Lipatov's effective action in Euclidean space

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Based on:
S.Bondarenko,
Eur. Phys. J. C **80** (2020) no.5, 356 arXiv:1907.01756;
S.Bondarenko, L.Lipatov and A.Prygarin,
Eur. Phys. J. C **77**, no. 8, 527 (2017) arXiv:1706.00278 ;
S. Bondarenko, L. Lipatov, S. Pozdnyakov and A. Prygarin,
Eur. Phys. J. C **77**, no. 9, 630 (2017), arXiv:1708.05183 ;
S. Bondarenko, M. Zubkov,
Eur. Phys. J. C **78** (2018) no.8, 617 arXiv:1801.08066.

Effective action setup

- Consider QCD action with added sources of longitudinal gluons:

$$S_{eff} = -\int d^4 x \left(\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + tr \left[v_+ J^+(v_+) - A_+ j^+_{reg} + v_- J^-(v_-) - A_- j^-_{reg} \right] \right)$$

For the currents $J^\pm_a(v_\pm)$ we request that under a variation on the gluon fields they behave as

$$\delta \left(v_\pm J^\pm(v_\pm) \right) = (\delta v_\pm)\, j^{ind}_\mp(v_\pm) = (\delta v_\pm)\, j^\pm(v_\pm)$$

where induced currents are covariant:

$$\left(D_\pm j^{ind}_\mp(v_\pm) \right)^a = \left(D_\pm j^\pm(v_\pm) \right)^a = 0$$

Additionally there are the following boundary conditions for the gluon's classical solutions to LO precision:

$$v_{+\,cl} = A_+ , \quad v_{-\,cl} = A_-$$

That is enough (almost) for determination of the form of $J^\pm_a(v_\pm)$ currents.

Effective action setup

- The construction above is equivalent to the construction of the Lipatov's gauge-invariant action in rapidity interval $(y_0 - \eta, y_0 + \eta)$ with the gluon-reggeon interactions with $A_\pm$ reggeon fields and corresponding boundary conditions at the edges of the interval (similarly to the Gribov's RFT). The action allows to calculate scattering amplitudes in multi-Regge kinematics when $|t|/s \rightarrow \infty$.

$$S_{eff} = - \int d^4 x \left(\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + tr \left[\left(\mathcal{T}_+(\mathsf{v}_+) - A_+ \right) j^+_{reg} + \left(\mathcal{T}_-(\mathsf{v}_-) - A_- \right) j^-_{reg} \right] \right)$$

$$\mathcal{T}_\pm(\mathsf{v}_\pm) = \frac{1}{g} \partial_\pm O(x^\pm, \mathsf{v}_\pm) = \mathsf{v}_\pm O(x^\pm, \mathsf{v}_\pm) \,,$$

$$j^\pm_{reg\,a} = \frac{1}{C(R)} \partial_i^2 A^\pm_a \,, \quad \partial_- A_+ = \partial_+ A_- = 0 \,,$$

where $C(R)$ is the eigenvalue of Casimir operator in the representation R, $C(R) = N$ in the case of adjoint representation. See in:

L. N. Lipatov, Nucl. Phys. B 452, 369 (1995); Phys. Rept. 286, (1997) 131; B.Ioffe, V.Fadin and L.Lipatov, "Perturbative QCD".

Effective action setup

- There is an another definition of the Liaptov's action, the additional part to the QCD Lagrangian is defined as induced part of the effective action, S_{ind} and it has a form of the interacting Eikonal lines:

$$
\begin{aligned}
S_{ind} \;&=\; \Big(\frac{1}{2\,C(R)}\int d^4x\,\mathcal{T}_+\,\partial_\perp^2\,\mathcal{T}_- \;+ \\
&+\; \frac{1}{2\,C(R)}\int d^4x\,J_-(x^-,\,x_\perp)\,\mathcal{T}_+ \;+\; \frac{1}{2\,C(R)}\int d^4x\,J_+(x^+,\,x_\perp)\,\mathcal{T}_-\Big)
\end{aligned}
$$

where again

$$
\mathcal{T}_\pm(v_\pm) \;=\; \frac{1}{g}\,\partial_\pm\,O(v_\pm) \;=\; v_\pm\,O(v_\pm)
$$

S. Bondarenko, M. Zubkov, arXiv:1801.08066; E. Verlinde and H. Verlinde, QCD at high energies and two-dimensional field theory, preprint PUPT 1319, IASSNS-HEP 92/30, 1993; I. Ya. Aref'eva, Large N QCD at high energies as two-dimensional field theory, preprint SMI-5-93.

