that the total current remains zero with the interaction corrections

$$\bar{\sigma}_1(g) = \bar{\sigma}_2(0)$$

$$\bar{\sigma}_1(0) = \bar{\sigma}_1(g)$$

no interaction corrections





α is zero in the part II, E(I) = - E(II)

$$I(g) = \int \frac{d^3R}{\beta S} \int \frac{d^3p}{(2\pi)^3} \mathrm{Tr}\, G_{g,W}(R,p) \star \frac{\partial}{\partial p_x} Q_W(R,p)$$

$$G_{g,W} = G_W + G_W \star \Sigma_W \star G_W + \dots$$

$$I(g) = \sum_{n=0}^{\infty} I^{(n)}$$

$$I^{(n)} = \int \frac{d^3R}{\beta S} \int \frac{d^3p}{(2\pi)^3} \mathrm{Tr}\, (G_W \star \Sigma_W \star)^n G_W \star \frac{\partial Q_W}{\partial p_x}$$

an example: 1-loop

$$\mathcal{I}_1 = -\int \frac{d^3R}{\beta S} \int \frac{d^3p}{(2\pi)^3} \mathrm{Tr}\Big[\int \frac{d^3q}{(2\pi)^3} G_W(R,p-q)\mathcal{D}(q)\Big] \star \frac{\partial}{\partial p_x} G_W(R,p)$$

=0

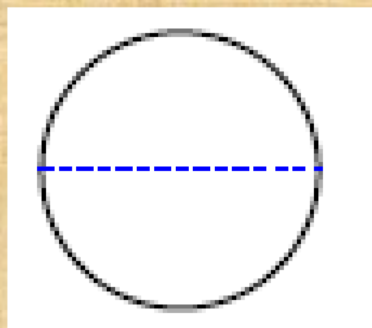
$$-\int \frac{d^3R}{\beta S} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \mathrm{Tr} \frac{\partial}{\partial p_x}\Big[G_W(R,p-q) \star G_W(R,p)\Big]\mathcal{D}(q) = 0$$

$$I(g) = \sum_{n=0}^{\infty} I^{(n)}$$

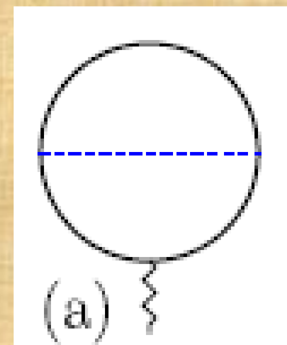
$$I^{(n)} = \int \frac{d^3R}{\beta S} \int \frac{d^3p}{(2\pi)^3} \mathrm{Tr}\left(G_W \star \Sigma_W \star\right)^n G_W \star \frac{\partial Q_W}{\partial p_x}$$

an example: 1-loop diagram

$\frac{\partial}{\partial p_x}$



= 2



$$\mathcal{I}_1 = -\int \frac{d^3R}{\beta S} \int \frac{d^3p}{(2\pi)^3} \mathrm{Tr}\Big[\int \frac{d^3q}{(2\pi)^3} G_W(R, p-q)\mathcal{D}(q)\Big] \star \frac{\partial}{\partial p_x} G_W(R,p)$$

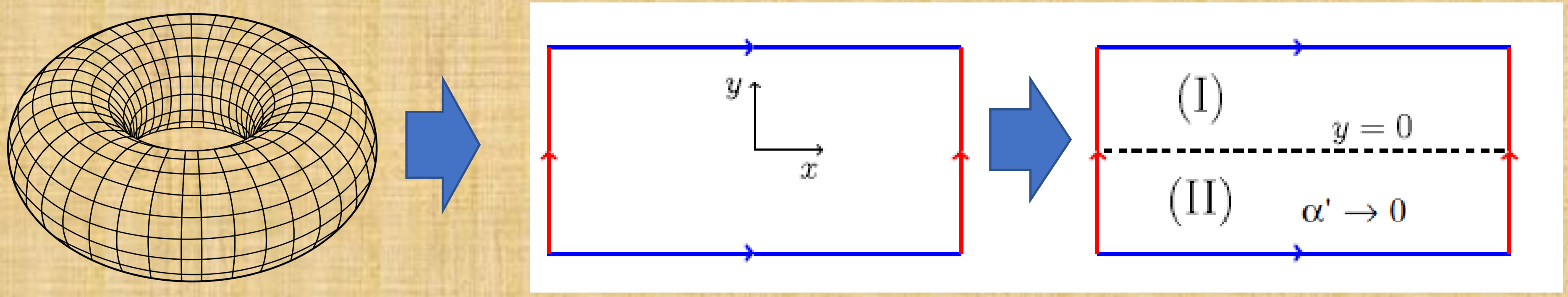
=0

$$-\int \frac{d^3R}{\beta S} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \mathrm{Tr}\frac{\partial}{\partial p_x}\Big[G_W(R, p-q) \star G_W(R,p)\Big]\mathcal{D}(q) = 0$$

