Reasoning processes in mathematics education and its connection to natural sciences – A theoretical paper

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Abstract. In this paper, school mathematics is interpreted as empirical theories with the help of scientific structuralism. The focus is on reasoning processes that show similarities to reasoning in the natural sciences. This is discussed on the basis of theoretical considerations and explicated by means of a mathematics textbook example.

1 Introduction

This paper deals with reasoning processes of students in an empirical mathematics education. This is characterized by the development of mathematical knowledge on the basis of empirical objects (e.g., figures drawn on paper in geometry or graphs of functions in calculus). This linking of mathematical knowledge to empirical objects is typical for school mathematics and can be observed in many mathematics textbooks. It is a central distinguishing feature from formalistic mathematics at universities ([1], [2]).

The assumption is that students develop an empirical belief-system of mathematics in such a context, where mathematical concepts are ontologically bound in a way similar to concepts in the natural sciences. Thus, student knowledge developed in this sense can be described as empirical theories with the help of scientific structuralism ([3]).

2 Justification in an empirical science – a structuralistic view

The discovery and justification of a theorem within an empirical theory can be simplified into three (non-temporal) phases according to scientific structuralism. In the first phase, hypotheses are developed in an exploratory manner. A hypothesis is an assumed relationship whose validity has not yet been justified. The justification then takes place through two additional phases. In the phase of knowledge explanation, the description of the hypothesis takes place within the framework of the underlying theory. The previously assumed connection is traced back to the fundamental and special laws of the theory by deduction in the context of a proof. In this way, the hypothesis is related to already known knowledge. However, this does not yet confirm an empirical statement. In the phase of knowledge validation, the hypothetical correlations are verified experimentally. The adequacy of a theorem for the description of a certain empirical phenomenon can only be inductively concluded by measuring corresponding correlations between empirical objects of the intended applications within the framework of an experiment (see also [4]).

Similar considerations can be found in many other models for describing knowledge development processes in the natural sciences, such as Galileo Galilei's experimental method or Einstein's EJASE model ([5]).

3 Justification in mathematics at school

As described at the beginning, school mathematics is described in this paper as empirical theories. The phases of hypothesis generation, knowledge explanation, and knowledge validation described in the previous section can also be found in the mathematics classroom. In this regard, we consider an excerpt from a German mathematics textbook on the sum of interior angles in a triangle (see Fig. 1) [4].

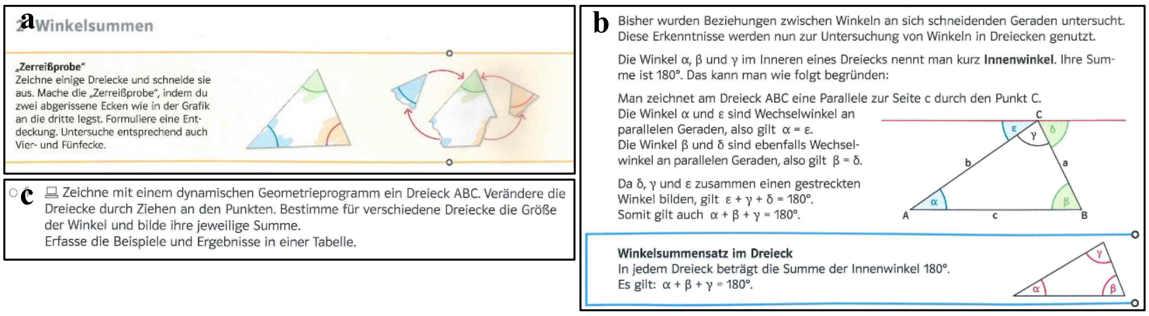

Fig. 1: Excerpt of the German textbook Lambacher Schweizer ([6])

At the beginning of the chapter on the sum of interior angles in a triangle, an experiment is described (Fig. 1, a) which the students are supposed to carry out. For this, several different triangles are drawn on a sheet of paper and cut out. Two vertices of each triangle are to be torn off and placed against the third. A discovery is to be formulated. This is followed by a short justification text with a sketch in which the parallel to one side through the opposite vertex of the triangle has been drawn (Fig. 1, b). The resulting angles to the triangle sides are alternate angles and consequently equal. Thus, the angle sum theorem is deductively traced back to the alternate angle theorem. The angle sum theorem in triangles is finally formulated in a box. Students are then asked to draw a triangle using dynamic geometry software (Fig. 1, c). By dragging the points, a variety of example triangles can be examined in a short time. The sizes of the individual angles and their sum should be recorded in a table.
If one describes this procedure with the background of an empirical theory (which was probably not aimed by the textbook authors), one recognizes first the phase of hypothesis formation, in which the students can also develop initial ideas for justification. This is followed by the knowledge explanation which means the tracing back to already known knowledge (alternate angle theorem). Finally, the knowledge is validated by systematically comparing a large number of triangles in an experiment.

4 Conclusion

The paper could show that the reconstruction of school mathematics as empirical theories provides interesting insights into reasoning processes in mathematics classes. Several empirical studies could confirm the importance of hypothesis formation, knowledge explanation, and knowledge validation for mathematical learning processes (e.g., [4]). If one takes these results seriously, one should consciously provide students with opportunities for corresponding activities in mathematics classes ([7]).

References

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