This reformulation of the Lipatov's effective action is convenient for the Euclidean formulation of the approach.

Effective action setup

- In the simplest case:

$$O_x \,=\, P\, e^{g\int_{-\infty}^{x^+} dx'^+\, v_+(x'^+)}\,,\;\; O_x^T \,=\, P\, e^{g\int_{x^+}^{\infty} dx'^+\, v_+(x'^+)}\,.$$

are usual ordered exponentials (Wilson lines operators) and

$$\delta\,\left(v_+\,J^+\right) \,=\delta\, tr[\,(\,v_{+x}\,O_x\,\partial_i^2\,A^+\,)\,] \,=\, -\,\delta v_+^a\, tr[\,T_a\,O\,T_b\,O^T\,]\,\left(\partial_i^2\,A_b^+\right)\,.$$

That determines the form of the effective induced currents.

Correspondingly there are new gluon-reggeon vertices in the action which can be used for the diagrammatic construction of the amplitudes with gluon and reggeon fields: *L. N. Lipatov, Nucl. Phys. Proc. Suppl.* **99A**, *175 (2001); M. A. Braun and M. I. Vyazovsky, Eur. Phys. J. C* **51**, *103 (2007); M. A. Braun, L. N. Lipatov, M. Y. Salykin and M. I. Vyazovsky, Eur. Phys. J. C* **71**, *1639 (2011); M. Hentschinski and A. Sabio Vera, Phys. Rev. D* **85**, *056006 (2012); G. Chachamis, M. Hentschinski, J. D. Madrigal Martínez and A. Sabio Vera and etc.*

Effective action: RFT calculus

- Expanding around the classical solutions (light-cone gauge, for example)

$$v_i^a \,\rightarrow\, v_{i\,cl}^a(A_-, A_+) + \varepsilon_i^a\,, \quad v_+^a \,\rightarrow\, v_{+\,cl}^a(A_-, A_+) + \varepsilon_+^a$$

and integrating over the fluctuations we obtain Regge Field Theory (RFT) effective action:

$$S_{eff}(A_+,\, A_-) \,=\, \Gamma \,=\, \int d^4 x \,(\, s_1[g,\, A_+,\, A_-] \,+\, g\, s_2[g,\, A_+,\, A_-] \,+\, \cdots)$$

which can be considered as generating function of the Reggeon interactions vertices:

$$\Gamma \,=\, \sum_{n,m\,=\,0} \left(A_+^{a_1} \,\cdots\, A_+^{a_n}\, K_{b_1\,\cdots\, b_m}^{a_1\,\cdots\, a_n}\, A_-^{b_1}\,\cdots\, A_-^{b_m} \right)$$

All QCD contribution in the Γ determines the precision of the K vertices, these vertices are kernels of the BFKL approach.

Effective action: what do we want

- We want to know the behavior of the correlators. For example, to leading order

$$< A_+ A_- > \propto s^{1+\omega_R(t)}$$

 The LO perturbative value of the trajectory ω_R is too large to be correct. We do not know, in general, the behavior of many Reggeon correlators, namely, there is a Pomeron like correlator for example:

$$< A_+ A_+ > \propto s^{1+\omega_P(t)}$$

 but, again, the LO perturbative value of the ω_P is too large.