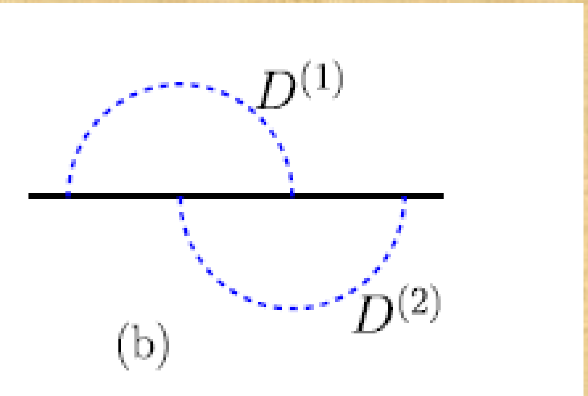
$$\sigma_{xy} = \frac{\mathcal{N}}{2\pi}$$

$$\mathcal{N} = \frac{T}{S3!4\pi^2}\epsilon_{ijk}\int d^3x \int d^3p \, \mathrm{Tr}\, G_{\mathrm{W}}(p,x) * \frac{\partial Q_{\mathrm{W}}(p,x)}{\partial p_i} * \frac{\partial G_{\mathrm{W}}(p,x)}{\partial p_j} * \frac{\partial Q_{\mathrm{W}}(p,x)}{\partial p_k}.$$

In the presence of interactions the sum of the currents in the two pieces is zero ➔ the electric conductivity receives no corrections in the part I



Another example of diagram technique



$$\int\int [G_1(R,p)\circ_1 \star G_2(R,p-k_1)\circ_2 \star G_3(R,p-k_1-k_2)\star_1\circ G_4(R,p-k_2)\star_2\circ G_5(R,p)] D_W^{(1)}(R,k_1)D_W^{(2)}(R,k_2)dk_1dk_2.$$

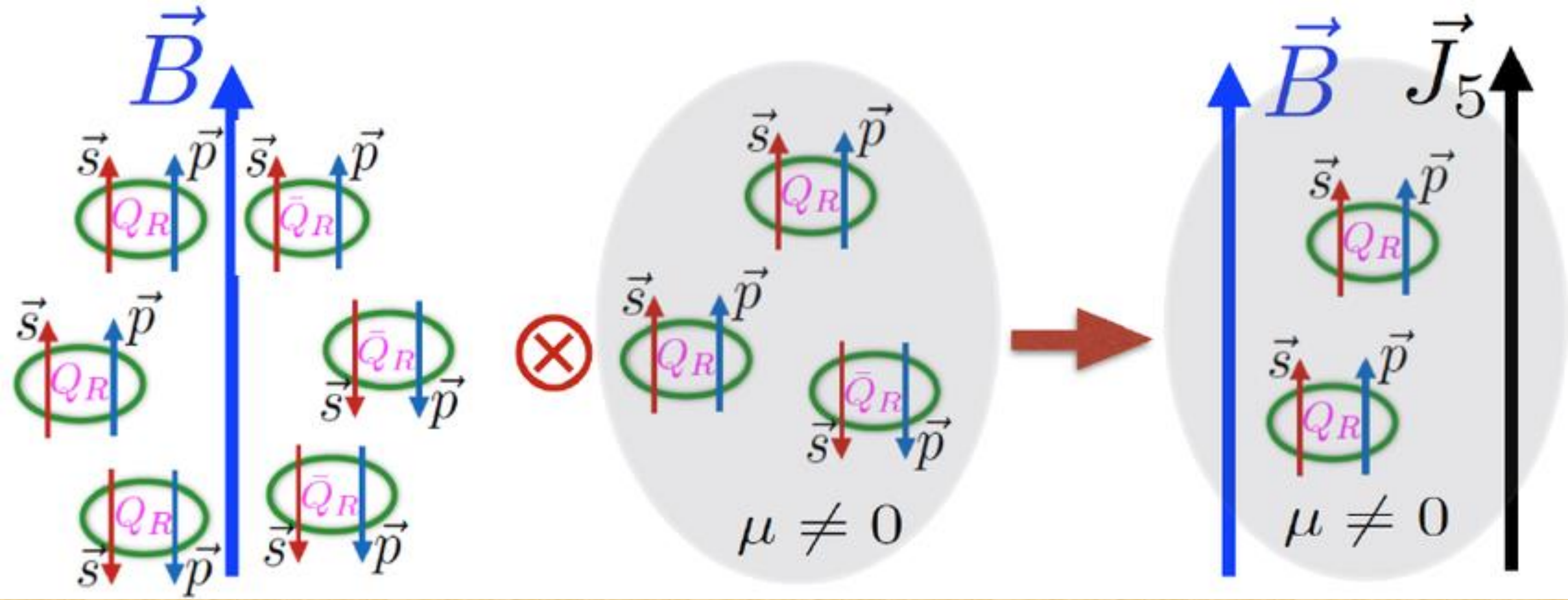
$\circ_j = e^{-i\overleftarrow{\partial}_p \partial_R^{(j)}/2}$ and ${}_j\circ = e^{i\partial_R^{(j)}\overrightarrow{\partial}_p/2}$. $\partial_R^{(j)}$ acts on $D^{(j)}$ only.