- There is a hard question about the region of applicability of the approach. It is a perturbative one and there is a question about the boundary between the perturbative and non-perturbative contributions into the amplitudes. Namely, for low momenta the coupling becomes large and we have more and more contributions from the many-legs Reggeon diagrams with complex topology. Also, is there a momenta which can fix the separation between the perturbative and low momenta contributions, i.e. there is a question about a value of the so called saturation momenta.
- Contributions of many-loops diagrams: Pomeron kernel is known till NNLO, check of the answers for the reggeized gluon kernel.

Effective action: what can we do?

- What can we do? Effectively account some type of the unitarity corrections (Balitsky hierarchy or Kovchegov equation for example), which, at least partially, accounts corrections to the ω. In general it leads to the calculations of correlators of Wilson lines operators instead correlators of the Reggeons.
- Try to "guess" the correct answer basing on some complimentary theories.
- Perform a numeric calculations of the non-truncated system of Dayson-Schwinger like equations (Balitsky hierarchy for example or the hierarchy obtained from the Lipatov's action).
- Lattice calculations– requires a reformulation of the problem in Euclidean space.

Effective action in Euclidean space: kinematics

- The usual four-momentum vectors in Minkowski space used for the description of the high energy scattering kinematics:

$$p_1^{\mu} = \frac{\sqrt{s}}{2}(1,1,0_{\perp}), \;\; p_2^{\mu} = \frac{\sqrt{s}}{2}(1,-1,0_{\perp})$$

$$s = (p_1 + p_2)^2 , \;\; t = (p_1 - p_1^{'})^2$$

- In the light cone coordinate frame these vectors can be rewritten as

$$p^{\mu}_{1\,L.C.} = \sqrt{\frac{s}{2}}(1,\beta,0_{\perp}), \;\; p^{\mu}_{2\,L.C.} = \sqrt{\frac{s}{2}}(\beta,1,0_{\perp})$$

with β as some regularization introduced in order to regularize the rapidity divergences of the corresponding integrals.

Effective action in Euclidean space: kinematics

- These vectors provides for the effective currents

$$\sqrt{\frac{2}{s}}\, p_{1\,L.C}^{\mu}\,\partial_\mu \stackrel{\beta\to 0}{\to} \partial_+ \,, \quad \sqrt{\frac{2}{s}}\, p_{2\,L.C.}^{\mu}\,\partial_\mu \stackrel{\beta\to 0}{\to} \partial_-$$

The Eikonal interaction term, correspondingly can be written in fully covariant form as

$$\frac{2}{s}\,\left(p_{1\,L.C.}^{\mu}\,\partial_\mu\, O(v_+)\right)\,\partial_\perp^2\,\left(p_{2\,L.C.}^{\nu}\,\partial_\nu\, O(v_-)\right)$$

or in the usual Minkowski coordinates as

$$\frac{2}{s}\,\left(p_{1}^{\mu}\,\partial_\mu\, O(v_+)\right)\,\partial_\perp^2\,\left(p_{2}^{\nu}\,\partial_\nu\, O(v_-)\right)$$

Now we are ready to perform a continuation of the effective action to the Euclidean space. Nevertheless, it is not easy to work with β as parameter performing the continuation for the arbitrary angle between particles trajectories in the Euclidean space.

Effective action in Euclidean space: kinematics

- Consider the hyperbolic angle between the light cone directions in the Minkowski space, and introduce the following vectors of the directions of the relativistic particles motion:

$$n_1^{\mu} = \frac{1}{\sqrt{2}}(1, \tanh(\gamma/2), 0_{\perp}), \;\; n_2^{\mu} = \frac{1}{\sqrt{2}}(1, -\tanh(\gamma/2), 0_{\perp})$$

- Correspondingly in the light cone coordinates we have:

$$n^{\mu}_{+\,L.C.} = \frac{1}{2}(1+\tanh(\gamma/2), 1-\tanh(\gamma/2), 0_{\perp}),$$

and

$$n^{\mu}_{-\,L.C.} = \frac{1}{2}(1-\tanh(\gamma/2), 1+\tanh(\gamma/2), 0_{\perp})$$

Effective action in Euclidean space: kinematics

- For the

$$p^2 = m^2$$

we have at high energy

$$\gamma \approx \ln(s/m^2)\,, \;\; \beta = m^2/s\,.$$

The Wick rotation of the vectors to the Euclidean space can be done now by

$$\gamma \to 2\imath\phi$$

continuation, where 2ϕ is an angle between trajectories of the two particles in the c.m.f. in Euclidean space.

- We obtain correspondingly

$$n^{\mu}_{1\,E} = \frac{1}{\sqrt{2}}\,(1, \imath\tan(\phi), 0_{\perp}),$$

$$n^{\mu}_{2\,E} = \frac{1}{\sqrt{2}}\,(1, -\imath\tan(\phi), 0_{\perp})$$

Effective action in Euclidean space: continuation

- The following transforms must be performed further as well :

$$\partial_0 \to \imath\, \partial_{4\,E}\,,\quad V_0^a \to \imath\, V_{0\,E}^a$$

and we obtain

$$n_1^{\mu}\,\partial_\mu \to \imath\, n^{\mu}_{+\,E}\,\partial_{\mu\,E}\,,\quad n_2^{\mu}\,\partial_\mu \to \imath\, n^{\mu}_{-\,E}\,\partial_{\mu\,E}$$

with

$$n^{\mu}_{+\,E} = \frac{1}{\sqrt{2}}\,(1, \tan(\phi), 0_\perp),\;\; n^{\mu}_{-\,E} = \frac{1}{\sqrt{2}}\,(1, -\tan(\phi), 0_\perp)$$

as vectors in the Euclidean space.

Induced action in Euclidean space

- Now the Eikonal interaction term in Minkowski space can be rewritten in Euclidean space as

$$
\begin{aligned}
- \quad & \imath \int d^4x \left(\left(n_1^{\mu}\, \partial_{\mu}\, O(v_+) \right) \partial_{\perp}^2 \left(n_2^{\nu}\, \partial_{\nu}\, O(v_-) \right) \right) \rightarrow \\
\rightarrow \quad & \int d^4x_E \left(\left(n^{\mu}_{+\,E}\, \partial_{\mu\,E}\, O_{E\,+} \right) \partial^2_{\perp\,E} \left(n^{\mu}_{-\,E}\, \partial_{\mu\,E}\, O_{E\,-} \right) \right)
\end{aligned}
$$

 with

$$
O(v_{\pm}) = P\, e^{g \int_{-\infty}^{1} d\lambda\, (n^{\mu}_{1,2}\, v_{\mu})} \rightarrow O_{E\,\pm} = P\, e^{\imath\, g \int_{-\infty}^{1} d\lambda\, (n^{\mu}_{\pm\,E}\, v_{\mu\,E})}, \quad v_{\mu\,E} = \imath\, T^a\, v_{\mu}^a\,.
$$

 The remaining part of the effective Lagrangian is continued to the Euclidean space as usual.

- We note, that the exponential is real in the case of the fundamental representation of the gluon field but remains imaginary for the adjoint representation of the field with $T^a_{b\,c} = -\imath\, f_{abc}$. In this case an another continuation in the Euclidean space must be performed.

Induced action in Euclidean space

- For the adjoint representation we define:

$$\partial_i \,\to\, -\imath\, \partial_{i\,E}\,,\;\; V_i^a \,\to\, -\imath\, V_{i\,E}^a\,,\;\; \partial_0 \,\to\, \partial_{4\,E}\,,\;\; V_0^a \,\to\, V_{4\,E}^a$$

The continuation of the action into the Euclidean space is still correct:

$$S \,=\, \int d^3x_i\, dt \left(\frac{1}{2}\, G_{0\,i}^a\, G_{0\,i}^a \,-\, \frac{1}{4}\, (\, G_{ij}^a\,)^2 \right) \,\to\, \imath\, S_E$$