5.

Wigner – Weyl calculus may also be used for the investigation of the other non – dissipative transport phenomena.

An example is
Chiral Separation Effect

Axial current along magnetic field in the presence of chemical potential



D.E. Kharzeev, J. Liao, S.A. Voloshin, G. Wang,
Progress in Particle and Nuclear Physics, Volume 88, 2016, Pages 1-28,

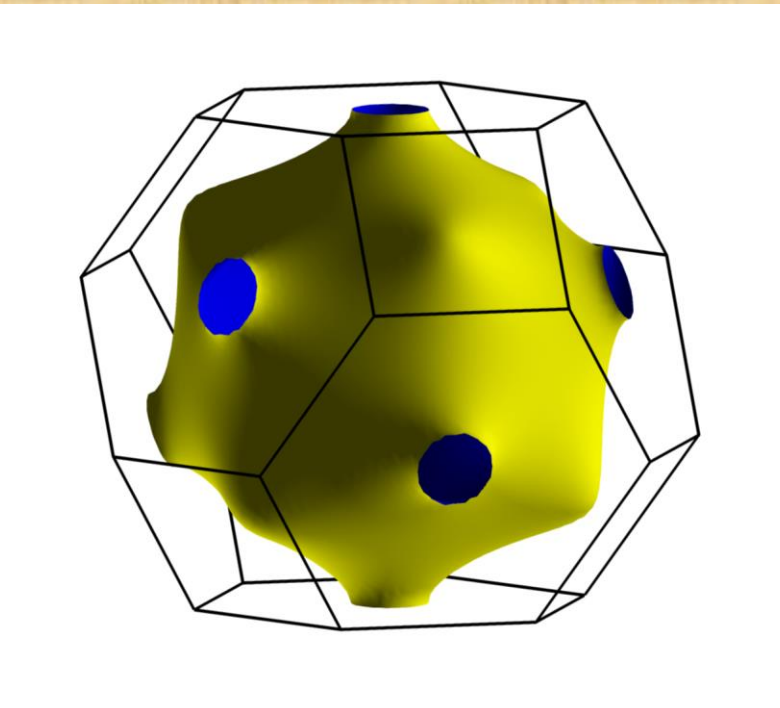
$$J_5^k = -\frac{1}{4\pi^2}\epsilon^{ijk0}\mu F_{ij}$$

A. Metlitski and Ariel R. Zhitnitsky,Phys. Rev. D 72 (2005), 045011

Lattice Dirac operator Q is 4 x 4 matrix expressed through the Gamma matrices

$$j_k^5(x) = -\int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \mathrm{tr}\left[\gamma^5 G_W(x,p) \partial_{p_k} Q_W(x,p)\right]$$

The system with Fermi surface of arbitrary complicated form



Fermi surface of gold

from

http://exciting.wikidot.com/nitrogen-fermisurface

Lattice Dirac operator Q is 4 x 4 matrix

expressed through the Gamma matrices

$$j^5_k(x) = -\int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \,\mathrm{tr}\, \left[\gamma^5 G_W(x,p) \partial_{p_k} Q_W(x,p)\right]$$

The system with Fermi surface of arbitrary complicated form

$$\bar{J}^k_5 = -\frac{\mathcal{N}}{4\pi^2} \epsilon^{ijk0} \mu F_{ij}$$

$$\mathcal{N} = -\frac{1}{48\pi^2 \mathbf{V}} \int_{\Sigma_3} \int d^3x \,\mathrm{tr} \left[\gamma^5 G_W \star dQ_W \star G_W \wedge \star dQ_W \star G_W \star \wedge dQ_W\right]$$

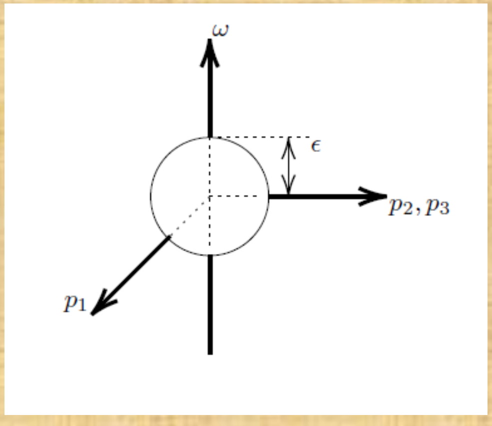
Surface Σ_3 surrounds the singularities

of $\left[\gamma^5 G^{(0)}_W \star dQ^{(0)}_W \star G^{(0)}_W \wedge \star dQ^{(0)}_W \star G^{(0)}_W \star \wedge dQ^{(0)}_W\right]$

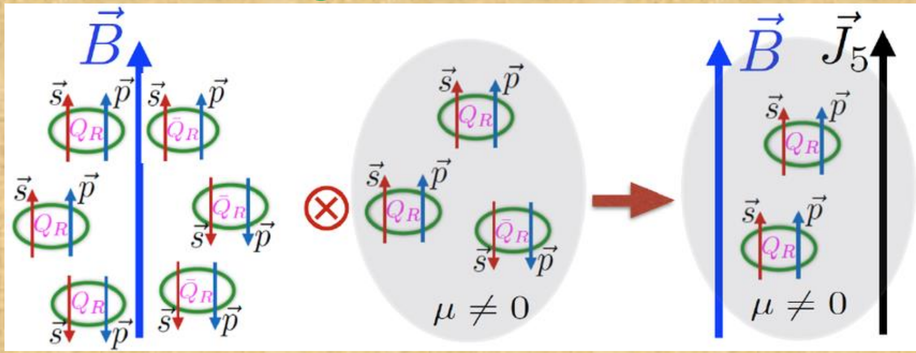
γ^5 commutes/anticommutes with Q

in small vicinity of Σ_3

M.Suleymanov, M.Zubkov, Physical Review D 102 (7), 076019



Lattice Dirac operator Q is 4 x 4 matrix expressed through the Gamma matrices



The system with Fermi surface of arbitrary complicated form

$$\bar{J}^k_5 = -\frac{\mathcal{N}}{4\pi^2}\epsilon^{ijk0}\mu F_{ij}$$

Irrespective of the form of the Fermi surface the value of $\mathcal{N}$ is equal to the number of chiral 4 – component Dirac fermions

M.Suleymanov, M.Zubkov, Physical Review D 102 (7), 076019

Conclusions

- Wigner – Weyl calculus allows to represent in compact form the conductivities of non – dissipative transport phenomena **in non – uniform systems**.
- Equilibrium systems at zero temperature: QHE conductivity is given by topological invariant composed of the Wigner transformed two-point Green functions.
- Equilibrium systems at finite temperatures: CME response of electric current to magnetic field is the topological invariant in phase space.

Conclusions

- Non – equilibrium systems, Keldysh technique and Wigner – Weyl calculus allow to express in compact form electric conductivity.
- Out of equilibrium already in one loop the interaction corrections to Hall conductivity do not vanish. In equilibrium theory at zero temperature these corrections disappear to all orders.
- This technique may be applied also to the other non – dissipative transport effects, say, to CSE.