The effective currents are real as well:

$$n_1^{\,\mu}\,\partial_\mu \,\to\, n_{+\,E}^{\mu}\,\partial_{\mu\,E}\,,\;\; n_2^{\,\mu}\,\partial_\mu \,\to\, n_{-\,E}^{\mu}\,\partial_{\mu\,E}\,;$$

$$- \quad \imath \int d^4x \left(\left(n_1^{\,\mu}\,\partial_\mu\, O(V_+) \right) \partial_\perp^2 \left(n_2^{\,\nu}\,\partial_\nu\, O(V_-) \right) \right) \to$$

$$\to \quad \int d^4x_E \left(\left(n_{+\,E}^{\mu}\,\partial_{\mu\,E}\, O_{E\,+} \right) \partial_{\perp\,E}^2 \left(n_{-\,E}^{\mu}\,\partial_{\mu\,E}\, O_{E\,-} \right) \right);$$

$$O_{E\,\pm} \,=\, P\, e^{\,g\, \int_{-\infty}^{1} d\lambda\, (n_{\pm\,E}^{\mu}\, V_{\mu\,E})}\,,\quad V_{\mu\,E} \,=\, f^a\, V_\mu^a$$

Lipatov's action in Euclidean space: first possibility

- In the case of the adjoint representation of the gluon fields we have the following expression for the action:

$$
\begin{aligned}
Z[J] &= \frac{1}{Z'}\int D\mathrm{v}\exp\Big(-\,S_{YM}[\mathrm{v}] + \frac{1}{2\,g^2\,C(R)}\int d^4x\,\Big(n_+^\mu\,\partial_\mu\,O_+\Big)\,\partial_\perp^2\,\Big(n_-^\mu\,\partial_\mu\,O_-\Big) - \\
&- \frac{1}{2\,g\,C(R)}\int d^4x\,J_-\,\Big(n_+^\mu\,\partial_\mu\,O_+\Big) - \frac{1}{2\,g\,C(R)}\int d^4x\,J_+\,\Big(n_-^\mu\,\partial_\mu\,O_-\Big)\Big)
\end{aligned}
$$

the external currents $J_\pm$ here are the auxiliary ones, we can take them zero at the end. Introducing Lipatov's operators

$$
\mathcal{T}_\pm = \frac{1}{g}\,n_\pm^\mu\partial_\mu\,O_\pm = \Big(n_\pm^\mu\,\mathrm{v}_\mu\Big)\,O_\pm
$$

we rewrite it as

$$
\begin{aligned}
Z[J] &= \frac{1}{Z'}\int D\mathrm{v}\exp\Big(-\,S_{YM}[\mathrm{v}] + \frac{1}{2\,C(R)}\int d^4x\,\mathcal{T}_+\,\partial_\perp^2\,\mathcal{T}_- - \\
&- \frac{1}{2\,C(R)}\int d^4x\,J_-\,\mathcal{T}_+ - \frac{1}{2\,C(R)}\int d^4x\,J_+\,\mathcal{T}_-\Big)
\end{aligned}
$$

Lipatov´s action in Euclidean space: second possibility

- With the help of some auxiliary fields $A_{\pm}$, in Minkowski space these fields are Reggeon, we write the same generating functional as:

$$
\begin{aligned}
Z[J] \;=\;& \frac{1}{Z'}\int D\mathsf{v}\, DA \exp\Big(- S_{YM}[\mathsf{v}] - \frac{2}{C(R)}\int d^4x\, A_+(x)\,\partial_\perp^2\, A_-(x) + \\
+\;& \frac{1}{C(R)}\int d^4x\, \mathcal{T}_+\,\partial_\perp^2\, A_- + \frac{1}{C(R)}\int d^4x\, \mathcal{T}_-\,\partial_\perp^2\, A_+ + \frac{1}{C(R)}\int d^4x\; J_-\, A_+ + \\
+\;& \frac{1}{C(R)}\int d^4x\, J_+\, A_- - \frac{1}{2\,C(R)}\int d^4x\, J_+\, (\partial_\perp^2)^{-1}\, J_- \Big)
\end{aligned}
$$

- The classical equations of motion

$$
(\,D_\mu\, G^{\mu\nu}\,)_a \;=\; \partial_\mu\, G_a^{\mu\nu} \;+\; g\, f_{abc} \mathsf{v}_\mu^b\, G^{c\,\mu\nu} \;=\; j_a^\nu
$$

are the same in case of both representation if in the fundamental one we redefine the Reggeon fields as

$$
A_+ \;\to\; \imath\, A_+\,,\quad A_- \;\to\; \imath\, A_-
$$

Lipatov's action in Euclidean space: what we can do

- We can calculate an energy dependence of the correlators of Reggeon fields. This is BFKL approach to the high energy processes. There are also complimentary theories (supersymmetry, quantum spectrum curve) which determine the many form of the reggeized gluon kernel in both perturbative and non-perturbative regimes, it is interesting to compare these results with the numerical ones. There are also interesting questions about the behavior of the theory in respect to the rapidity space divergences.
- We can calculate an energy dependence of the correlators of the Wilson lines operators, by redefinition of the auxiliary current

  $$J_{\pm}\,\partial_{\mu}\,O_{\mp}\;\rightarrow\;-\,O_{\mp}\partial_{\mu}\,J_{\pm}$$

  we obtain a generating functional for the Wilson lines operators. The correlators are main subject of interest in the Balitsky-Kovchegov-JIMWLK approach to the high energy scattering processes.
- The energy dependence in the calculations of the correlators, i.e. interpolation of their s dependence by the dependence of the Euclidian amplitudes on the scattering angle ϕ, it is an interesting question from the point ofvies of unitarization of the approaches. Also, it could be interesting to calculate the real physical amplitudes in the Regge kinematics.

Lipatov's action in Euclidean space: instanton contribution

- The gluon fields in Euclidean space we can write as as classical solution of homogenous equation plus classical solution of non-homogenous equation (the Reggeon solution) plus fluctuations around it:

$$
\begin{aligned}
V_4 &= v_4^{cl} + \varepsilon_4 = V_4^{inst} + V_4^{cl}(A_+,\, A_-,\, V^{inst}) + \varepsilon_4 \\
V_i &= v_i^{cl} + \varepsilon_i = V_i^{inst} + V_i^{cl}(A_+,\, A_-,\, V^{inst}) + \varepsilon_i
\end{aligned}
$$

Here the dependence of the $V_{4\,i}^{cl}$ classical Reggeon solution on the instanton fields will arise from the NLO of the perturbation theory due the non-linearity of the equations of motion.

- The generating functional:

$$
\begin{aligned}
Z[V^{inst},\, A_+,\, A_-] &= \frac{1}{Z'}\int D\varepsilon \exp\Big(- S_{YM}[V] - \frac{2}{C(R)}\int d^4x\, A_+(x)\, \partial_\perp^2\, A_-(x) + \\
&+ \frac{1}{C(R)}\int d^4x\, \mathcal{T}_+(V)\, \partial_\perp^2\, A_- + \frac{1}{C(R)}\int d^4x\, \mathcal{T}_-(V)\, \partial_\perp^2\, A_+ \Big)
\end{aligned}
$$

This functional effectively determines the vertices of interactions of $A_\pm$ fields with the instanton fields in the framework of high energy Euclidean QCD RFT.

Lipatov's action in Euclidean space: summary

- There is a possible implementation of the numerical (lattice) calculation of the Reggeon's or Wilson lines correlators in the framework with the Euclidean action that will allow to trace their high energy behavior in Minkowski space. These non-perturbative calculations of the high energy asymptotic behavior of the different correlators with the unitarity corrections included are important due the role of the BFKL calculus in the high-energy QCD.
- There is a definition of the correct interaction vertices of the correlators of $A_{\pm}$ fields with the instantons. An integration of the effective action with respect to the classical instanton fields will provide instanton induced corrections to the Reggeon fields correlators. An opposite is also true, there is a possibility to define a contribution of the Reggeon fields into the instanton solutions by initial integration of the Reggeon fields in the generating functional.
- Clarification of important problems of boundaries of the approaches in respect to the perturbative and non-perturbative contributions and low-momenta behavior of the amplitudes.
- Calculation of full scattering amplitudes in high-energy Regge kinematics.